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*essentials of*

# **VECTOR & PHASE ANALYSIS**

*by* ALAN ANDREWS



A basic textbook on the principles  
necessary for a thorough knowledge  
of AC circuit operation.

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# Essentials of Vector and Phase Analysis

by  
Alan Andrews



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ESSENTIALS OF VECTOR AND  
PHASE ANALYSIS

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# Preface

When working with AC circuits, the phases of the currents and voltages must be considered. This applies both to analyses of circuit operation and to calculations involving the various characteristics of AC circuits. This volume reviews the fundamentals of AC, with special emphasis on phase relationships. Included is an extensive coverage of trigonometry and vectors, the basic tools for AC calculations. AC network calculations are thoroughly presented and explained, followed by examples of how vectors can be used in analyzing phase relationships and circuit operations.

The discussion of AC in this book is necessarily different from others, concentrating on the specific aspects of phase angles and phase relationships. For some readers the information in the first two chapters may be understood well enough to omit them in this study. However, other readers may want a more comprehensive review of the fundamentals of both electronics and mathematics. Such readers are referred to the two-volume set, *Electronics Math Simplified*. For the average reader, however, the information contained in this volume should provide sufficient background.

This text has been prepared to provide a deeper understanding of AC circuits, clearly explaining how to use the vector approach to simplify circuit analyses that may otherwise involve long and tedious calculations.

ALAN ANDREWS

May, 1963

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## CHAPTER 1

# Alternating Current and Phase

Several problems are encountered when a transition is made from DC to AC circuits. For a given DC circuit the applied voltage usually retains the same polarity, and in many cases it also has the same amplitude. If the voltage does change, the current changes in step with it. Applying AC to a circuit introduces three new problems:

1. The AC voltage periodically reverses polarity.
2. The voltage is continually changing in amplitude.
3. The current and voltage may not always change in step with each other.

When the circuit includes only resistance to oppose the flow of electrons, the AC circuit calculations are not much different from those for DC circuits. When inductance, capacitance, or both are introduced, additional factors must be considered. Changes of current in inductive or capacitive circuits do not occur in step with changes of applied voltage; therefore the characteristic of phase must be considered. The word phase has several different meanings, but the one used in connection with electrical circuits has to do with current or voltage changes with reference to some other change or to some fixed reference.

Let's clarify this definition in order to understand the use of the term more fully. If current and voltage change in step, or

synchronization, with each other in an AC circuit, they are said to be in phase. When the changes do not occur in step with each other, the current and voltage are said to be out of phase. Usually such an expression will also indicate how far out of phase they may be (expressed in some number of degrees), and it may also indicate which is ahead of the other.

A full cycle of AC is considered to occupy  $360^\circ$  of change along a time axis. That is, each cycle can be divided into 360 equal parts of time, during which the voltage or current undergoes one complete set of variations. A cycle normally consists of changes from zero to maximum amplitude, back to zero, to maximum amplitude with opposite polarity, and then back to zero again. Each cycle consists of two alternations (or half-cycles), one with positive polarity and the other negative. Each alternation occupies  $180^\circ$  along the time axis.

The term phase can be used to denote the instantaneous position of voltage or current with reference to the complete AC cycle, beginning at zero and ending at  $360^\circ$ . For example, the voltage is at the phase angle of  $90^\circ$ ; or, as previously indicated, phase can be used to express a time difference (in terms of degrees) between changes of current and voltage, and it can also be used to express the difference between two currents or between two voltages. These ideas of phase will be further clarified in later portions of the text with reference to particular circuits and circuit conditions.

Vectors are graphical tools which can be used to represent phase relationships. As will be shown more clearly later, the vector not only represents a quantity in a given problem, but it represents an angle or direction as well. In addition to their use with electric circuits, vectors can be used in a variety of other applications—for example, with various types of forces. After reviewing some other ideas relating to alternating current and some ideas relating to angular measurement, we will investigate how vectors can be used in analyzing AC circuits.

## SINE WAVES

The basic method of generating an AC voltage is to move a conductor through a magnetic field or to move the field past the conductor. When the motion is in one direction, a voltage of a certain polarity is induced. When the movement is reversed, the polarity of the voltage is also reversed. As an example we can use the simplified diagrams of a generator shown in Figs. 1-1A and 1-1B, correlating its rotation with the sine-wave output of Fig. 1-1C.

In Fig. 1-1A the single loop of wire is in a vertical position as it rotates in the horizontal permanent-magnet field. As it passes through the vertical position, the motion is parallel to the magnetic lines of force, and the induced voltage is zero. This value of induced voltage corresponds to the zero position of the sine-wave diagram (Fig. 1-1C). As the rotation continues, the induced voltage increases until the loop is as shown in Fig. 1-1B. At this instant, the motion of the loop is perpendicular to the direction of the lines of force so that maximum voltage is induced. This voltage corresponds to the 90° position on the sine wave with the polarity of the output voltage as indicated on the coil leads.

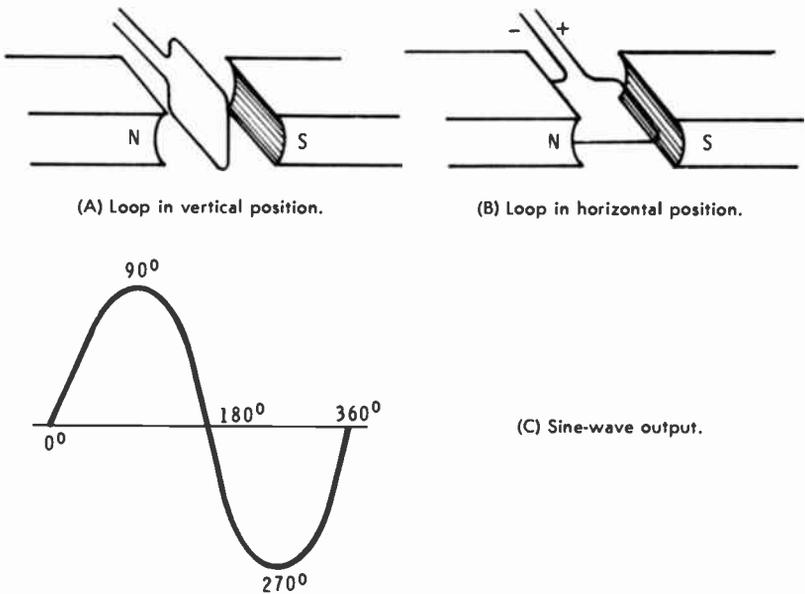


Fig. 1-1. Simplified diagrams of a generator and its sine-wave output.

The voltage decreases to zero when the coil once again reaches the vertical plane. This corresponds to 180°; however, even though the coil is vertical, the leads have been reversed from the position at 0°. On the next quarter-turn the coil produces maximum voltage as it reaches the horizontal plane. But the leads are reversed from the positions shown in Fig. 1-1B so that the output is maximum, but with opposite polarity. This is the 270° point on the sine wave. After another quarter-turn the cycle is completed.

The basic waveform produced by the generator is that of a sine wave, which is also the basic waveform on which all others are based. A sine wave is one that follows a specific set of variations, those of the sine, a trigonometric function. That is, the amplitude of the wave at any point is proportional to the sine of that particular angle. By multiplying the maximum value of the voltage or current by the sine of the angle, the actual voltage at that instant can be determined. There are many, many different sets of possible variations, but the sine wave is a specific one, and has exactly the same shape in every instance. Notice also that the rise in voltage in Fig. 1-1C is not linear; for example, half of the maximum amplitude occurs at  $30^\circ$  in the cycle, but it requires  $60^\circ$  to rise the other half. Maximum voltage is reached at  $90^\circ$ , after which the drop in voltage is symmetrical to the rise which preceded it. In  $60^\circ$  the voltage drops to half the maximum; then only  $30^\circ$  are required in which to drop the remaining half, decreasing to zero at  $180^\circ$ .

One half of a sine wave does not have the shape of a half-circle, as is sometimes depicted in printed matter. Nor are there any straight-line portions, even though the curve comes more nearly to a straight line each time it passes through zero amplitude. This has to do with the rate of change of the waveform on which the phase difference sometimes depends. The rate of change is not linear at any point, but it is continually changing. Maximum rate of change (or slope) occurs each time the wave passes through  $0^\circ$  or  $180^\circ$ ; that is, the change of voltage per unit of time is greatest at these points. The rate of change is minimum at the maximum-amplitude points on the curve,  $90^\circ$  and  $270^\circ$ . At these points the rate of change is zero because the wave reverses direction at these points. From  $270^\circ$  on one cycle to  $90^\circ$  on the next the voltage is changing from maximum negative to maximum positive. Mathematically this is a positive slope, or rate of change. From  $90^\circ$  to  $270^\circ$  within the same cycle there is a negative slope, or rate of change as the voltage goes from maximum positive to maximum negative. These are important points in analyzing how inductance and capacitance act in AC circuits. There are other characteristics of sine waves which are also important in analyzing AC circuit action. One of these is frequency. If the speed of rotation of the coil in Fig. 1-1A were increased, more sine waves would be produced in any given unit of time. Frequency of an AC wave is defined as the number of cycles that occur during one second of operation, hence the basic unit is cycles per second (cps). If more cycles are produced each second, the time duration of each single cycle is de-

creased proportionally. Frequency and time are inversely proportional as shown by the relationships:

$$t = \frac{1}{f} \qquad f = \frac{1}{t}$$

where,

frequency is in cycles per second,  
time is in seconds.

For example, with a 60-cps waveform each cycle has a time duration of 1/60 second. But if the frequency were doubled, the time of each cycle would be halved. However, each cycle is still divided into 360°, regardless of the time required for the cycle to occur. When frequency is increased, each degree represents a smaller unit of time, and vice versa. In some cases the time duration of a cycle is referred to as its period.

The average amplitude of a sine wave is zero because the positive alternation is exactly like the negative, except for polarity. This does not mean that there is zero voltage, although a DC voltmeter might indicate zero. It only means that the average amplitude over a full cycle is zero. Average amplitude of any one alternation is 0.636 of the peak (maximum) value. The effective value of an alternation is the square root of the average of the instantaneous values and is also called the root-mean-square value (rms). Numerically, the rms value is equal to 0.707 of the peak value of the wave.

The instantaneous value (the amplitude at any instant of time) can be determined by multiplying the peak value of the wave by the sine of the angle for that instant of time. And finally, the peak value may be determined by multiplying the rms value by 1.41, or the average value of one alternation by 1.57. It should be stressed, however, that these relationships are true only for sine waves and not for any other wave shape that might occur. Also, these relationships are for sine waves at any frequency.

In some cases the peak-to-peak value of a signal may be of interest. This is the total voltage swing, from maximum positive to maximum negative (or vice versa), and it is equal to twice the peak value of an alternation. This relationship is generally true regardless of the waveshape of the signal in question. All these relationships and the conversion factors are summarized in Table 1-1.

As previously stated for an AC circuit having a resistive load, the current and voltage are in phase with each other. That is, the changes of waveform occur in step, even though the amplitudes and units of measurement are different. This

**Table 1-1**  
**Voltage Relationships and Conversion Factors**

GIVEN	PEAK TO PEAK =	PEAK =	RMS =	AVERAGE =
PEAK TO PEAK		$0.5 \times P \text{ to } P$	$0.3535 \times P \text{ to } P$	$0.318 \times P \text{ to } P$
PEAK	$2 \times \text{PEAK}$		$0.707 \times \text{PEAK}$	$0.636 \times \text{PEAK}$
RMS	$2.82 \times \text{AVERAGE}$	$1.41 \times \text{RMS}$		$0.9 \times \text{RMS}$
AVERAGE	$3.14 \times \text{AVERAGE}$	$1.57 \times \text{AVERAGE}$	$1.11 \times \text{AVERAGE}$	

condition is illustrated in Fig. 1-2A. When reactive components (L or C) are also included in the circuit, the current and voltage are out of phase by some number of degrees. For an inductive circuit the voltage peaks occur ahead of the current peaks (Fig. 1-2B) because inductance tends to oppose a change of current. Using voltage as the reference, current is said to be lagging the voltage. With capacitive circuits the opposite conditions hold true, and current leads the voltage. If the labeling of the waveforms in Fig. 1-2B were reversed, it would represent a capacitive circuit. In the remainder of this chapter it will be shown how these phasing conditions are caused.

The term phase angle (usually  $\theta$ ) represents the angular difference between current and voltage; it may also be referred to as phase difference. However, before proceeding, several possible causes for confusion should be pointed out. First, notice the relationships in Fig. 1-2B. The horizontal axis is calibrated in terms of time; thus, as shown, the positive peak of voltage occurs at an earlier time than the positive peak of current. So voltage is leading the current, or conversely, current is lagging the voltage. The confusion here arises because, when looking at the waveforms, the current seems to be out

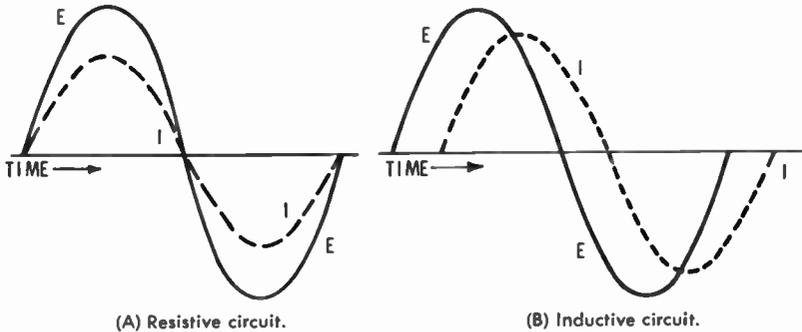


Fig. 1-2. Voltage and current phase relationships in resistive and inductive circuits.

ahead of the voltage. Erroneous results are obtained from this type of analogy. And a final warning, the term phase has meaning only when the frequencies of the waveforms being compared are the same. Otherwise, the time differences are continually changing, and a phase condition cannot be stated definitely. When frequencies are the same, the time difference, and thus the phase angle, are the same at any points of comparison. If the relative phases were shifting, even though the frequencies were still the same, it would be equally difficult to state a definite phase relationship.

### INDUCTIVE CIRCUITS

A coil of wire has the tendency to oppose any change of current through it. This occurs because the coil builds up a counter-electromotive force (CEMF) in opposition to the applied voltage. Opposing a change of current means that if there is no current through the coil, there is opposition in trying to start a current. But if there is current already flowing, the tendency of the coil is to keep that current flowing, even though the excitation voltage may have been removed. Also, a coil opposes any existing increase, or decrease, of current.

The CEMF is built up only during the time when the current through the coil is changing in value. The actual voltage induced is directly proportional to the rate of change of the current. When DC is applied, the current builds up to the maximum value some time after the voltage is applied. At the first instant the voltage is applied current is zero, since the CEMF equals the applied voltage. Total voltage affecting the circuit at any instant is the applied voltage minus the CEMF. So when the latter is high, the total circuit voltage is low, causing low current. As the applied voltage stops changing in value, the induced CEMF decreases, allowing the current to increase to a higher value. The current increases at an exponential rate, as shown in Fig. 1-3.

The time required for the current to increase in an inductance is determined by the ratio of inductance (L) to the resistance (R) of the coil.

$$TC = \frac{L}{R}$$

where,

TC is the time constant in seconds,

L is the inductance in henrys,

R is the resistance in ohms.

During one time constant the current increases to approximately 63.2% of the maximum value. In the next time constant the current increases by 63.2% of the remaining change (63.2% of 36.8% = 23.3%). So at the end of the second time constant the current has increased to 63.2% plus 23.3%, or 86.5%, of the maximum value obtainable. Theoretically the current never reaches 100% of the maximum possible value, but in a practical sense it can be assumed that the maximum is reached in five time constants. The percentage at the end of each time constant is given in Table 1-2, using only the left side. During each time constant the current changes by 63.2% of the remaining possible change.

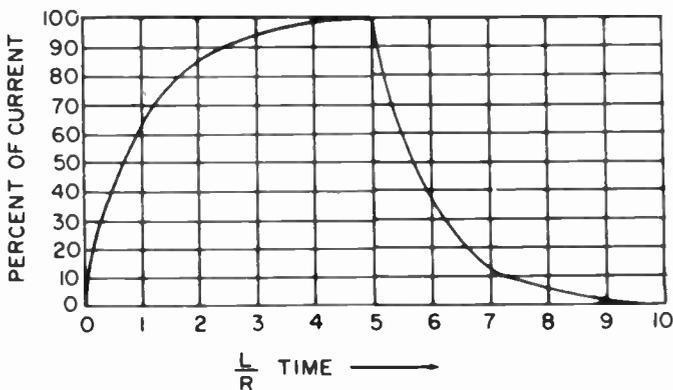


Fig. 1-3. Current increases exponentially in an inductive circuit.

As an example, suppose that a particular coil has an inductance of 10 henrys and a resistance of 100 ohms.

$$TC = \frac{10}{100} = 0.1 \text{ second}$$

If the applied voltage is 100 volts, at 0.1 second after the DC voltage is applied the current increases to 63.2% of the maximum value.

$$\text{Maximum current} = \frac{100V}{100\Omega} = 1 \text{ ampere}$$

So at the end of one time constant the current is 63.2% of 1 ampere, or 0.632 ampere. At the end of the second time constant the current will rise to 0.865 ampere, etc., until the maximum is approximately reached at the end of five time constants. Notice that at that time the actual current is 0.993

**Table 1-2**  
**Percentages of Current Increase and Decrease**  
**at the End of Each Time Constant**

CURRENT INCREASE		CURRENT DECREASE	
AT END OF	CURRENT REACHES	AT END OF	CURRENT REACHES
1 TC	63.2%	1 TC	36.8%
2 TC	86.5%	2 TC	13.5%
3 TC	95%	3 TC	5%
4 TC	98.2%	4 TC	1.8%
5 TC	99.3%	5 TC	0.7%

ampere, which is maximum for all practical purposes. And even if the time were extended to ten time constants, the additional rise would be only a very small amount.

If, after the current has reached maximum, the applied voltage were disconnected, a CEMF would again be induced. But this time the polarity of the CEMF is opposite, but again opposing a change of current. The tendency here is to keep the current flowing, but since the CEMF cannot remain indefinitely, the net result is that the current is delayed in its decrease to zero. The rate of decrease is the same as the rate of increase, and the actual time involved depends on the L/R time constant. Referring to Table 1-2 (the right side), it can be seen that at the end of 1 time constant of decrease the current has dropped 63.2% of the total amount of which the drop can be. Therefore, at the end of 1 time constant the current has decreased to 36.8% of the maximum value. Then, the current drops to zero, for all practical purposes, at the end of five time constants.

For purposes of using vectors and phase, one important point from this explanation can be learned. That is, in an inductive circuit current changes lag voltage changes by some amount of time. The current changes after the voltage does, or we can say simply that current lags voltage. Or, if you prefer, we could say that voltage is leading the current.

With DC applied the time of lag is determined by the L/R time. However, with an AC sine-wave input to a pure inductance the angle of lag is always  $90^\circ$ , as indicated in Fig. 1-4. This lag of  $90^\circ$  holds for any frequency. The actual lag time is inversely proportional to the frequency. For example, if the frequency is increased, the time duration of each cycle is de-

creased so that the lag time is also decreased, even though it is still  $90^\circ$  of the operating frequency.

Any practical circuit also contains resistance in which the current and voltage are in phase with each other. In an LR circuit the actual phase angle is always less than  $90^\circ$  but greater than  $0^\circ$ ; the actual amount depends on the relative values of the circuit components. This also means that any calculations involving an inductive circuit must consider the phase in order to obtain a correct solution.

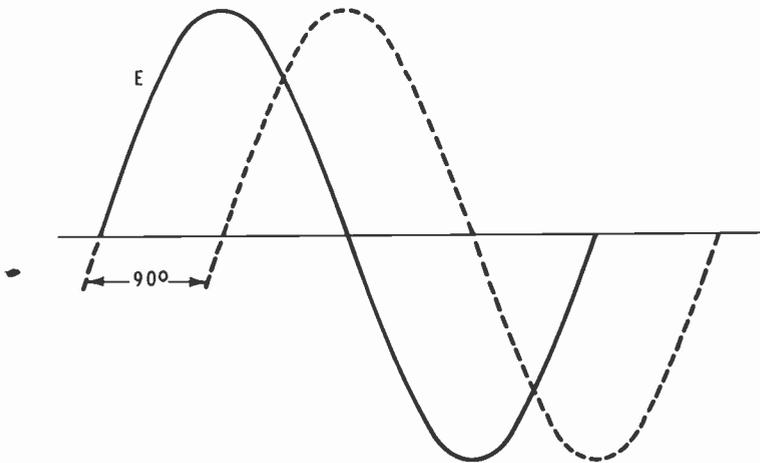


Fig. 1-4. The angle of lag in a pure inductance circuit is always  $90^\circ$ .

The opposition that inductance exhibits in an AC circuit is called inductive reactance ( $X_L$ ) and can be determined by:

$$X_L = 2\pi fL$$

where,

$X_L$  is the inductive reactance in ohms,  
 $f$  is the frequency in cycles per second,  
 $L$  is the inductance in henrys.

This means that circuit opposition is directly proportional to operating frequency and also to inductance. If either, or both, are increased, the inductive reactance is also increased.  $X_L$ , however, acts at  $90^\circ$  with reference to resistance so that  $X_L$  and  $R$  cannot be added arithmetically, for example in a series circuit. They must be added vectorially, which creates another need for a knowledge of vectors and phase for AC circuit calculations. These vectorial methods are investigated in later chapters.

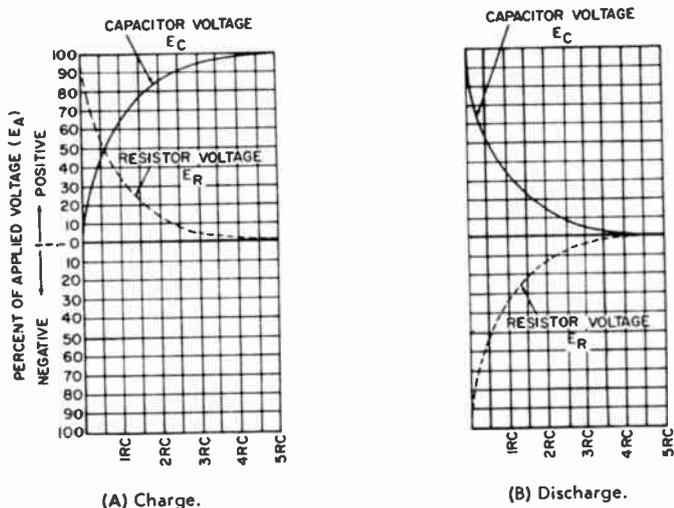


Fig. 1-5. Charge and discharge curves of an RC circuit.

### CAPACITIVE CIRCUITS

A capacitor has the tendency to oppose any change of voltage across it, and in some ways it can be said to act oppositely from inductance. If a DC voltage is applied to a capacitive circuit, the current through the circuit is maximum at the first instant, but there is zero voltage across the capacitor. Then the capacitor accepts voltage at an exponential rate. As the voltage across the capacitor increases, it opposes the applied voltage, causing current to decrease. As soon as the voltage across the capacitor equals the applied voltage, the circuit current is decreased to zero. The rise in capacitor voltage is at the same exponential rate as the rise in current in an inductive circuit. Then if the applied voltage is cut off, the capacitor acts as a source of voltage, creating maximum current at the first instant. As the capacitor loses voltage at the exponential rate the current decreases at the same rate until both reach zero. These changes are shown in graphic form in Fig. 1-5.

The time required for these changes to occur is determined by the values of resistance  $R$  and capacitance  $C$  in the circuit.

$$TC = R \times C$$

where,

$TC$  is the time constant in seconds,

$R$  is the resistance in ohms,

$C$  is the capacitance in farads.

**Table 1-3**  
**Percentage of Voltage Increase and Decrease**  
**Across a Capacitor at the End of Each Time Constant**

VOLTAGE INCREASE		VOLTAGE DECREASE	
AT END OF	VOLTAGE REACHES	AT END OF	VOLTAGE REACHES
1 TC	63.2%	1 TC	36.8%
2 TC	86.5%	2 TC	13.5%
3 TC	95%	3 TC	5%
4 TC	98.2%	4 TC	1.8%
5 TC	99.3%	5 TC	0.7%

Just as with inductance, the time constant is the time required for 63.2% of the total change to occur. For a series RC circuit the voltage across the capacitor rises to 63.2% of the applied voltage during one time constant. During the same time the current decreases to 36.8% of the maximum value. Other values are given in Table 1-3, and just as with inductance the total change for all practical purposes is assumed to occur in the time of five time constants. Also shown in this Table are the percentages of capacitor voltage and current for each time constant of capacitor discharge.

These actions show that, for a capacitor, the current occurs ahead of the voltage. The time difference is determined by the product of R and C. Therefore, for a capacitive circuit the current leads the voltage. When a sine wave of voltage is applied, instead of DC, the phase angle between current and voltage is 90°, assuming that circuit resistance is zero. This 90° phase difference holds for any frequency, and the time represented by the 90° is one-fourth of the time of one cycle at the operating frequency. For example, if frequency were increased, 90° of a cycle would represent a shorter period of time.

In a practical circuit containing R and C the phase angle is less than 90° but greater than 0°, depending on the relative circuit values. The opposition offered by the capacitor is capacitive reactance ( $X_C$ ) and can be determined by:

$$X_C = \frac{1}{2\pi fC}$$

where,

$X_C$  is the capacitive reactance in ohms,  
 f is the frequency in cycles per second,  
 C is the capacitance in farads.

The reactance is inversely proportional to both the frequency and the capacitance. If, for example, either, or both, were increased, the capacitive reactance would decrease.  $X_C$  acts at an angle of  $90^\circ$  as compared to  $R$ ; therefore in any RC circuit calculation the phase must be considered. But with a capacitor the current leads the voltage by  $90^\circ$ ; whereas with an inductor the current lags by  $90^\circ$ . This means that inductive and capacitive currents are  $180^\circ$  out of phase with each other. And, as will be seen later, in some circuits their effects directly oppose each other. A combination RCL circuit brings in other uses of vectors and phase.

Harmonics are whole-number multiples of some fundamental frequency, and any waveshape other than a sine wave is composed of a number of sine waves at different frequencies, phases, and amplitudes. For example, a square wave is composed of a fundamental sine wave plus sine-wave components at odd-number multiples of the fundamental frequency. The fundamental in any wave is the lowest frequency contained and is also called the first harmonic. Two times the fundamental is the second harmonic frequency, etc. In music these harmonics are often referred to as overtones.

With regard to phasing, the complex waveshapes introduce additional problems because the actual time of lead or lag is different for each frequency. This is caused by two things. First,  $90^\circ$  of a cycle represents a different time duration for every frequency, decreasing as frequency is increased. Second, the values of  $X_L$  and  $X_C$  vary with frequency. This changes the magnitude of the phase angle, giving a different number of degrees of lead or lag for each separate frequency.

With this brief review of reactive circuits we have shown how the different phases originate, and why vectors and phase must be considered to be extremely valuable tools in solving or analyzing AC circuits. Subsequent chapters pursue the subject further, covering the various phase and vectorial methods needed.

### PRACTICE PROBLEMS

1. How many degrees are contained in a complete cycle of rotation?
2. A vector rotates two and a quarter revolutions. How many degrees does this include?
3. A signal has a frequency of 500 cps. What is the time duration of each cycle?
4. If a signal has a period of 0.05 second, what is its frequency?
5. The rms value of a sine wave is 100 volts. What is the peak value? What is the peak-to-peak value?

6. An AC signal has a peak amplitude of 250 volts. What is the rms value? What is the average value of a single alternation?
7. By what factor would peak-to-peak voltage be multiplied in order to determine rms voltage?
8. An RL series circuit includes an inductance of 5 henrys and a resistance of 200 ohms. What is the time constant?
9. In this same circuit the peak current is 2 amperes. What is the current at the end of three time constants after a DC voltage is applied?
10. Will a coil discharge its field faster with a short across it or with its terminals open-circuited?
11. A 0.5-mfd capacitor is connected in series with 60,000 ohms. What is the time constant?
12. If a DC voltage of 200 volts is applied, what is the capacitor voltage after one time constant of charge?

## CHAPTER 2

# Angular Measurements

As already indicated in the preliminary discussion of phase, the subject is involved with angles and various characteristics of angles. And, as we shall see later, the study of angles leads to triangles and some study in trigonometry (trig). This chapter reviews the pertinent data from geometry and trig that may be necessary in working with vectors and phase in practical applications.

### ANGLES

An angle is formed whenever two straight lines meet at a point. This point is called the vertex of the angle, and the lines are referred to as the sides. Several such angles are shown in Fig. 2-1. The angles may be lettered, numbered, or identified in some other way, especially if two or more angles enter into a given problem. In the previous chapter,  $\theta$  (theta) was used to indicate the phase angle in an AC circuit. This designation is used quite often. The Greek letter  $\phi$  (Phi) is also used, as well as several other letters from the Greek alphabet. An angle could be lettered as angle A,  $\theta$ , or  $\phi$ , numbered as angle 1, or three letters could be used as in Fig. 2-1. Here each of the angles can be referred to as angle ABC. The middle letter usually indicates the letter at the vertex of the angle.

In the previous chapter it was considered that a full cycle of AC occupies  $360^\circ$  of angular position. This comes from the fact that a complete circle includes  $360^\circ$  of angular rotation, which value was an arbitrary choice. It is said that the circle was divided into 360 equal parts several thousand years ago when the basic year was considered to consist of 360 days. In subdividing the circle, that number of days was merely used for the number of subdivisions, called degrees.

Referring again to Fig. 2-1, notice that the angle in Fig. 2-1A is quite small as compared to the other two. This is an acute angle, a term used for any angle less than  $90^\circ$ . Fig. 2-1B shows a right angle, consisting of exactly  $90^\circ$ , or one-fourth of a

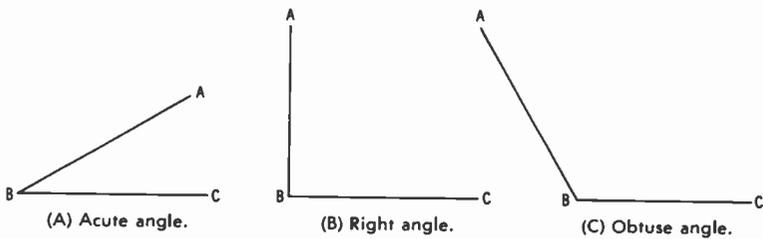


Fig. 2-1. Three different angles.

circle. An obtuse angle is shown in Fig. 2-1C, and this term is applied to any angle greater than  $90^\circ$ . An angle of  $180^\circ$  is called a straight angle, and it represents the condition of two vector quantities in direct opposition to each other.

When working with vectors and phase we can consider that angles can be generated by rotating a line about a vertex which becomes the center of a circle. Each time a line is rotated one complete revolution it has covered a full circle (or cycle), or  $360^\circ$ . It should be stressed that every circle contains  $360^\circ$ , no matter how large the circle may be. Also, the number of degrees in any angle depends only on the relative directions of the lines forming that angle, not on the lengths of the lines. For example, suppose we assume that the acute angle (Fig. 2-1A) includes  $30^\circ$ . This angle is still  $30^\circ$ , even if the lengths of the lines are shortened or lengthened. And for any angle, the sides are separated by greater distances as the point of measurement is moved out from the vertex, although the number of degrees in the angle remains the same.

Fig. 2-2 is still used to illustrate some additional ideas regarding rotation of lines and generation of angles. The angle is shown in what is known as standard position, with one side on the horizontal axis. Zero degrees is also standardized and

is shown, on the horizontal axis and to the right. The radius line (R) can be considered to be a vector since it has the fundamental characteristics of one. It expresses magnitude (length) and is drawn at some specific angle (direction).

The vector line can be rotated in either direction, but conventional notation assumes that counterclockwise rotation is positive; clockwise rotation is negative. And unless otherwise specified, we can normally assume positive rotation for the generation of angles, as was done in marking the angles in Fig. 2-2. For example, the  $90^\circ$  angle is one-quarter of a ro-

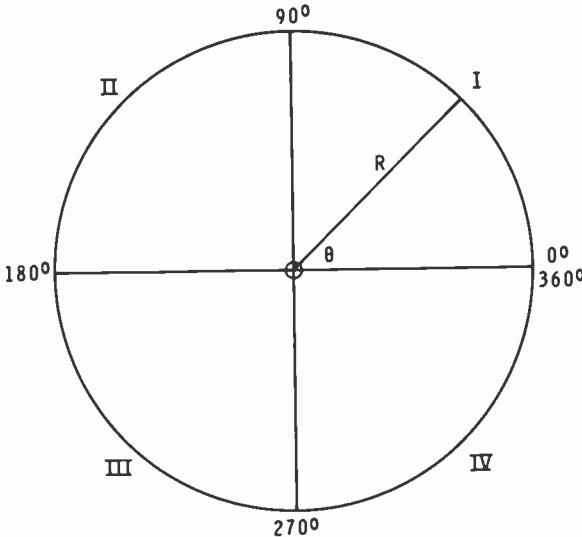


Fig. 2-2. Illustrating rotation of lines and generation of angles.

tation from the zero-degree reference line. With negative rotation of  $270^\circ$  (actually  $-270^\circ$ ) the vector would be at the same location as if it had rotated  $90^\circ$  in the positive direction. These may be referred to as counterpart angles, and the sum of their numerical values (neglecting the signs) is  $360^\circ$ . For example,  $135^\circ$  has the same vector location as  $-225^\circ$ , the only difference being that opposite rotations were used to arrive at the angle. And as a check,  $135^\circ + 225^\circ$  totals  $360^\circ$ . Rotation can be more than  $360^\circ$ ; for example, a rotation of  $720^\circ$  would include two complete revolutions.

The Roman numerals in Fig. 2-2 show the standard numbering of quadrants, the conventional name for quarter-circles (or  $90^\circ$ ). These are also numbered with reference to positive (CCW) rotation. The first quadrant extends from  $0^\circ$  to  $90^\circ$ ,

and the second quadrant from  $90^\circ$  to  $180^\circ$ . Third-quadrant angles range from  $180^\circ$  to  $270^\circ$ , and from  $270^\circ$  to  $360^\circ$  are fourth-quadrant angles. An angle is considered to be in the quadrant in which the generator vector lies. A  $60^\circ$  angle would be in first quadrant,  $135^\circ$  in the second quadrant,  $230^\circ$  in the third quadrant, etc., using these as examples. Be careful with negative angles, however, for example  $-45^\circ$  is in the fourth quadrant (not the first) because the quadrants are always numbered for positive rotation.

In the production of sine waves a vector is assumed to rotate continuously around the circle. For other vector applications, however, it may be assumed that the vector is stationary at some particular angle. Or, in the case of rotating vectors we may be interested in some specific angle in a given problem. In discussing angles there are several other terms which are used to some extent and which should be learned.

Two angles are said to be complementary when their sum is  $90^\circ$ . Thus,  $35^\circ$  and  $55^\circ$  are complementary, and each can be said to be the complement of the other. Two angles are supplementary when their sum is  $180^\circ$ . Thus,  $120^\circ$  and  $60^\circ$  are supplementary and each is the supplement of the other. Counterpart angles have already been mentioned; their sum is  $360^\circ$  and each is the counterpart of the other.

## UNITS OF MEASUREMENT

It has been mentioned that a circle can be divided into  $360^\circ$ , with each degree representing a specific amount of angle. Quite often, however, it is necessary to express angles in smaller dimensions, and this involves subdividing each degree into smaller units of angle. Two systems are in general use. What might be termed the older of the two systems divides each degree into 60 minutes and each minute into 60 seconds, thus dividing each degree into 60 times 60, or 3,600 subdivisions. This is referred to as the *sexagesimal system* because it is based on sixes. In this system, for example, an angle may be expressed as  $42^\circ 35' 10''$ , read as 42 degrees, 35 minutes, and 10 seconds.

The *decitrig system* is being used to a greater extent all the time, and in electronics it is used more than the sexagesimal system. This method of subdivision is based on the decimal system and divides each degree into tenths, or even hundredths or thousandths if necessary. Thus, an angle may be expressed as  $38.72^\circ$  in the decitrig system. This would be read as 38 and 72 hundredths of degrees. In the calculations in this book

whole-number degrees will be considered in all the example and practice problems. This may not always be true, however, in practical work, and it may be necessary to convert from one system of subdivision to the other.

In making these conversions several things should be kept in mind. Whole numbers of degrees are the same in both systems, and conversion is required only for the fractional parts of the angles. There are 60 minutes in each degree, meaning that for every tenth of a degree there are 6 minutes. To convert from decimal portions of degrees to minutes, multiply by 60, as in these examples.

$$\begin{aligned}0.6^\circ &= 0.6 \times 60 = 36' \\0.32^\circ &= 0.32 \times 60 = 19.2'\end{aligned}$$

To convert fractional parts of minutes to seconds (as in the previous example) multiply by 60.

$$\begin{aligned}0.2' &= 0.2 \times 60 = 12'' \\ \text{so, } 19.2' &= 19' 12''\end{aligned}$$

To convert from minutes to degrees divide by 60 as in this example.

$$42' = \frac{42}{60} = 0.7^\circ$$

To convert from seconds to minutes also divide by 60, for example:

$$\begin{aligned}48'' &= \frac{48}{60} = 0.8' \\42' 48'' &= 42.8' \\42.8' &= \frac{42.8}{60} = 0.71^\circ \text{ (rounded off)}\end{aligned}$$

therefore,

$$42' 48'' = 0.71^\circ$$

Another unit for measuring angles is the radian, which is the basic unit in what might be called circular, or pi, measurement. It is based on the relationship between the radius (R), which is half the diameter (D), and the circumference of the circle. The circumference is the circular distance around the outer edge of the circle, and the relationship is expressed by:

$$C = \pi D = 2\pi R$$

An angle of one radian is formed when the arc (part of the circle) between the two sides of the angle has a dimension equal

to the radius of the circle. This is illustrated in Fig. 2-3 where  $\theta$  is one radian of an angle. There are  $2\pi$  radians in a complete circle ( $360^\circ$ ) because the circumference is  $2\pi$  times the radius. Numerically one radian is equal to approximately  $57.3^\circ$ .

These conversion formulas can be used to change degrees to radians, or vice versa :

$$\text{radians} = \frac{\pi}{180} \times \text{degrees}$$

$$\text{degrees} = \frac{180}{\pi} \times \text{radians}$$

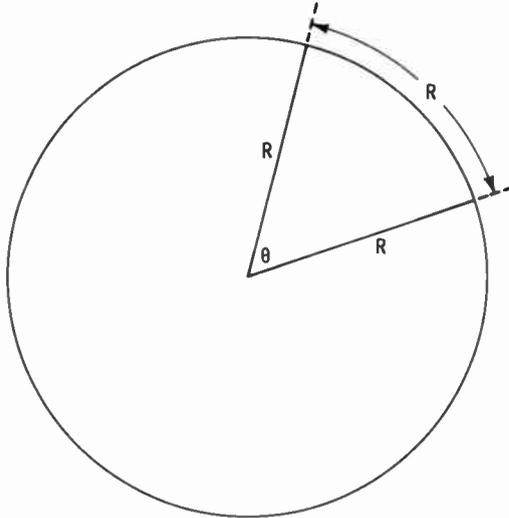


Fig. 2-3. An angle of one radian.

These examples show how these formulas can be used.

1. Convert  $210^\circ$  to radians.

$$\text{radians} = \frac{\pi}{180} \times 210 = \frac{7\pi}{6}$$

2. Convert  $\frac{3\pi}{2}$  radians to degrees.

$$\text{degrees} = \frac{180}{\pi} \times \frac{3\pi}{2} = 270^\circ$$

This circular, or pi, system is especially meaningful in determining angular velocity, hence the reactance of a coil or a capacitor. In covering a full circle a vector rotates through an

angle of  $2\pi$  (approximately 6.28) radians. The number of revolutions made per second is a function of the frequency ( $f$ ) expressed in cycles per second. Therefore  $2\pi f$  is the angular velocity of a vector expressed in radians per second. Because it occurs so often in electronics calculations, the Greek letter Omega ( $\omega$ ) is often used in place of  $2\pi f$ . As used in the reactance formulas:

$$X_L = 2\pi fL = \omega L$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{\omega C}$$

### TRIG FUNCTIONS

The study of trigonometry (trig) is concerned with the relationships of sides and angles in a triangle, which is any figure bounded by three sides which are straight lines. The trig functions express the numerical relationships of any two sides of a right triangle, which is any triangle containing one right ( $90^\circ$ ) angle. And in such a triangle the sum of the other two angles is  $90^\circ$  so that they are complementary.

In defining the trig functions the triangle in Fig. 2-4 should be used.  $\theta$  is the phase angle, O is the side opposite that angle, A is the side adjacent to it, and H is the hypotenuse, or diagonal, of the triangle. The three functions most often used are as follows:

$$\sin \theta = \frac{O}{H}$$

$$\cos \theta = \frac{A}{H}$$

$$\tan \theta = \frac{O}{A}$$

These are abbreviations of sine, cosine, and tangent, respectively, and each of these has a reciprocal function. These are cosecant, secant, and cotangent, respectively. But these reciprocal functions are not used to nearly the extent of the other three, and consequently they are not considered here.

It should be stressed that these functions are ratios of the lengths of sides and are different for each angle. For example, the sine of  $30^\circ$  is 0.5. This means that for any size triangle the side opposite the  $30^\circ$  angle is one-half the length of the hypotenuse. This is true for any lengths of lines, as long as the angle is  $30^\circ$ . The idea holds similarly for the other functions—they

are ratios, determined by the angle, not by the actual lengths of the sides.

Trig tables usually give sine, cosine, and tangent values for any angle from  $0^\circ$  to  $90^\circ$ . A set of decitrig tables is included in an appendix of this book. Sines and cosines are listed together; tangents and cotangents are listed together. To determine the sine of an angle, locate the angle in the left-hand column and then read the value of the function under the proper heading of the tenths columns. For cosines, use the degree listings in the right-hand column, then read the cosine in the proper tenths column. Notice that for cosines the tenths are listed at the bottom of the columns, not at the top as for sines. Also notice that the angle markings for cosines progress upward,

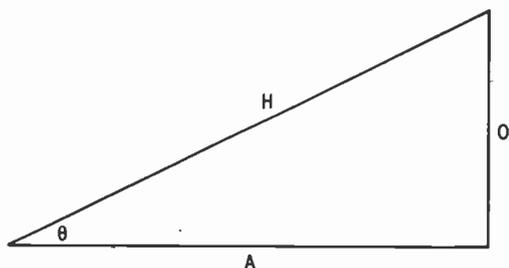


Fig. 2-4. Right triangle used in defining trig functions.

the tenths progress toward the left. The tables for tangents and cotangents are read similarly although the cotangent section may not be used so often.

Other trig tables may be arranged differently but the basic idea remains the same. Once the arrangement is understood, it is easy to see how other tables are made up and how to use them. You may check these examples to see whether or not your use of the tables is correct.

$\sin 21^\circ = .3584$	$\cos 54.2^\circ = .5850$
$\sin 38.5^\circ = .6225$	$\tan 24.8^\circ = .4621$
$\sin 64.3^\circ = .9011$	$\tan 72.4^\circ = 3.152$
$\cos 16.5^\circ = .9588$	$\tan 87.3^\circ = 21.2$
$\cos 39.7^\circ = .7694$	$\cot 21.4^\circ = 2.552$

These tables can be arranged this way because cofunctions of complementary angles are equal. Thus, one set of tables can be used for both a function and its cofunction. As an example, notice that the sine of  $34^\circ$  is the same value as the cosine of  $56^\circ$ .

And this holds true for any angles; a function of any angle is equal to the cofunction of  $90^\circ$  minus that angle. For vector analysis of AC circuits the sine function is used more often than any of the others. Notice that the sine starts at zero for  $0^\circ$  and progresses to 1 for  $90^\circ$ . It can never have a value greater than 1.

Sometimes in working with trig the value of a function may be known but not the angle. These are referred to as arc functions. For example:

$$\begin{aligned}\sin 30^\circ &= .5 \\ \text{arc sin } .5 &= 30^\circ\end{aligned}$$

The basic process is the same as in determining trig functions, except in reverse. In determining arc sin .3502 locate .3502 in the body of the table and read the angle on the same line in the left-hand column. Then read the number of tenths at the top of the column in which .3502 occurs. The same procedure is used for arc cosine or any of the other trig functions. For arc cosine, however, be careful to read the angle in the right-hand column and the tenths along the bottom of the tables. For example:

$$\begin{aligned}\text{arc cos } .9018 &= 25.6^\circ \\ \text{arc cos } .6401 &= 50.2^\circ\end{aligned}$$

Arc sin may also be written  $\sin^{-1}$ , both having exactly the same meaning. Similarly arc cos may be written  $\cos^{-1}$ , or arc tan as  $\tan^{-1}$ . Both terminologies should be understood because both are in general use.

### SOLUTION OF TRIANGLES

Probably one of the best tools for solution of right triangles is based on the Pythagorean relationship:

$$H^2 = O^2 + A^2$$

The lettering is the same as that used in Fig. 2-4, and the relationship literally states that the square of the hypotenuse is equal to the sum of the squares of the other two sides. Suppose that in a given right triangle the O side were 3 inches long and the A side 4 inches. The H side could be calculated by:

$$\begin{aligned}H^2 &= O^2 + A^2 \\ H &= \sqrt{O^2 + A^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \text{ inches}\end{aligned}$$

Or, if necessary, the basic formula could be rearranged to solve for O or A.

$$O = \sqrt{H^2 - A^2}$$

$$A = \sqrt{H^2 - O^2}$$

The Pythagorean theorem, however, does not take the angles into account—only the lengths of the sides. It is assumed though that one angle is  $90^\circ$  because the formula does not hold for any except right triangles. In later chapters considerable use of this relationship will be made. Other solutions of triangles involve combinations of lengths of sides and trig functions, because the Pythagorean relationship can be used only when the lengths of two sides are known. Then the third side can be determined. To find the number of degrees in any of the angles (except the  $90^\circ$  one) trig must be used.

For the triangle just solved, angle  $\theta$  can be determined from any two of the side lengths by using arc functions and reading to the closest tenth of a degree.

$$\theta = \text{arc sin } \frac{3}{5} = 36.9^\circ$$

$$\theta = \text{arc cos } \frac{4}{5} = 36.9^\circ$$

$$\theta = \text{arc tan } \frac{3}{4} = 36.9^\circ$$

The other angle is the complement of  $36.9^\circ$  so it is  $90^\circ - 36.9^\circ$ , or  $53.1^\circ$ . Or it too could be determined by arc functions, remembering that for that angle the opposite side is the same as the adjacent side of  $\theta$ , and vice versa. In cases where an angle and one side are known, a trig function can be used to determine another of the sides. For example, if  $\theta$  is  $35^\circ$ , and the opposite side is 6 units in length, the hypotenuse can be solved by using the sine function.

$$\sin \theta = \frac{O}{H}$$

$$\sin 35^\circ = \frac{6}{H}$$

$$.5736 = \frac{6}{H}$$

$$H = \frac{6}{.5736} = 10.46 \text{ units}$$

Then the adjacent side can be determined by the Pythagorean theorem or by using the cosine or the tangent function. Or, we could have used the tangent function and solved for A first.

$$\begin{aligned}\tan \theta &= \frac{O}{A} \\ \tan 35^\circ &= \frac{6}{A} \\ .7002 &= \frac{6}{A} \\ A &= \frac{6}{.7002} = 8.57 \text{ units}\end{aligned}$$

Notice that for every problem there are several possibilities which can be used. And, if given a choice, the Pythagorean relationship is better because it does not require a trig table. But, as previously stated, two side lengths must be known. In using the trig functions, the choice is determined by what is given. For example, in the previous problem we could not solve for the cosine first because neither A nor H were known. It was necessary to start with a function which included O, either sine or tangent. If neither of the smaller angles is given, one of the arc functions can be used, as shown previously, to determine one of the angles.

### THE CO-ORDINATE SYSTEM

The idea of dividing the circle into four equal sectors (quadrants) was introduced earlier in this chapter along with the numbering of the quadrants. It was also shown (Fig. 2-2) that the quadrant numbering is based on counterclockwise rotation beginning with the zero-degree reference on the horizontal and to the right. When working with angular quantities it is necessary to divide the quadrants into smaller divisions, both horizontally and vertically, for additional accuracy. Such a system of laying out the quadrants is known as the co-ordinate system. This enables us to set up all three sides of a triangle on the same surface.

So far we have described rotating vectors as moving about a circle, but actually there are no circumference limitations involved. So co-ordinates normally involve a square or rectangular surface subdivided into small segments each of which is a square. These are called rectangular co-ordinates and must include both positive and negative areas because both polarities

occur in practical situations. Or, as we shall see later, the same surface can be calibrated in degrees with radii extending out from the center of the surface. This is polar graph paper. Both types are used because a vector can be expressed in terms of its rectangular co-ordinates in two dimensions, or in polar (or diagonal) form at some specific angle. Vectors could also be in three dimensions, but they do not concern us in this coverage.

To plot rectangular co-ordinates the cross-hatched paper is laid out as shown in Fig. 2-5. The O at the center is called the origin, and all measurements on the paper are expressed with reference to it. The horizontal axis is called the X axis, and the vertical axis is the Y axis. To the right of the Y axis the num-

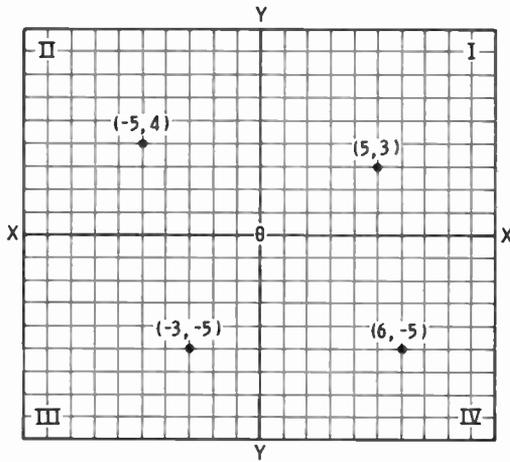


Fig. 2-5. Plot of rectangular co-ordinates.

bers are considered to be positive. Negative numbers are on the left of the Y axis. Also, above the X axis the numbers are positive, below it they are negative. The origin is zero in both directions.

In order to locate a point on the graph, two dimensions are involved. First, the point is a particular number of units to the right or to the left of the Y axis; this is the X dimension. The point is a particular number of units above or below the X axis; this is the Y dimension. In every case the horizontal dimension is specified first and referred to as X on the graph even though it may represent some other quantity, for example resistance, in a given problem. In every case the vertical dimension is referred to as Y on the graph, even though it also may represent some other quantity, for example reactance. Thus, (5,3) repre-

sents a specific point on the graph. It is 5 units to the right of the Y axis and 3 units above the X axis, as has been shown in Fig. 2-5.

Notice that the horizontal distance is measured with respect to the vertical axis; also, the vertical distance is measured with respect to the horizontal axis. Remember that this may eliminate one possible point of confusion. Also notice that each of the axes has a value of zero in one dimension all along its entire length. For example, for plotting in the horizontal dimension the Y axis is zero for its entire length.

Three other points are also plotted on the rectangular coordinates of Fig. 2-5, one in each of the other quadrants. The point  $(-5,4)$  in the second quadrant is located 5 units to the left of zero and 4 units up from the reference. Any point in the second quadrant has a negative value for X and a positive value for Y. For third-quadrant points both X and Y are negative in every case as shown in Fig. 2-5, where the point  $(-3, -5)$  is shown. In the fourth quadrant the X value is always positive, and the Y value negative. Point  $(6,-5)$  is shown in Fig. 2-5.

**Table 2-1**  
**Polarity of Points in Each Quadrant**

QUADRANT	X	Y
I	+	+
II	-	+
III	-	-
IV	+	-

The polarities (positive or negative) are very important in plotting graphic data, also in evaluating trig functions in all the quadrants. In every case the signs indicate the quadrant in which a point is located, and for ready reference Table 2-1 has been included. However, after working with the subject for even a short time, the polarities will be evident without having to refer to a table.

Using these ideas of polarity and those previously learned for trig functions, we can determine the functions for angles greater than  $90^\circ$ . Previously we considered the lengths of all three sides to be positive, and this is the case in most trig calculations when only the absolute values (without polarity) are considered. The triangle can be assumed to be in the first

quadrant with  $\theta$  less than  $90^\circ$ . However, vectors can exist in any direction of rotation, so they can appear in any one of the four quadrants. This means that we must be able to consider angles greater than  $90^\circ$  in our trig calculations.

Although on first thought we may consider the functions of angles greater than  $90^\circ$  to be quite different from those of angles under  $90^\circ$ , that is not the case. The ranges of values are the same, except that some of the values are negative since the sides involved also have negative values. We can use the same trig tables as used for functions of angles under  $90^\circ$ .

In the first quadrant all the sides of a triangle have positive values, so all the trig functions are also positive. And we must consider that the hypotenuse is positive in all the quad-

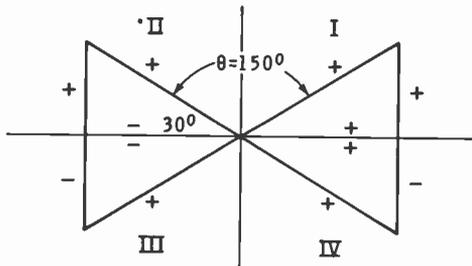


Fig. 2-6. Illustrating which sides of the triangle in each quadrant are negative and which are positive.

rants because it is laid out diagonally with respect to the origin. Therefore we cannot conceive of it having a negative value. The opposite and adjacent sides, however, are still either positive or negative, depending on the actual quadrant. Fig. 2-6 shows which sides are positive and which are negative.

Still using Fig. 2-6, consider the functions of  $150^\circ$ , a second-quadrant angle, as shown. Numerically the functions of  $150^\circ$  are the same as those for  $30^\circ$  (except for polarity of cosine and tangent). So  $30^\circ$  could be considered as the working angle. For any angle in the second quadrant the working angle is  $180^\circ$  minus the angle in question ( $150^\circ$  in our example). Thus, we have the following:

$$\sin 150^\circ = +.5000$$

$$\cos 150^\circ = -.8660$$

$$\tan 150^\circ = -.5774$$

Except for polarity signs, the functions are those of the working angle. To see why these changes come about let's assume values so that  $H = 2$ ,  $O = 1$ , and  $A = -\sqrt{3}$ , therefore:

$$\sin 150^\circ = \frac{1}{2} = .5000$$

$$\cos 150^\circ = \frac{-\sqrt{3}}{2} = -.8660$$

$$\tan 150^\circ = \frac{1}{-\sqrt{3}} = -.5774$$

Any angle in the second quadrant has a positive sine, and negative cosine and tangent. And this also holds true for the reciprocal functions; cosecant is positive, secant and cotangent are both negative. However, as previously stated, these functions are not used nearly as much as the other three.

For angles in the third quadrant the idea is similar except that the working angle is  $\theta$  minus  $180^\circ$ . For example, in Fig. 2-7 an angle of  $210^\circ$  is considered. The working angle is  $210^\circ - 180^\circ$ , or  $30^\circ$ , and the sides could have values as shown. Notice that both the opposite and adjacent sides are negative in

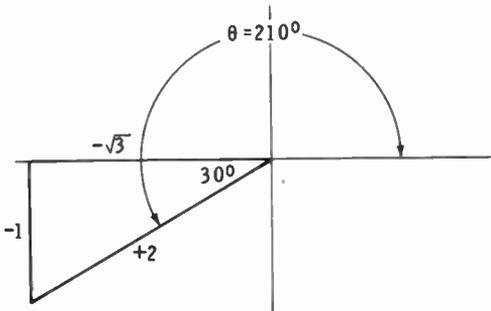


Fig. 2-7. A  $210^\circ$  angle.

value, while the hypotenuse is positive. So the functions of  $210^\circ$  are as follows:

$$\sin 210^\circ = \frac{-1}{2} = -.5000$$

$$\cos 210^\circ = \frac{-\sqrt{3}}{2} = -.8660$$

$$\tan 210^\circ = \frac{-1}{-\sqrt{3}} = +.5774$$

Notice that the sine and cosine are negative and that the tangent is positive. This is true for all angles which terminate

in the third quadrant. Now let's look at one more example, this one an angle in the fourth quadrant, say  $330^\circ$ . This is shown in Fig. 2-8 and the working angle is  $360^\circ$  minus  $\theta$ , or in this case  $30^\circ$ . The side lengths have the polarities shown.

$$\sin 330^\circ = \frac{-1}{2} = -.5000$$

$$\cos 330^\circ = \frac{\sqrt{3}}{2} = +.8660$$

$$\tan 330^\circ = \frac{-1}{\sqrt{3}} = -.5774$$

For simplification all the working angles in the examples were  $30^\circ$ , but the idea holds for any angles. Table 2-2 gives the

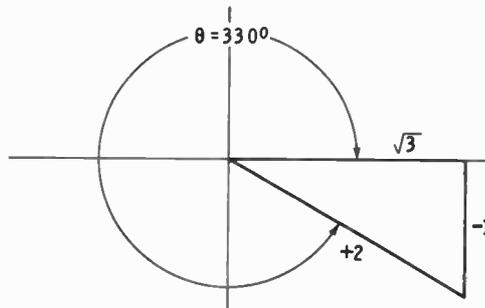


Fig. 2-8. A  $330^\circ$  angle.

polarity for the three trig functions in all four quadrants and can be used for ready reference. However, by understanding how the signs are determined we have a means of determining the correct sign in case the information is not readily available. Notice that the sine function is positive in the first and second quadrants and negative in the third and fourth quadrants. This

**Table 2-2**  
**Polarity of Trig Functions in Each Quadrant**

QUADRANT	SIN	COS	TAN
I	+	+	+
II	+	-	-
III	-	-	+
IV	-	+	-

agrees with the characteristic of a sine wave shown previously, that the positive alternation ( $0^\circ$  to  $180^\circ$ ) is positive, and the negative alternation ( $180^\circ$  to  $360^\circ$ ) is negative.

Notice that each function has two quadrants in which it is negative and two in which it is positive. This causes no problems when determining the values of trig functions, but it does for arc functions. For example, what is the arc  $\sin .5$ ? This could be either  $30^\circ$  or  $150^\circ$ , depending on the quadrant. In most cases, however, the first-quadrant angle is the one considered but a point of confusion could arise in a problem in which both answers could be used.

### PRACTICE PROBLEMS

1. What is the complement of  $12^\circ$ ?
2. What is the supplement of  $75^\circ$ ?
3. What is the counterpart angle of  $110^\circ$ ?
4. Convert  $32^\circ 18'$  to an angle in the decitrig system.
5. Convert  $43.7^\circ$  to degrees and minutes.
6. Convert an angle of  $15^\circ 10' 42''$  to an angle in the decitrig system.
7. An angle has a magnitude of  $3\pi/2$  radians. How many degrees does this represent?
8. Convert  $240^\circ$  to radians.
9. Evaluate these trig functions:  $\sin 31.8^\circ$ ,  $\cos 15.7^\circ$ , and  $\tan 39.3^\circ$ .
10. Evaluate these trig functions:  $\sin 72.3^\circ$ ,  $\cos 68.6^\circ$ , and  $\tan 76.0^\circ$ .
11. Determine these arc functions: arc  $\sin .6455$ , arc  $\cos .8878$ , arc  $\tan .2180$ .
12. Determine these arc functions:  $\sin^{-1} .8738$ ,  $\cos^{-1} .1219$ ,  $\tan^{-1} 2.872$ .
13. A right triangle has short-side lengths of 3 and 6 feet. What is the length of the hypotenuse?
14. In the same right triangle, what are the magnitudes of the two smaller angles?
15. In a right triangle  $\theta$  is  $38^\circ$  and the side opposite it is 12 units in length. What are the lengths of the other two sides?
16. The hypotenuse of a right triangle is 8 feet long and the base angle is  $56^\circ$ . What are the lengths of the other two sides?
17. A right triangle has side lengths of 5, 9, and 10.3 units. What are the magnitudes of the angles?
18. Determine these functions:  $\sin 220^\circ$ ,  $\cos 310^\circ$ ,  $\tan 125^\circ$ .
19. Determine these functions:  $\sin 105^\circ$ ,  $\cos 238^\circ$ ,  $\tan 265^\circ$ .
20. What is the arc cosine of  $.3420$ ? List two answers.

## CHAPTER 3

# Vectors

The basic idea of vectors has already been introduced in the two previous chapters. Stated simply, a vector is a line that represents quantity and direction. This direction may be compared with a known reference, or with some other vector. It can be said that vectors are directed lines because the line points in a specific direction. The term directed-line segment is also used, assuming that the line itself extends indefinitely, but in the problem at hand only a small part (segment) is being used.

Vectors can be compared with scalars, also called scalar quantities, which are the quantities with which we work most often. Such quantities may be a temperature, a number of specific items, area or volume of a figure, population, attendance at an event, or many other similar examples. These are all specific amounts of what is being measured, but they do not involve direction. To say that the temperature is  $72^\circ$  implies only a magnitude, no sense of direction. Temperature is stated completely by specifying the number of degrees, assuming the measure of a degree is known. So it can be said that a scalar quantity has only magnitude; it does not imply any direction of action.

Other quantities with which we deal can be expressed in terms of magnitude, but are more meaningful if direction is also stated. These include statements of velocity, number of

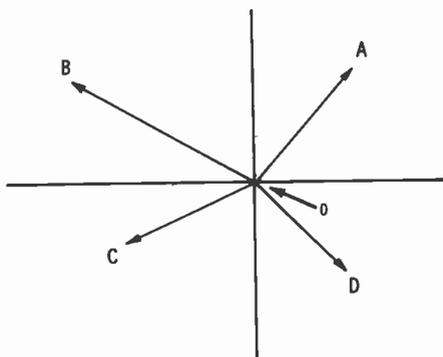


Fig. 3-1. Four different vectors, one in each quadrant.

pounds of push or pull, or other cases in which action is performed in a specific direction. Technically, speed is considered to be a scalar quantity; velocity is vectorial and includes both speed and direction. Although not explicitly directional, the vector idea is used in certain aspects of electrical circuits, for example the phases of voltages and currents. This was shown briefly in the latter part of the preceding chapter and will be used to a much greater extent in the last two chapters.

Fig. 3-1 includes four different vectors, one in each of the four quadrants. The length of each line is proportional to the magnitude of the quantity being represented. The arrow of each vector points in the direction in which the represented quantity is acting, and in every case the vectors originate at the origin (O). That end of the vector at which the arrow is located can be called the terminal, or point, end. The opposite end can be called the initial, or starting, end (or point). As long as all the vectors in Fig. 3-1 begin at the origin, they could be labeled OA, OB, OC, and OD. It is evident that these are vectors representing vector quantities. But without the diagram the designations OA, OB, etc. may be confusing. So to indicate that they are vectors  $\overline{OA}$ ,  $\overline{OB}$ , etc. could be used, and in some cases only the line is used above the letter as  $\overline{OA}$ ,  $\overline{OB}$ , etc. The first letter normally indicates the initial point; the second letter is the terminal point.  $\overline{OA}$ , for example, means that the vector lies in the direction of A.

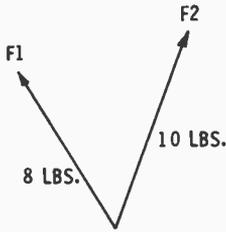
Other forms of notation are also used, and one of the most popular is a single letter, such as A, B, etc. Later, you will see that R,  $X_L$ , Z, and other electronics abbreviations are used as vector symbols. And in some texts boldface type is used such as **A**, **B**, etc. In addition to expressing a vector in terms of letter designations, they may also be expressed in ways that not only

designate them as vectors, but which also give their magnitude and angular direction. One of these is  $A/\theta$ , which indicates a vector length of  $A$  acting in the direction of angle  $\theta$ . This is known as the polar form of a vector because this length radiates from the origin which could be called a pole. Along this same line, circular paper calibrated in degrees of angle is called polar graph paper. Or, as will be explained later, a vector can be expressed in terms of rectangular co-ordinates including an  $X$  and a  $Y$  dimension. Other symbols may be used for vectors, but regardless of what symbol is used, the writing should make it clear that vectors are being represented and what the symbols indicate. In this writing vector quantities are considered almost entirely so there should be no confusion along that line.

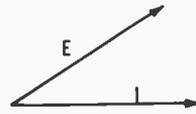
In performing mathematical operations with two or more vectors, they are not treated the same as scalar quantities. Scalars are simply numbers and are handled as any other numbers in legitimate mathematical operations. Vectors, however, cannot be treated so simply. Not only must the magnitudes be considered, but the directions must also be included in order to obtain correct results. Much of this chapter and all of the next chapter is devoted to mathematical operations involving vectors. This is necessary because in order to use vectors in AC circuitry we must be able to add, subtract, multiply, and divide by using vector notation. The word resultant will occur many times, and it will be used to indicate the single total vector that is obtained from a specific combination of individual vectors.

At this point in the text it may be well to introduce several other terms related to vectors, because they may be encountered in various texts on electrical and electronic principles. First, it is reasonably standard to assume that two forces pushing or pulling an object can be represented by vectors. Such a condition is shown in Fig. 3-2A in which  $F_1$  is a force of 8 pounds attempting to pull an object in a certain direction.  $F_2$  is a force of 10 pounds that is pulling in another direction as shown. The resultant force (also a vector) would be at an angle between  $F_1$  and  $F_2$ , and would have a specific number of pounds of pull. A vector diagram could also be three-dimensional, for example one vector pulling to the right, one to the left, and one either up or down. Those shown here are plane vectors; they are not three-dimensional. And in this writing only plane vectors are considered.

Fig. 3-2B shows specific types of vectors which occur very often in electrical and electronic work. In this case they repre-



(A) Pulling forces.



(B) Voltage and current relationship.

Fig. 3-2. Vectors representing relationships of forces.

sent the voltage and current relationship in an electrical circuit. They may be referred to as phasors (rather than vectors) because they represent phase relationships of the circuit. Phasors can be defined as quantities representing sine waves in amplitude and phase position by length and direction from a given reference line. Comparing this diagram with Fig. 3-2A we find that there are differences in the quantities represented. It is difficult to say that voltage and current are acting in a specific direction; but we can say that if voltage is at some instantaneous point in its sine-wave cycle, the current is a specific number of degrees behind the voltage in reaching that same angle of rotation. Thus, a phasor is used instead of a vector.

With these ideas in mind, you can see that Fig. 3-3A would be a phasor diagram and Fig. 3-3B would be a vector diagram. We have to agree that there are definite relationships between the oppositions in a circuit and the current-voltage relationship in that same circuit. So the use of two terms here (phasor and vector) could create confusion. Still another definition states that phasors are any complex-number quantities associated with AC circuit theory. Using this definition, Fig. 3-3B would also be a phasor diagram as we will conclude after studying complex numbers in the next chapter.

The term sinor is also used to some extent and applies to sine-wave positions varying with respect to time. It could be said that the sinor is a special form of phasor, in fact one



Fig. 3-3. Phasor and vector diagrams.

definition states that "phasors that specify sinusoidal time functions are termed sinors." As an example, a sine wave represents a voltage or current varying at a specific rate and plotted against time. It could be said that a phasor diagram shows two or more separate sinors, but this statement is not a clear-cut definition.

In order to simplify this text, the term vector is used exclusively for several reasons:

1. The term vector is readily understood, and its use extends back many years, even for electrical quantities.
2. Use of the single term prevents confusion, especially when there seems to be no sharp dividing line between the various terms.
3. Actually, a phasor is a specific form of vector, using the latter in a general sense.
4. Regardless of what is being represented by the directed lines, the methods of working with them are the same.

As long as it is known what is being represented by the directed lines, these differences in terminology should present no real problem, hence the use of the single term vector.

### VECTOR COMPONENTS

Any vector can be described (or defined) in several different ways. Symbols for vectors were introduced earlier, also the polar form of notation, but there is still another way which lends itself quite well to trigonometric solutions. For example, consider the vector in Fig. 3-4; it starts at the origin and terminates at point (8, 4). Plotted on the co-ordinate axes, as shown, the vector can be expressed in terms of its X and Y values, called rectangular components. The process of determining the horizontal and vertical components from the vector is referred to as resolving the vector, or resolution of the vector.

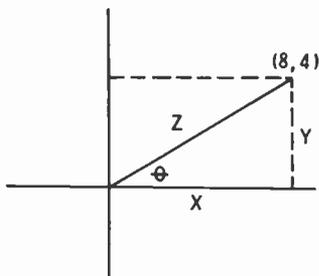


Fig. 3-4. A vector expressed by rectangular components.

If a triangle is completed by drawing a line from the end of the vector to the X axis (shown as dashed line), we find that the side opposite angle  $\theta$  is 4 units in length. The adjacent side is 8 units in length. So originating at the origin the rectangular components are (8,4). Expressing the vector in this form is referred to as the rectangular form of vector notation because the measurements are with respect to the rectangular co-ordinates. The point numberings being used were introduced along with rectangular co-ordinates in the preceding chapter.

Depending on what is given, either form can be converted to the other by trigonometric calculations. In Fig. 3-4 the rectangular form is given, and in this form the rectangular co-ordinates give the lengths of the two short sides of a right triangle. The actual vector length is the hypotenuse of that same triangle and it forms the angle  $\theta$  with the adjacent side. Using the symbols given, the vector length can be determined by substituting into the Pythagorean formula :

$$Z^2 = X^2 + Y^2$$

$$Z = \sqrt{X^2 + Y^2}$$

$$Z = \sqrt{8^2 + 4^2} = \sqrt{80} = 8.94 \text{ units}$$

Angle  $\theta$  may be determined by  $\theta = \text{arc tan } \frac{Y}{X} = \text{arc tan } 0.5 = 26.6^\circ$ . So the actual vector, in polar form, is  $8.94 / 26.6^\circ$ .

Both  $Z$  and  $\theta$  are stated in terms of the closest digits rather than by large number of decimal places. This practice is also followed in succeeding sections of the text because methods are more important than extreme accuracy when learning the basic processes.

In practical calculations either form may be given, and it may be necessary to convert from one form to the other. When the rectangular form is given, the polar form is determined by :

$$\sqrt{X^2 + Y^2} \quad / \quad \text{arc tan } \frac{Y}{X}$$

using the lettering of Fig. 3-4.

When the polar form is given, the horizontal component is  $Z$  times  $\cos \theta$ ; the vertical component is  $Z$  times  $\sin \theta$ , again using the lettering of Fig. 3-4. Bear in mind, however, that these two forms and the calculations involved apply only to a single vector and are based on the relationships existing in a right triangle. As we shall see in the next section of this chapter, when working with more than one vector, the angle between them must be considered, and this is not necessarily a right angle.

However, each individual vector can be resolved and this is one of the methods of solution which are introduced.

As shown in the preceding chapter, unless otherwise specified, angle  $\theta$  is assumed to be in standard position, that is, measured with respect to  $0^\circ$ . Normally  $0^\circ$  is on the horizontal line and to the right of the origin. The  $0^\circ$  line may be called the initial side of the vector and the actual vector the terminal side, especially when considering rotating vectors as in AC. These terms were also introduced previously.

### VECTOR ADDITION

As previously indicated, vectors can be combined in various ways, just as other types of math expressions can be, as long as the angles are considered along with the magnitudes. Scalar quantities can be added directly, for example:

$$15 \text{ tubes} + 10 \text{ tubes} = 25 \text{ tubes}$$

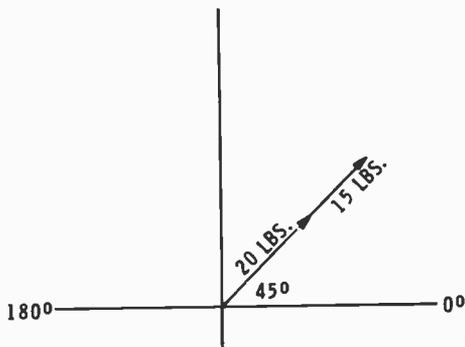
$$3 \text{ books} + 2 \text{ books} = 5 \text{ books}$$

But unless vectors are at the same angle, the magnitudes cannot be added directly. The fact that the quantities are acting in different directions prevents that. Also remember that only vectors representing like quantities can be added. For example, to combine voltage and current into a resultant quantity has no meaning. Two voltages, however, can be combined to give a resultant voltage; this idea applies similarly to other vector quantities.

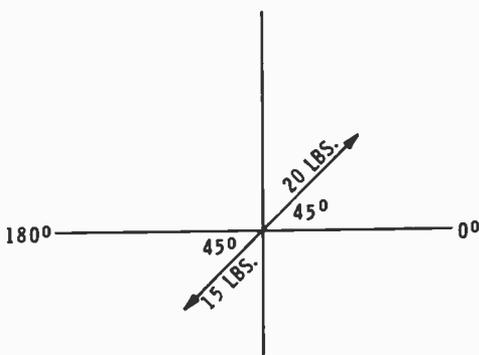
#### Addition of Two Vectors

If two vectors are parallel (at the same angle), they are acting in the same direction so that their magnitudes can be considered like scalar quantities and added directly. The resultant is still a vector acting at the same angle as the two separate quantities. As an example, suppose that two forces, 20 pounds and 15 pounds, are acting at an angle of  $45^\circ$  with respect to some reference. The resultant is 35 pounds acting at the same angle,  $45^\circ$ . They can be plotted on the same vector line as shown in Fig. 3-5A, and parallel vectors in any direction are handled similarly. A vector would be considered equal to another vector only when both have the same magnitudes and are acting at the same angle.

Vector lines may be parallel with respect to each other, but acting in exactly opposite directions. There is then a  $180^\circ$  angular difference between them, and they are not considered to be parallel vectors. The effects of such vectors are cancellable



(A) Acting in same direction.



(B) Acting in opposite directions.

Fig. 3-5. Parallel vectors.

since they directly oppose each other. In Fig. 3-5B a force of 20 pounds is acting at an angle of  $45^\circ$  and a force of 15 pounds is acting at an angle of  $225^\circ$ . The resultant vector is 5 pounds ( $20 - 15$ ) acting at an angle of  $45^\circ$ . The magnitudes are subtractive, and the resultant vector is in the direction of that vector having the larger magnitude. If both vectors had exactly the same magnitude the resultant force would be zero. Bear in mind that the individual angles need not be  $45^\circ$  and  $225^\circ$ . Vectors are subtractive whenever they directly oppose each other, at any angles.

It may seem that the  $0^\circ$  and  $180^\circ$  angular differences would not necessarily be useful, but that is not the case. In electrical and electronic circuits these angles occur quite often. For example, two inductive reactances act at the same angle in a circuit and are directly additive. But inductive and capacitive reactances add in opposition to each other and are subtractive.

Also, in amplifier circuits, signals may be in phase or  $180^\circ$  out of phase with each other. So the vectors representing them would be either additive or subtractive. These subjects, however, are covered in greater detail in a subsequent chapter.

Now consider a situation in which the vectors are not at  $0^\circ$  nor  $180^\circ$ , as in Fig. 3-6. Here  $F_1$  is a force of 10 pounds acting on an object at the origin, and acting at an angle of  $80^\circ$ .  $F_2$  is a force of 15 pounds acting on the same point but at an angle of  $30^\circ$ . The resultant is the single force acting at a specific angle that would act exactly like the two single forces applied at the two separate angles. The resultant here is at some angle between  $30^\circ$  and  $80^\circ$  and is labeled  $R$ .

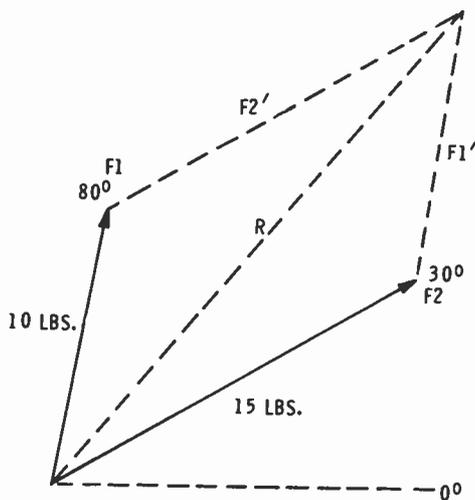


Fig. 3-6. Adding two vectors that are not at  $0^\circ$  or  $180^\circ$ .

The resultant in this case, including force and angle, has been determined by drawing a parallelogram with the two given forces as adjacent sides. A parallelogram is a four-sided figure in which opposite sides are parallel and equal in length, and in which opposite angles are equal to each other. Line  $F_1'$  is the same length and at the same angle as line  $F_1$ . Similarly line  $F_2'$  is the same length and at the same angle as line  $F_2$ . The angle in each case would be measured with respect to where the dashed line (extended) would cross the X axis. The resultant vector is the diagonal of the parallelogram drawn from the point of origin of the two original vectors. Its length can be measured to determine the number of pounds of effective force, and the angle can be measured (with a protractor) to de-

termine the angle at which the resultant force is acting. From Fig. 3-6 we can conclude that the resultant is a force of 22.5 pounds acting at an angle of  $50^\circ$ . In drawing the resultant vector be careful to draw only that diagonal that starts at the point of origination of the two separate vectors. This is necessary because each parallelogram has two possible diagonal lines.

We can draw certain definite conclusions regarding such an arrangement. When the vectors are at the same angle, the sum is greater than either of the two separate vectors. At  $180^\circ$  the difference vector is shorter than the longer of the two separate vectors. For angles between  $0^\circ$  and  $180^\circ$  the resultant is between the two separate angles, and the resultant vector length depends on the angular difference. As the two individual angles come closer to each other, the resultant length increases, reaching its maximum when the angles are equal. Actual resultant length is also determined by the length of the individual vectors, becoming longer as the separate vectors are made longer. If the forces represented by  $F_1$  and  $F_2$  are the same, then the resultant angle would be exactly half way between the two separate angles. However, if they are not equal, the resultant is closer to that vector having the larger magnitude. In Fig. 3-6  $F_2$  represents a larger force than  $F_1$  so that the resultant is closer to  $30^\circ$  than to  $80^\circ$ . If  $F_2$  were increased in length ( $F_1$  remaining the same), the resultant angle would be even closer to  $30^\circ$ . Any convenient scale can be used to determine the lengths of the lines, and each inch of vector length represents a specific number of pounds of force. Generally, the larger the drawing, the more accurate will be the results obtained by this graphic method of solution. When the drawing is too small, even the width of the pencil line may decrease the accuracy.

Another similar example is that shown in Fig. 3-7. In this case,  $F_1$  is a force of 15 pounds at  $110^\circ$ , and  $F_2$  is a force of 25 pounds at  $50^\circ$ . The resultant is a force of 35 pounds acting at  $72^\circ$ . Notice that in every example of this type the results are only as accurate as the graphical presentation, and they may not always be exact. This is a definite disadvantage of any graphical means of solution, but in the presentation of the subject they are invaluable for a visual description of the process involved.

A similar but different method of solution is that of Fig. 3-8, which is a representation of the same problem solved in Fig. 3-6. Instead of starting both vectors at the origin, only one is started there. Then the second begins at the terminal point of the first. The dashed line completing the triangle is the

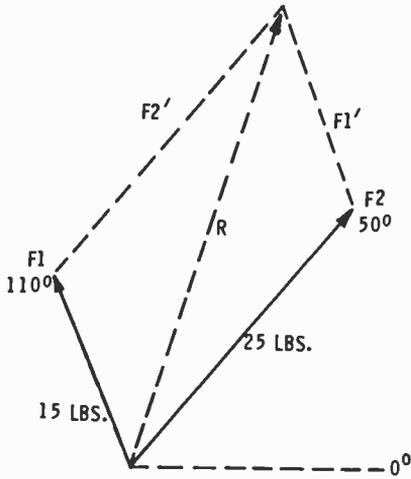


Fig. 3-7. Another example of adding vectors that are not at  $0^\circ$  or  $180^\circ$ .

resultant and it is the same, both in magnitude and direction, as the diagonal of the parallelogram of Fig. 3-6.

It does not matter which vector starts at the origin; the same resultant is obtained either way. But they must be arranged so that the initial point of the second vector joins the terminal point of the first. Then the resultant is the vector that connects the initial point of the first vector to the terminal point of the second. This is sometimes called the head-to-tail method, and it is especially useful when more than two vectors are involved. It involves a smaller number of lines, and therefore does not appear as complicated as the parallelogram method. This method is also graphic, however, and subject to the same inaccuracies as other graphic methods.

If two forces are at right angles to each other, the problem is simplified because trig can be used to give a more exact resultant. One angle of the triangle is then a right angle. Fig. 3-9 shows an example in which the parallelogram method of

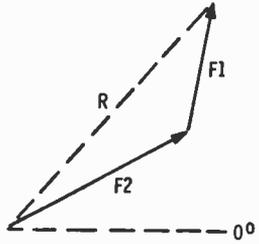
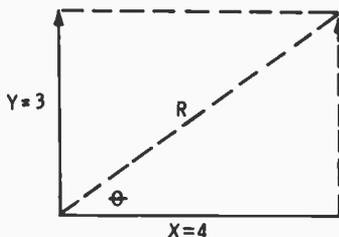


Fig. 3-8. Adding vectors by placing them end to end.

Fig. 3-9. Adding two vectors that are at right angles.



solution has been used. The head-to-tail method could also have been used, in which case the Y vector would appear as the right side of the parallelogram. In either method of solution the resultant magnitude can be measured with a scale and a protractor. When the two vectors to be added are at  $90^\circ$  (Fig. 3-9), the length of the diagonal is the same regardless of which diagonal is used. To avoid confusion with respect to the resultant angle, the diagonal which begins at the origin should be used.

As long as the right triangle does exist in this case, a trig solution can be made as follows. The Pythagorean relationship is used to determine the length of the resultant:

$$R = \sqrt{X^2 + Y^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

$\theta$  can be determined by:

$$\theta = \text{arc tan } \frac{3}{4} = 36.9^\circ$$

In this problem the resultant is 5 units acting at an angle of  $36.9^\circ$ . Either arc sine or arc cosine could also be used once the length of the resultant has been determined.

Another possibility for using rectangular components is the solution of Fig. 3-10. F1 and F2 are vectors whose resultant solution has been determined by the parallelogram method. But these vectors were drawn so that the X and Y components would be whole numbers to better illustrate another method of solution. F1 terminates at point (3,9); F2 terminates at point (13,6). The resultant vector terminates at point (16,15). If the X values (3 and 13) are added, the Y values (9 and 6) are added. The X result is 16; the Y result is 15. This provides an algebraic method of vector addition, first the X components are added, and then the Y components are added. The rectangular components of the resultant vector are the sums of the rectangular components of the individual vectors.

Fig. 3-11 shows the same idea applied to two other vectors, one in the first quadrant, the other in the second. F1 terminates

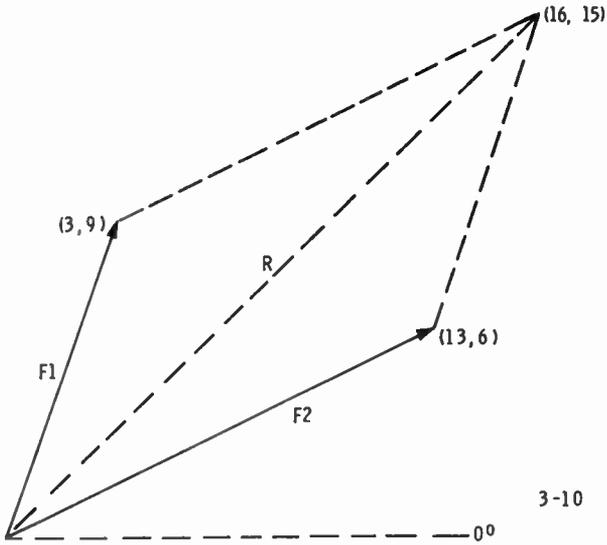


Fig. 3-10. Using rectangular components for two vectors in the first quadrant.

at  $(-4, 12)$ , F2 terminates at  $(8, 7)$ . Adding the X components  $(-4$  and  $8)$  gives  $4$ , and adding the Y components  $(12$  and  $7)$  gives  $19$ . The resultant terminates at point  $(4, 19)$ , and a vector presentation such as that shown gives the same result. This resolution method is especially important in electrical calculations because of the  $90^\circ$  phase shift of current and voltage in reactive components.

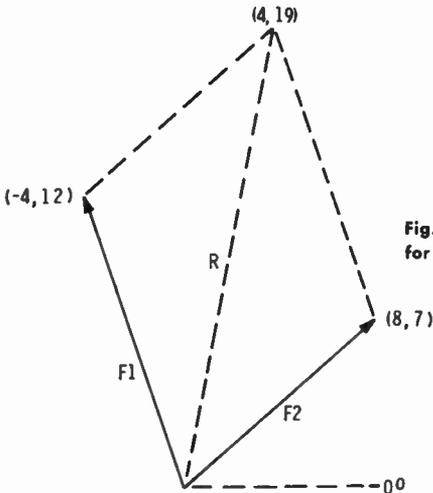


Fig. 3-11. Using rectangular components for one vector in the first quadrant and the other in the second.

When a vector is given in polar form, it can be converted to rectangular co-ordinates by:

$$X = \text{length} \times \cos \theta$$

$$Y = \text{length} \times \sin \theta$$

There is no algebraic method of adding polar-form vectors unless their angles are the same, for example:

$$3/20^\circ + 2/20^\circ = 5/20^\circ$$

Otherwise, the rectangular co-ordinates must be added, and then, if necessary, the resultant converted back to polar form. More use will be made of these ideas in later chapters with reference to complex numbers. These are numbers which use specific notations to denote vectors, either in polar or rectangular form.

Three methods of vector addition have been shown. These are the parallelogram, head-to-tail, and resolution methods of which the last is preferred because it is an algebraic method, does not require drawing utensils, and probably gives the best accuracy. There is, however, a trigonometric method that can be used to solve for the resultant when the individual vectors are not at right angles to each other. This involves the study of oblique triangles and the Law of Cosines. But because it is used to such a limited degree in vectors associated with electronics, that method is not included here. Volume 2 of *Electronics Math Simplified* contains a section on oblique triangles and the use of the Law of Cosines.

### **Addition of Three Vectors**

The basic ideas involved in adding more than two vectors are the same as those shown for the two-vector additions, except that more operations are involved. In this section the addition of three vectors is considered, but the ideas can be applied to even more. Most practical problems do not include more than three so that is the limitation here. Any of the methods previously shown can be used for three vectors—the algebraic method is usually the one preferred.

In Fig. 3-12 the parallelogram method has been used to add three vectors which start from the origin. The process used is to find the resultant of any two, then to add that resultant to the third vector and thus obtain a resultant for the entire combination. F1 is 6 units in length at  $100^\circ$ , F2 is 8 units at  $60^\circ$ , and F3 is 10 units at  $-20^\circ$ . In Fig. 3-12, F1 and F2 were added and the diagonal of the parallelogram is the resultant (R1). Then F3 and R1 were added by constructing a second parallel-

ogram, the diagonal of which is  $R_T$  the total resultant. As measured, the resultant is approximately 15.5 units in length at an angle of about  $37.5^\circ$ .

Plotting any other combination would give the same result. As an example, F2 and F3 could be added and then the resultant combined with F1 to obtain the total resultant vector. F1 and F3 could be added and the resultant added to F2.

The head-to-tail method has been used on the same problem in Fig. 3-13. F1 was started at the origin, F2 started at the terminal end of F1, and then F3 started at the terminal end of

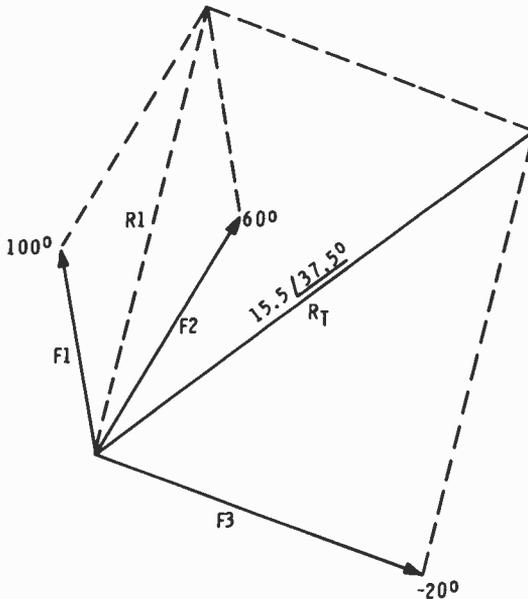


Fig. 3-12. Parallelogram method of adding three vectors which start from the origin.

F2. The  $R_1$  resultant (same as in Fig. 3-12) has been drawn, but it is not necessary in the solution of the problem.  $R_T$  is again the total resultant and is drawn from the origin to the terminal end of F3. This is sometimes referred to as closing the polygon. When three vectors are added head-to-tail, the resultant is that vector required to close the four-sided figure. The figure constructed, however, is not normally a parallelogram. From Fig. 3-13, resultant  $R_T$  is about the same as previously found, 15.5 units at  $37.5^\circ$ .

The same problem can also be solved by the resolution method of obtaining the X and Y components of the vector.

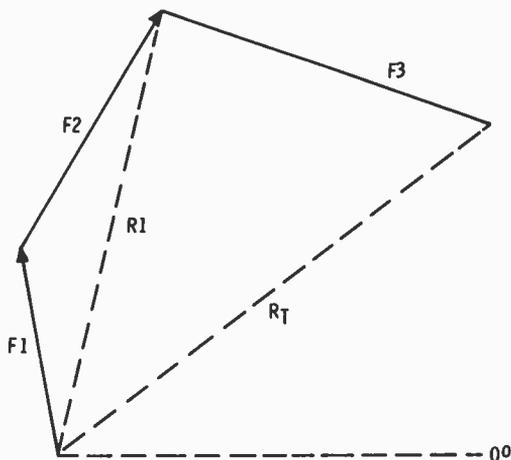


Fig. 3-13. Using the head-to-tail method for adding three vectors.

Using the same scale as in Figs. 3-12 and 3-13, the X and Y components are approximately as shown here:

$$\begin{array}{l}
 \text{F1} \quad X = -1, \quad Y = 6 \\
 \text{F2} \quad X = 4, \quad Y = 7 \\
 \text{F3} \quad X = 9.3, \quad Y = -3.5 \\
 \text{R}_T \quad X = 12.3, \quad Y = 9.5
 \end{array}$$

Adding the X's and the Y's separately produces  $X = 12.3$  and  $Y = 9.5$  for the total resultant vector. The length of  $R_T$  can be determined by the Pythagorean relationship.

$$\begin{aligned}
 R_T &= \sqrt{X^2 + Y^2} = \sqrt{12.3^2 + 9.5^2} = \sqrt{151.29 + 90.25} \\
 &= \sqrt{241.54} = 15.54
 \end{aligned}$$

This result is very close to that obtained by graphic means.

$$\begin{aligned}
 \theta &= \text{arc tan } \frac{Y}{X} = \text{arc tan } \frac{9.5}{12.3} \\
 &= \text{arc tan } .7724 = 37.7^\circ \text{ (approx.)}
 \end{aligned}$$

This too is about the same as determined by the graphic method. But let's consider even another possibility. If each of the polar-form vectors were converted to rectangular form by algebraic means and then combined by addition, results which are more accurate would be obtained. As we shall see later, most practical vector problems are solved by algebraic means.

### VECTOR SUBTRACTION

In algebra a number is subtracted by changing its sign and then performing algebraic addition. This is also true for subtracting vectors. When the sign of a vector is changed, it is the

$$3/25^\circ + 2/25^\circ = 5/25^\circ$$

same as changing the angle by  $180^\circ$ .

When adding,

When subtracting,

$$3/25^\circ - 2/25^\circ = 1/25^\circ$$

Effectively, the direction of the second vector was reversed, causing it to directly oppose the action of the first. So the second expression could also be shown as:

$$3/25^\circ + 2/205^\circ = 1/25^\circ$$

The parallelogram method can be used in the subtraction of vectors as well as in addition. Two vectors are added by determining the diagonal of the parallelogram of which the two original vectors are adjacent sides. Referring back to Fig. 3-6,  $F_1$  and  $F_2$  were added to give the resultant ( $R$ ), as shown:

$$\overline{F_1} + \overline{F_2} = \overline{R}$$

Considering the same problem values let's assume a subtraction such that:

$$\overline{R} - \overline{F_1} = \overline{F_2}$$

As shown in Fig. 3-14,  $R$  is  $22.5/50^\circ$ , and  $F_1$  is  $10/80^\circ$ . In this method of subtraction a parallelogram is set up such that the vector being subtracted from (the minuend) is the diagonal. As a first step, the ends of the vectors are connected (dashed lines have been used). Then the parallelogram has been completed as shown. Measuring  $F_2$  gives  $15/30^\circ$ . Comparing this with Fig. 3-6, it agrees, and the diagram is the same

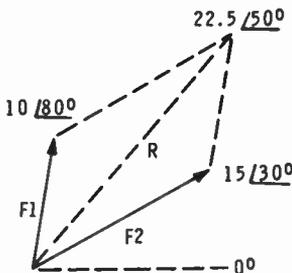


Fig. 3-14. Using the parallelogram method to subtract vectors.

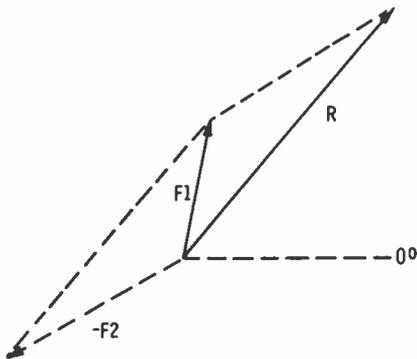


Fig. 3-15. Subtracting the resultant vector (R) from vector F1.

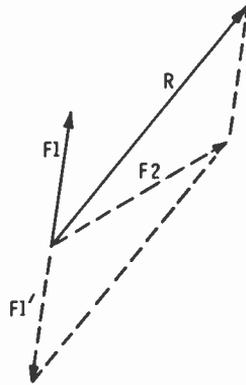


Fig. 3-16. Subtraction method by changing the sign of the vector being subtracted.

parallelogram as obtained in the addition problem. So effectively we have determined the vector that would combine with F1 (addition) to give the resultant R. This shows that subtraction in vectors is the inverse of addition, just as it is in any other form of mathematics.

Notice that it makes a difference as to what is being subtracted. The vector being subtracted from is the diagonal in every case. For example, in Fig. 3-14 if vector F2 had been subtracted from R ( $\bar{R} - \bar{F}_2$ ), the diagram would have been the same as shown. F2 would have been given in the original problem and F1 would have been the difference of the subtraction process.

If the process had been altered to  $\bar{F}_1 - \bar{R}$  the difference would not be the F2 value found previously because F1 would be the diagonal. The other value would be that as shown by F2 in Fig. 3-15. This new vector is approximately  $14/210^\circ$ .

Another subtraction method which can be used is based on the changing of signs mentioned at the beginning of this section on subtraction. It is possibly much simpler than the parallelogram method because there is less chance of performing the wrong subtraction. Referring back to Fig. 3-15 we find that  $10/80^\circ$  was subtracted from  $22.5/50^\circ$ . Then after changing the sign of the vector being subtracted the problem becomes:

$$22.5/50^\circ - 10/80^\circ$$

The graphic representation of this problem is shown in Fig. 3-16. Changing F1 by  $180^\circ$  causes the problem to read:

$$22.5/50^\circ + 10/260^\circ$$

This is true because  $-10/80^\circ$  is the same vector as  $+10/260^\circ$  shown as  $F1'$ . By completing the parallelogram and drawing the diagonal,  $F2$  is found to be  $15/30^\circ$ , which agrees with the previous solution. When using this method, change only the sign of the vector being subtracted. Any other change produces erroneous results.

The X and Y components can be subtracted algebraically in much the same way as used in vector addition. The only difference is the changing of signs of the vector being subtracted. An example is the problem shown in Fig. 3-17. Here  $F1$  has

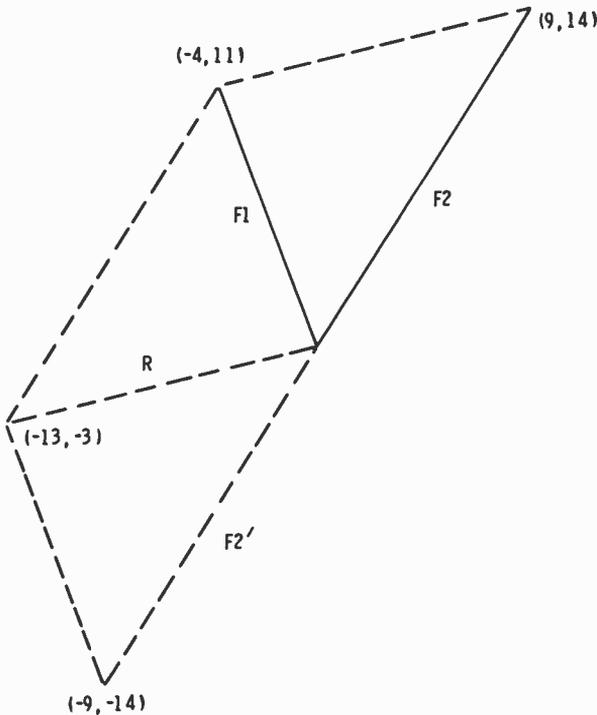


Fig. 3-17. Subtracting X and Y components algebraically.

rectangular components of  $X = -4$ , and  $Y = 11$ .  $F2$  values are  $X = 9$  and  $Y = 14$ . If  $F2$  is being subtracted from  $F1$ , the signs of the  $F2$  components are changed and they become  $X = -9$  and  $Y = -14$ . Adding them to the  $F1$  values gives  $X = -13$  and  $Y = -3$ . In Fig. 3-17  $F2$  has been shifted by  $180^\circ$  as denoted by  $F2'$ , and the diagonal of the parallelogram is labeled  $R$ . Notice that it terminates at point  $(-13, -3)$ .

One word of caution before leaving subtraction. In most algebraic (and vectorial) expressions subtraction is indicated in the problem itself. For example,  $8/15^\circ - 3/15^\circ$  shows subtraction; do not change any of the signs. But if the problem is to subtract  $3/15^\circ$  from  $8/15^\circ$  then the sign of the  $3/15^\circ$  must be changed. The problem then reads  $8/15^\circ - 3/15^\circ$ , as previously indicated.

### MULTIPLICATION AND DIVISION

Multiplication and division of vectors do not have as clear a meaning in graphical form as do addition and subtraction. For example, the addition of two vectors to determine a resultant vector is a logical process. We can see its meaning. Multiplication and division are also logical processes, but not as clear as to actual meaning. Suppose that vector F1 ( $3/20^\circ$ ) is multiplied by F2 ( $4/45^\circ$ ). The resultant product is:

$$(3/20^\circ) (4/45^\circ) = 12/65^\circ$$

In the process the magnitudes are multiplied together, and then the angles are added. These vectors are shown in Fig. 3-18. With addition and subtraction, only like quantities can be combined. For example, volts and amperes cannot be added or subtracted because no meaningful unit would result, even though the vectors themselves could be combined. With multiplication and division, however, different units can be combined as long as the same scale is used for both basic units. This does not mean that just any combination can be multiplied together because there is not a corresponding unit for every combination. But for example, Ohm's law states that current multiplied by resistance equals circuit voltage. So current (in amperes) times resistance (in ohms) equals circuit voltage (in volts). This is an example of a product combination that is meaningful with respect to the units involved. However, if we were graphing such a product it would be necessary to let the same length of line represent an ampere, an ohm, and a volt. Otherwise the magnitude of the product would be incorrect.

In the example of Fig. 3-18 only the basic units of the terms are considered, there is no statement of what the units may be. The two original vectors are  $3/20^\circ$  and  $4/45^\circ$ ; suppose that each unit on the graph is  $\frac{1}{4}$  inch in length. F1 would be  $\frac{3}{4}$  inch long on the graph, and F2 would be 1 inch long. R is  $12/65^\circ$  so its length would be 3 inches on the  $65^\circ$  line. As you can see, the resultant was obtained by algebraic, not graphical, means.

However, there is one interesting relationship that can be used even though it is not used to any extent in practical work. If a vector of unit length ( $\frac{1}{4}$  inch in our example) is laid out on the  $0^\circ$  line, an angle  $\theta$  is formed by connecting the ends of the unit line and F1. As shown, the same angle ( $\theta$ ) exists when a line is drawn between the ends of F2 and R.

As is probably obvious, multiplication and division are best performed by algebraic means, either in rectangular or polar form. All of these processes are described and illustrated more

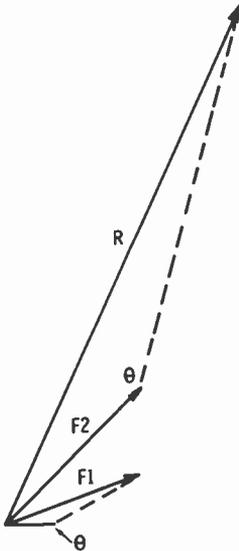


Fig. 3-18. Finding the resultant by considering only the basic units of the terms.

thoroughly in the next two chapters. But for now we can assume that in polar-form multiplication the magnitudes are multiplied together and the angles added. This means that:

$$(F1/\theta1) (F2/\theta2) = F1F2/\theta1 + \theta2$$

In division the magnitudes are divided and the angles subtracted, as follows:

$$\frac{F1/\theta1}{F2/\theta2} = \frac{F1}{F2} / \theta1 - \theta2$$

In some cases multiplication (or division) may be used as part of an entire problem, but it has no real meaning in itself. One example can be found in the product-over-the-sum method for determining total impedance:

$$Z_T = \frac{Z1Z2}{Z1 + Z2}$$

Multiplying  $Z_1Z_2$  gives a specific product, but it has no specific meaning unless it is combined with the rest of the equation. Then  $Z_T$  has definite meaning. In electronics problems concerning vectors a number of such examples exist.

### ROTATING VECTORS

The basic idea underlying a rotating vector was described previously, but in this chapter it is carried further. In Chapter 1 a simple generator was used to generate a sine wave and the amplitude at any instant of time was shown to be dependent on the angle at which the magnetic lines of force were cut. The same idea can be illustrated vectorially, using the diagram of Fig. 3-19. The radius of the circle is assumed to be a rotating vector, and several different instantaneous positions are shown. Positive (CCW) rotation is assumed, beginning at zero degrees.

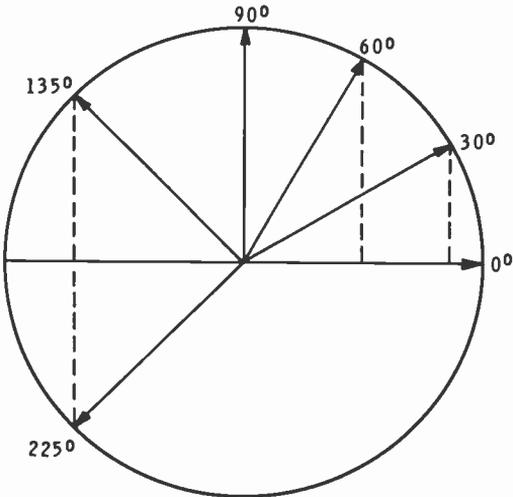


Fig. 3-19. Representing a rotating vector and several different instantaneous positions.

The length of the vector represents the maximum amplitude of the vector and is constant for any given problem, even though the instantaneous amplitude is maximum at only two angles in the complete sine-wave cycle. At any instantaneous position of rotation a vertical line drawn from the end of the rotating vector to the horizontal axis represents the instantaneous amplitude. The length of this line is proportional to the sine of that angle, because with a given length of vector

$\sin \theta = O/H$ . The vector is the hypotenuse so that this proportionality holds true.

Several examples are illustrated in Fig. 3-19. At the  $30^\circ$  angle the dashed line is one-half the length of the vector (hypotenuse). At  $60^\circ$ , the dashed line is .866 of the vector length; since both are above the horizontal axis, they are positive. A second-quadrant angle of  $135^\circ$  is also shown, and  $\sin 135^\circ = .707$ . The instantaneous amplitude at that angle is .707 of the maximum amplitude. At  $225^\circ$  the sine is  $-.707$ ; therefore the instantaneous amplitude is also negative, as it is throughout the third and fourth quadrants. This also verifies the statement made previously, that at any angle, the instantaneous amplitude of the sine wave is equal to the maximum amplitude multiplied by the sine wave of the angle.

When two sine waves are involved, they may or may not be

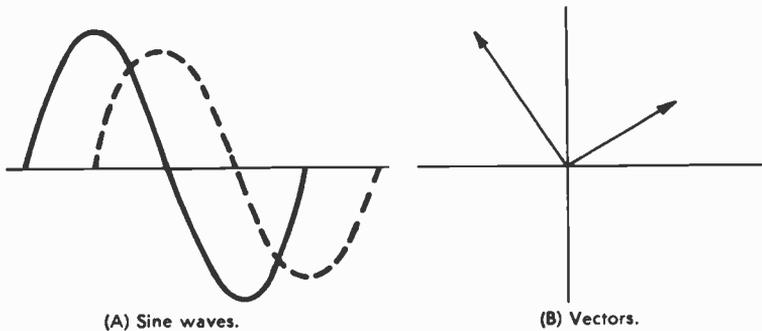


Fig. 3-20. Two ways of showing lead and lag.

in phase as previously indicated. Fig. 3-20 shows two ways of showing lead or lag. In Fig. 3-20A both sine waves are shown, and by proper labeling it can be indicated whether current is leading or lagging. The entire cycles are shown, because to show only small sections of cycles may prove to be a hazard in determining phase relationships. However, the diagram in Fig. 3-20B, with proper labeling, can show the same thing with two vectors, although some texts call this a phasor diagram, as previously indicated. The vectors can be shown at any angles in the cycles, but the angular difference between them should always be the same. By careful checking it is seen that the angular differences in both diagrams are the same.

In later chapters it will be pointed out that it is more or less standard to show one vector at  $0^\circ$  and the other at some other angle. This is easier, both for drawing and for reference purposes. The same type of diagram can represent phase relation-

ships of two voltages, two currents, or of a voltage and a current. All of the diagrams would be the same except for labeling. More than two vectors can be included in either type of diagram, as will be seen in the last two chapters.

### PRACTICE PROBLEMS

1. A vector is 12 units in length at  $35^\circ$ . What are the X and Y rectangular co-ordinates?
2. The terminal end of a vector lies at point (6,4). Express the vector in polar form.
3. By any convenient method add the vectors  $5/20^\circ$  and  $6/58^\circ$ .
4. By any convenient method add the vectors  $8/-35^\circ$  and  $6/40^\circ$ .
5. By any convenient method add the vectors which have terminal ends at points (-5,6) and (8,3).
6. Subtract vector  $7/18^\circ$  from  $15/72^\circ$ .
7. Subtract vector (4,6) from (-8,5).
8. Multiply vectors  $6/15^\circ$  and  $3/20^\circ$ .
9. Multiply vectors  $8/-38^\circ$  and  $5/32^\circ$ .
10. Divide vector  $16/48^\circ$  by  $8/18^\circ$ .

## CHAPTER 4

# Complex Numbers

In the previous chapters it was shown how vectors can be resolved in terms of horizontal and vertical components, considering them as X and Y co-ordinates. It was also shown that a vector can be expressed in two different ways with respect to the co-ordinate axes. As an example, a vector could be expressed in polar form, such as  $5/\underline{53.1^\circ}$ ; or the same vector can be expressed in rectangular form with an X value of 3 and a Y value of 4. With the rectangular form it is difficult to express the quantities with convenient algebraic notation. Therefore, in this chapter a type of algebra, in which these vectors can be expressed, is introduced.

This is called *Vector Algebra*. When it is used, vectors can be added, subtracted, multiplied, or divided, all in algebraic form. If required, the vector can be raised to a power, or a specific root of the vector can be determined. These operations include both polar and rectangular form, plus conversion from one form to the other when that may be required.

Most of the groundwork for Vector Algebra has been introduced in previous chapters. This includes the two forms, polar and rectangular, plus the basic idea of plotting vectors on co-ordinate axes. The numbers above the horizontal (X) axis are positive; those below it are negative. Those numbers to the right of the vertical (Y) axis are positive; those to the left

are negative. This enables us to plot angles in any of the four quadrants. Also X and Y can be positive or negative, depending on the quadrant in which the vector lies.

In this chapter the algebraic expressions for vectors are introduced, and then the various mathematical operations involving such numbers are explained. The main requirement for setting up such a system is that there be some way to show which is the X and which is the Y co-ordinate in the rectangular form. The standard notation for polar form has already been introduced, and it also forms a part of Vector Algebra. In addition, Vector Algebra draws both from conventional algebra and from right-angle trig in order to determine problem solutions.

### IMAGINARY NUMBERS

Algebraic notation for vectors in rectangular form is based on a concept called *imaginary numbers*. These numbers are not imaginary because they don't exist, but rather because they cannot be evaluated in real numbers. Imaginary numbers occur when the square root (or any even-numbered root) of a negative number is taken. When a number is squared (multiplied by itself), the resulting product is always a positive number. As examples:

$$3 \times 3 = 9$$

$$(-3) \times (-3) = 9$$

The results are the same regardless of whether a positive or a negative number is squared; the product is always positive. Conversely, to take the square root of a positive number gives either a positive or a negative root. In a given problem it may not be known which is correct, or even if there is a correct one.

For example,  $\sqrt{9} = 3$ , but  $\sqrt{9}$  could also be  $-3$ . Usually  $\sqrt{9}$  is shown as  $\pm 3$ , indicating that either root would satisfy the equation. In practice, however, the positive root is considered as the principal root because it has more meaning. To illustrate, suppose that a problem were solved and the answer showed 15 ohms of resistance. The negative value has no meaning, so the principal root, positive 15 ohms, would be assumed to be the answer. In the solution of an equation, however, say  $x^2 = 25$ , both  $+5$  and  $-5$  satisfy the equality so we say that there are two roots,  $\pm 5$ .

In some calculations it is necessary to determine the square root of a negative quantity, for example  $\sqrt{-9}$ . When determining the square root of a number, the process is to find the number which when squared produces the quantity under the radi-

cal sign. Squaring always produces a positive quantity, so there can be no value for  $\sqrt{-9}$ , nor for the square root of any negative number. Hence, these are called imaginary numbers. It can be considered that an imaginary quantity consists of a real number and an imaginary component multiplied together, as shown by this example:

$$\sqrt{-9} = \sqrt{9} \sqrt{-1} = \pm 3\sqrt{-1}$$

As long as this is true, the only imaginary number connected with this operation is  $\sqrt{-1}$ . Any other quantity can be resolved similarly, indicating that the only imaginary number is  $\sqrt{-1}$ . A small *i*, representing the word "imaginary," is used as a symbol for  $\sqrt{-1}$ . This was formulated by mathematicians many years ago, and it could be considered that imaginary was an unfortunate choice of adjective. The word often creates misconceptions, especially to the beginning student of mathematics.

When electricity and electronics became practical realities, the letter "i" was used for instantaneous current, so a possibility of confusion arose. For this reason, the lower-case *j* is used for  $\sqrt{-1}$  in electronics although most math books still use the conventional *i*. We can therefore assume that:

$$\sqrt{-1} = i = j$$

Since this text is primarily concerned with electronics, the *j* will be used to indicate the imaginary number in every case. This means that  $\sqrt{-9}$ , for example, would be written  $\pm 3\sqrt{-1}$ , or  $\pm 3j$ , with the latter form preferred.

Before proceeding, one minor modification should be noted. There is a possibility that  $3j$  could be confused with terms such as  $3a$ ,  $3x$ , etc. in an algebraic expression. So in electronics expressions the *j* is listed before its numerical coefficient, and  $3j$  is noted as  $j3$ . But a word of caution with respect to this; be careful in writing these or else  $j3$  could resemble  $j^3$ . Also, *j* should not be used in any algebraic expression unless the imaginary  $\sqrt{-1}$  is intended. Any other usage could introduce doubt on the part of the reader and definite difficulties in problem solution.

A binomial is an expression that contains two terms, such as  $x + y$ ,  $2a - b$ ,  $3x - 4y$ , etc. Other binomials can consist of a real number (one that can be evaluated) and an imaginary component. As long as the imaginary term cannot be evaluated, it cannot be combined with the real number. Examples of such binomials are  $3 + j5$ ,  $2 - j3$ ,  $-5 - j4$ ,  $-2 + j$ , etc. These are called complex numbers, and each consists of a real and an im-

aginary term. The numerical coefficient of the  $j$  term is always a real number, but an imaginary factor multiplied by a real one results in another imaginary term. We must consider also that either of these terms can have zero as its numerical coefficient. If the real-number coefficient is zero, the binomial is imaginary. If the  $j$  coefficient is zero, the complex quantity is a real number. Complex-number terms can also have literal coefficients, such as  $a + jb$ ,  $x - jy$ , etc. However, these occur more often in general expressions rather than those pertaining to specific problems.

The following assumptions must be made with regard to imaginary numbers:

$$\begin{aligned}j &= \sqrt{-1} \\j^2 &= \sqrt{-1} \sqrt{-1} = -1 \\j^3 &= j^2 \cdot j = -1 \cdot j = -j \\j^4 &= j^2 \cdot j^2 = (-1) (-1) = +1\end{aligned}$$

Notice that  $j^2$  equals  $-1$ . This could be considered as a departure from the usual rules of algebra. This meaning, however, should be better understood after studying the graphic analysis of imaginary numbers, which is presented in the next section of this chapter. Of the assumptions just made,  $j$  and  $j^2$  occur much more often in practical problems than do the others. In every case,  $j^2$  should be evaluated as  $-1$  before proceeding in the problem being solved and  $j$  should be substituted for  $\sqrt{-1}$ .

### Graphic Analysis

Two real numbers are plotted on the X axis in Fig. 4-1 and are labeled  $+A$  and  $-A$ . Counterclockwise rotation is assumed in designating the angles.  $A$  and  $-A$  are opposite each other and can be considered to be  $180^\circ$  apart. If  $A$  is multiplied by  $-1$ , the resultant product is  $-A$ . Then if  $-A$  is multiplied by  $-1$ , the resultant product is  $A$ . This means that each time a vector is multiplied by  $-1$  the vector has been rotated  $180^\circ$ . Then if  $(-1)(A)$  is a rotation of  $180^\circ$  and  $j^2$  equals  $-1$ , it can be assumed that a rotation of  $j$  turns the vector by  $90^\circ$ . This is also shown in Fig. 4-1, where  $A$  has been multiplied by  $j$  to give a product of  $jA$ .

Another multiplication by  $j$  gives  $j^2A$  or  $-A$ , placing the vector at  $180^\circ$ . Then  $-A$  times  $j$  gives  $-jA$  which represents an angular position of  $270^\circ$ . And  $-jA$  times  $j$  gives  $-j^2A$  or  $+A$ , after substituting  $-1$  for the  $j^2$  factor.  $A$  is then  $360^\circ$ , which is the same vector position as  $0^\circ$ . Note that real numbers

occur on the X axis and that imaginary numbers occur on the Y axis, providing a method for plotting complex numbers on co-ordinate axes.

A complex number such as  $3 + j5$  can be plotted as in Fig. 4-2. The diagonal drawn from the origin is the hypotenuse of a right triangle, and the rectangular components are the shorter sides of the same triangle. Such a binomial could also be plotted in any of the other quadrants except that at least one of the rectangular components would be negative. The hypotenuse (vector length) is assumed to be positive regardless of the quadrant in which it appears.

These ideas are used in electrical and electronics calculations by plotting resistance on the real-number axis and reactance on the imaginary axis. By using the Pythagorean relationship, impedance is found to be the diagonal (hypotenuse). Such a plot can be made because of the  $90^\circ$  phase difference between resistance and reactance. This does not mean that reactance is imaginary, but the idea provides a convenient method of notation, as will be seen in the next chapter.

### Working With Imaginaries

Conventional math methods hold for imaginaries as long as  $j^2$  is assumed to equal  $-1$ , and that substitution is made each time  $j^2$  appears. The usual rule in most problems is to change

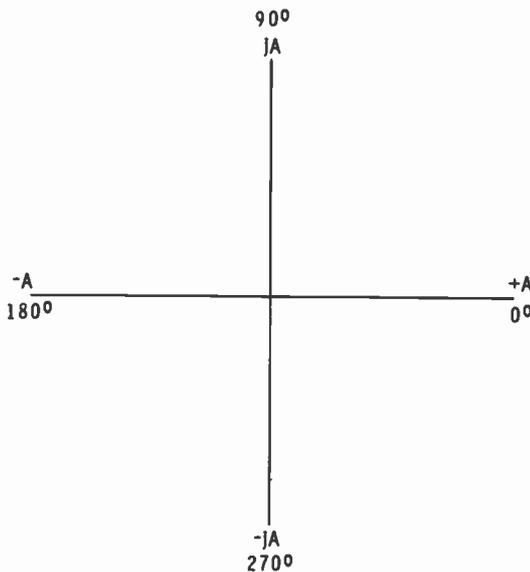


Fig. 4-1. Multiplying  $A$  by  $j$  to give  $jA$ .

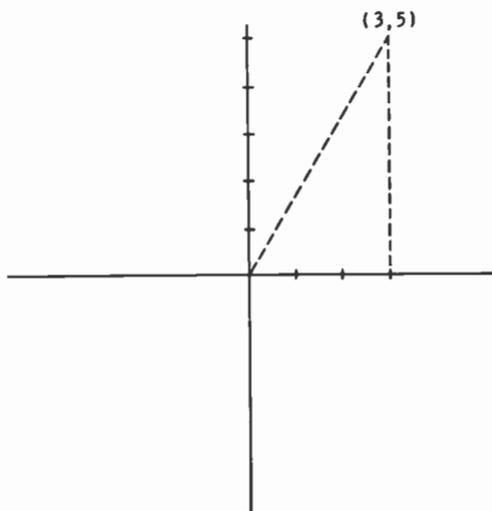


Fig. 4-2. Plotting a complex number.

$\sqrt{-1}$  to  $j$  each time it appears before attempting to perform any operations on the problem. This is especially evident in the example multiplication problems shown next.

$$1. \quad \sqrt{-3} \cdot \sqrt{-2} = j\sqrt{3} \cdot j\sqrt{2} = j^2\sqrt{6} = -\sqrt{6}$$

Notice that  $(-3)$  multiplied by  $(-2)$  within the radicals would have given a product of  $+6$ , which is incorrect. The conversion to the  $j$  form is also evident in the following examples of addition, subtraction, and division.

$$2. \quad \sqrt{-3} + \sqrt{-3} = j\sqrt{3} + j\sqrt{3} = j2\sqrt{3}$$

Here the  $j$  is treated as any other base number, and the numerical coefficients are added. But this may be illustrated better in the next example problem in which the radicands are perfect squares.

$$3. \quad \sqrt{-9} + \sqrt{-16} = j\sqrt{9} + j\sqrt{16} = j3 + j4 = j7$$

Subtraction is similar to addition, it merely involves algebraic addition of the numerical coefficients and uses the sign of the larger term.

$$4. \quad j6 - j4 = j2$$

$$5. \quad j3 - j8 = -j5$$

Example problem 6 involves division of imaginary terms in which the  $j$ 's are divided out and do not appear in the quotient.

$$6. \frac{\sqrt{-6}}{\sqrt{-2}} = \frac{j\sqrt{6}}{j\sqrt{2}} = \sqrt{3}$$

As with other algebraic expressions involving radicals it is not considered proper form to leave the radical in the denominator, and  $j$  is considered to be a radical because it is equal to  $\sqrt{-1}$ . The radical is eliminated from the denominator by rationalization, the process of multiplying the numerator and the denominator by the quantity that eliminates the radical from the denominator. An example of this is problem 7.

$$7. \frac{\sqrt{8}}{\sqrt{-2}} = \frac{\sqrt{8}}{j\sqrt{2}} = \frac{\sqrt{4}}{j} = \frac{2}{j} \cdot \frac{j}{j} = \frac{j2}{-1} = -j2$$

### CONVERSION OF FORMS

It is frequently necessary to convert a vector from polar to rectangular form, or vice versa, when solving a particular problem. We have already seen that the polar and rectangular forms of a vector express the same thing, except in different ways. In the example previously given,  $3 + j4$  is the same as  $5/53.1^\circ$ . Each form has its uses, also its limitations, for algebraic purposes. For example, addition or subtraction of polar-form vectors cannot be performed, except by graphic means. Vector algebra is desirable in problem solutions because it provides greater accuracy than the graphical methods, although graphical methods provide a visual image of the problem being solved.

For an illustration of conversion of forms, Fig. 4-3 is used. This shows a first-quadrant vector in which the X value is 3 and the Y value is 5. Expressed in rectangular form it is  $3 + j5$ . Probably the easiest way to convert to polar form is to use the Pythagorean formula to determine Z and the arc tangent to determine the angle  $\theta$ . The polar form, expressed generally, would be  $Z/\theta$ .

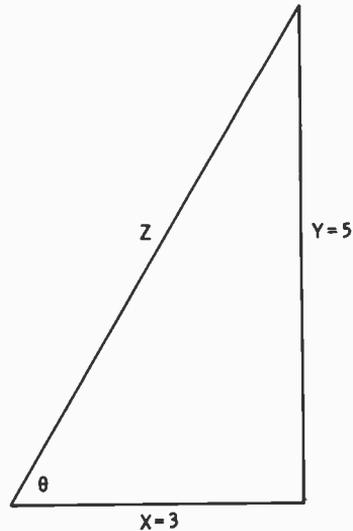
$$Z = \sqrt{X^2 + Y^2} = \sqrt{9 + 25} = \sqrt{34} = 5.83$$

$$\theta = \arctan \frac{Y}{X} = \arctan \frac{5}{3}$$

$$= \arctan 1.66 = 58.9^\circ$$

Therefore,  $3 + j5$  is the same as  $5.83/58.9^\circ$ . If the rectangular form had been  $3 - j5$ , a fourth-quadrant angle would be described, and the polar form would be  $5.83/-58.9^\circ$ . The length of Z is the same, only the angle is changed.

Fig. 4-3. Illustrating conversion of forms.



The usual method of converting from polar to rectangular form was mentioned in the previous chapter with regard to resolving a vector into its rectangular components. Here, the idea is expanded to include the  $j$  notation.

$$\cos \theta = \frac{X}{Z}, \text{ so } X = Z \cos \theta$$

$$\sin \theta = \frac{Y}{Z}, \text{ so } Y = Z \sin \theta$$

$Y$  is plotted on the imaginary axis so it would be noted as  $jZ \sin \theta$ . The full conversion then is:

$$\underline{Z/\theta} = Z \cos \theta + jZ \sin \theta$$

$$\text{or, } = Z(\cos \theta + j \sin \theta)$$

This is sometimes referred to as the trigonometric form, but it is merely a general statement of the rectangular form including the trig functions.

As an example, consider the vector  $6/32^\circ$ .

$$\begin{aligned} \underline{6/32^\circ} &= Z \cos \theta + jZ \sin \theta \\ &= 6 \cos 32^\circ + j6 \sin 32^\circ \\ &= 6(.8480) + j6(.5299) \\ &= 5.088 + j3.1794 \end{aligned}$$

Depending on the accuracy required, this answer may be rounded off to  $5.09 + j3.18$ . If the polar expression had been

$6/\underline{-32^\circ}$ , the rectangular equivalent would have been  $5.09 - j3.18$ . When the angle in any problem is greater than  $90^\circ$ , it means that the vector lies in other than the first quadrant. The calculations are the same, but the proper signs must be applied to the rectangular terms. As an example, consider  $6/\underline{148^\circ}$  shown in Fig. 4-4. In this quadrant the cosine is negative and the sine positive, therefore,

$$\begin{aligned} 6/\underline{148^\circ} &= 6 \cos 148^\circ + j6 \sin 148^\circ \\ &= -6 \cos 32^\circ + j6 \sin 32^\circ \\ &= -6(.8480) + j6(.5299) \\ &= -5.09 + j3.18 \end{aligned}$$

This is the same vector as solved in the previous problem, but it is in a different quadrant. Similarly, when converting from rectangular to polar form, the quadrants must be con-

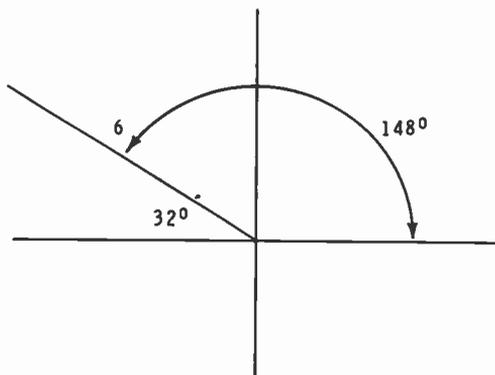


Fig. 4-4. Vector  $6/\underline{148^\circ}$  is the same as vector  $6/\underline{32^\circ}$ , except it is in a different quadrant.

sidered. The signs of the X and Y components indicate the quadrant. Vector length Z and the angle are calculated similarly for all quadrants. Bear in mind, however, that the angle determined from the trig table is always  $90^\circ$  or less. This is what we previously called the working angle and is not always the polar-form angle. For example, in the problem just solved, except working it in reverse, the rectangular form is  $-5.09 + j3.18$ . Using the arc tangent, an angle of  $32^\circ$  is read from the table. The polar-form angle (for second quadrant) is  $180^\circ$  minus the working angle. Therefore, in this problem the correct angle would be  $180^\circ$  minus  $32^\circ$ , or  $148^\circ$ . Careful attention to the signs will prevent errors in both conversions.

In most phase problems involving AC the angle is less than  $90^\circ$ , but in other vector problems angles greater than  $90^\circ$  occur often; therefore they should be considered in Vector Algebra. There is one other form of vector expression, the exponential form, which is used to a great degree in pure mathematics. However, it is not generally used in practical calculations and, consequently, is not considered in this text.

### VECTOR ALGEBRA

Complex numbers can be handled in the same manner as we do other types of binomials as long as the special significance of  $j$  is kept in mind and as long as the proper substitutions (for example  $j^2 = -1$ ) are made. These numbers can be added, subtracted, multiplied, divided, or worked with in terms of powers and roots. These are all the complex-number operations that are likely to be encountered in vector and phase analysis.

#### Addition and Subtraction

If complex numbers appear in rectangular form, they can be added or subtracted by combining the real parts and then the imaginary parts, taking the various plus and minus signs into account. Addition and subtraction are similar, but for the most part what is known as algebraic addition is used. This involves addition, but it also considers the signs. For example, add  $2 + j7$  and  $3 - j3$ :

$$\begin{array}{r} 2 + j7 \\ 3 - j3 \\ \hline 5 + j4 \end{array}$$

Actually, the  $j3$  was subtracted from  $j7$ , an example of what is meant by algebraic addition. If, however,  $3 - j3$  were subtracted from  $2 + j7$ , it would be necessary to change the signs of the number being subtracted. Thus:

$$\begin{array}{r} 2 + j7 \\ -3 + j3 \\ \hline -1 + j10 \end{array}$$

The difference is  $-1 + j10$ . This type of problem does not occur as often as the previous one because a problem is usually stated in mathematical terms rather than in the form of a sentence. In the previous chapter one of the subtraction methods presented was that of changing the signs of the number being subtracted. In rectangular form the changing of signs

represents a change of  $180^\circ$  in the direction of the vector. Several more examples of complex-number addition follow:

$$\begin{array}{r} 4 - j3 \\ 2 + j5 \\ \hline 6 + j2 \end{array} \qquad \begin{array}{r} -5 - j3 \\ 3 + j \\ \hline -2 - j2 \end{array} \qquad \begin{array}{r} 4 + j6 \\ -3 \\ \hline 1 + j6 \end{array} \qquad \begin{array}{r} 3 + j2 \\ -j3 \\ \hline 3 - j \end{array}$$

Polar-form vectors cannot be added or subtracted algebraically unless the lines representing the vectors are parallel. In all other cases the quantities must be converted to rectangular form, then added algebraically. If a polar resultant is needed, the rectangular sum must be converted into polar form.

### Multiplication

Multiplication of rectangular-form quantities can be performed simply as the multiplication of two binomials. Polar-form quantities can also be multiplied, as shown in the previous chapter, and this is probably the easier of the two methods. However, all the numbers being multiplied must be in the same form, either rectangular or polar. In this chapter multiplication in rectangular form is considered first. For example:

$$(3 - j2)(2 + j4)$$

Each term of the second factor must be multiplied by each term of the first, and the order of these operations is not important. Here is one form:

$$\begin{aligned} 6 - j4 + j12 - j^28 &= \\ 6 + j8 - (-1)8 &= \\ 6 + j8 + 8 &= 14 + j8 \end{aligned}$$

Notice that two terms result, one real and one imaginary as long as  $j^2$  is replaced by  $-1$ . Here is another example:

$$\begin{aligned} (2 + j4)(1 + j) &= \\ 2 + j4 + j2 + j^24 &= \\ 2 + j6 - 4 &= -2 + j6 \end{aligned}$$

When multiplying the sum and difference of the same two terms, the  $j$  term is eliminated and a real-number product occurs. For example:

$$\begin{aligned} (2 + j)(2 - j) &= \\ 4 + j2 - j2 - j^2 &= \\ 4 - (-1) &= 4 + 1 = 5 \end{aligned}$$

This type of combination is used when dividing by a complex number, as will be shown later. Such numbers are said to be conjugates of each other; they are alike, except for the signs separating them. Thus,  $2 + j$  is the conjugate of  $2 - j$ , and vice versa. Here are several other examples of conjugate numbers:

$$\begin{aligned} & -2 + j \text{ and } -2 - j \\ & \quad j4 \text{ and } -j4 \\ & \quad 3 \text{ and } 3 \end{aligned}$$

The product of conjugate numbers is always a real number; likewise, the sum of conjugate numbers.

Polar-form multiplication involves multiplying the magnitudes together and algebraically adding the angles, as previously explained. The magnitudes are always positive values, but the angles can be either positive or negative, and the products are in polar form. Here are four examples:

1.  $(2/40^\circ)(3/35^\circ) = 6/75^\circ$
2.  $(4/-30^\circ)(2/48^\circ) = 8/18^\circ$
3.  $(1.5/10^\circ)(2/-26^\circ) = 3/-16^\circ$
4.  $(3/-25^\circ)(4/-120^\circ) = 12/-145^\circ$

More than two vectors can be multiplied as evidenced by these next two examples:

5.  $(3/40^\circ)(2/25^\circ)(4/60^\circ) = 24/125^\circ$
6.  $(2/35^\circ)(5/-50^\circ)(6/15^\circ) = 60/0^\circ$

### Division

Division of polar-form quantities is similar to multiplication, except, of course, that they are inverse operations. The magnitudes are divided arithmetically to obtain the magnitude of the quotient. Then the angle of the divisor is subtracted from the angle of the vector being divided. This involves changing the sign of the divisor angle and adding the angles algebraically. The quotient is also in polar form, as seen in these examples:

$$\begin{aligned} \frac{15/45^\circ}{3/15^\circ} &= 5/30^\circ & , & & \frac{2/80^\circ}{6/-20^\circ} &= \frac{1}{3}/100^\circ \\ \frac{9/-50^\circ}{4.5/15^\circ} &= 2/-65^\circ & , & & \frac{12/-32^\circ}{4/-70^\circ} &= 3/38^\circ \end{aligned}$$

Dividing in rectangular form involves only rationalizing the denominator to eliminate the  $j$  term from it, and then simplifying the quotient as much as possible. If the denominator includes only the  $j$  term, multiply the numerator and the denominator by the denominator to determine the quotient, as in these examples:

$$1. \quad \frac{5}{j2} = \frac{5}{j2} \cdot \frac{j2}{j2} = \frac{j10}{j^24} = \frac{j10}{-4} = -j2.5$$

$$2. \quad \frac{3 + j2}{j4} = \frac{3 + j2}{j4} \cdot \frac{j4}{j4} = \frac{j12 + j^28}{j^216} = \frac{j12 - 8}{-16} = \frac{2 - j3}{4}$$

When dividing by only a real number, the quantity can usually be left as it is. Or else it can be broken into two fractions provided that it is reduced to lowest terms, as in these examples:

$$3. \quad \frac{3 + j}{4} = \frac{3 + j}{4} \text{ or } \frac{3}{4} + \frac{j}{4}$$

$$4. \quad \frac{4 + j3}{2} = \frac{4 + j3}{2} \text{ or } 2 + \frac{j3}{2}$$

Dividing by a complex number involves multiplying the numerator and the denominator by the conjugate of the denominator, as in this example:

$$5. \quad \frac{4 + j3}{1 - j} = \frac{4 + j3}{1 - j} \cdot \frac{1 + j}{1 + j} = \frac{4 + j3 + j4 + j^23}{1 - j^2}$$

$$= \frac{4 + j7 + j^23}{2} = \frac{1 + j7}{2} \text{ or } \frac{1}{2} + \frac{j7}{2}$$

### **Powers and Roots**

Determining powers and roots of polar vectors involves methods similar to those used in multiplication and division. Raising a number to a power involves a series of multiplications. As an example,  $2^4$  is the same as  $2 \cdot 2 \cdot 2 \cdot 2$ , using 2 as a factor 4 times. To raise a polar vector to a power, raise the magnitude of the vector to the indicated power. Then multiply the angle by the number representing that power. Here are three examples:

$$1. \quad (4/\underline{21^\circ})^2 = 16/\underline{42^\circ}$$

$$2. \quad (3/\underline{-15^\circ})^2 = 9/\underline{-30^\circ}$$

$$3. \quad (2/\underline{25^\circ})^3 = 8/\underline{75^\circ}$$

Raising rectangular-form vectors to a power can be done as a series of multiplications. The process is usually easier in polar form, especially when raising the vector to the third or higher power. For example, to raise a vector in the rectangular form to the fourth power involves three separate multiplications. Although it is not used to any great extent, DeMoivre's Theorem can be used. Stated in general form:

$$[Z(\cos \theta + j \sin \theta)]^n = Z^n (\cos n \theta + j \sin n \theta)$$

where,

$n$  is any real number.

This theorem will not be expanded on here because of its lack of general use.

Roots of vectors could be taken by the same formula, but in these cases  $n$  would be fractional. For example, in determining a square root,  $n$  has a value of  $\frac{1}{2}$ . But roots are almost always performed in polar form by determining the indicated root of the magnitude, then dividing the angle by the index of the root. These are examples:

1.  $\sqrt{9/42^\circ} = 3/21^\circ$
2.  $\sqrt{16/-20^\circ} = 4/-10^\circ$
3.  $\sqrt[3]{8/39^\circ} = 2/13^\circ$

It was previously indicated that square-root problems give two results. For ordinary numbers these are plus and minus values, with the positive value being considered as the principal root. This is also true when determining the square root of a polar-form vector. Only one example is considered here because of the limited usefulness of the idea in practical vector problems. In the first example problem it was shown that:

$$\sqrt{9/42^\circ} = 3/21^\circ$$

If  $360^\circ$  is added to the angle, the vector direction remains unchanged, but the new notation is:

$$\sqrt{9/402^\circ} = 3/201^\circ$$

The problem could also be written:

$$\sqrt{9/-318^\circ} = 3/-159^\circ$$

This is the same result as obtained previously because:

$$3/-159^\circ = 3/201^\circ$$

So in addition to the principal root each square root of a polar-form vector has another root having the same magnitude

but with the angle changed by  $180^\circ$ . Powers and roots of vectors, however, are not used to nearly the extent as the other mathematical processes in practical electronics problems involving vector relationships.

### PRACTICE PROBLEMS

1.  $\sqrt{-5} \cdot \sqrt{-2} =$
2.  $\sqrt{-4} \cdot \sqrt{-3} =$
3.  $2\sqrt{-5} + \sqrt{-5} =$
4.  $\sqrt{-25} - \sqrt{-9} =$
5.  $j7 - j3 =$
6.  $j8 + j5 =$
7.  $j15 - j18 =$
8.  $\frac{\sqrt{-8}}{\sqrt{-2}} =$
9.  $\frac{\sqrt{6}}{\sqrt{-3}} =$
10.  $\frac{\sqrt{-8}}{\sqrt{2}} =$

Change to polar form:

11.  $5 + j6$
12.  $3 - j7$

Change to rectangular form:

13.  $6/38^\circ$
14.  $4/160^\circ$

Perform the indicated operations:

15.  $(3 + j2) + (4 - j3)$
16.  $(5 - j4) - (3 + j)$
17.  $4 + j2 + j3$
18.  $(3 + j)(2 + j2)$
19.  $(2 - j3)(3 + j5)$
20.  $(3 + j)(3 - j)$
21.  $(3/36^\circ)(2/18^\circ)$
22.  $(4/160^\circ)(3/-20^\circ)$
23.  $\frac{18/72^\circ}{9/30^\circ}$
24.  $\frac{15/-30^\circ}{3/-48^\circ}$
25.  $\frac{2 + j3}{j2}$
26.  $\frac{2 + j}{1 + j}$
27.  $\frac{3 - j3}{2 + j3}$
28.  $(3/15^\circ)^2$
29.  $(2/20^\circ)^2$
30.  $\sqrt{25/82^\circ}$

## CHAPTER 5

# Network Calculations

In this chapter most of the information presented so far is combined and used to calculate various characteristics of AC circuits. Graphic methods can be used for many of the calculations, but, as previously shown, algebraic methods are more accurate and more convenient, so greater use is made of them. Chapter 1 introduced several basic properties of resistance and reactance which form the basis of the calculations presented. In a pure inductance the current lags the voltage by  $90^\circ$ ; in a pure capacitance the current leads the voltage by  $90^\circ$ ; and in a pure resistance current and voltage are in phase with each other. Of course, no component is perfect; a resistor, for example, also has some inductance, and even some capacitance. Similarly, inductors have resistance and capacitance, etc. These calculations are based on the lumping of R, L, and C under their respective categories in a given problem. As an example, if R and L are connected in series, the R in the problem is considered to be the total resistance in the resistors, inductors, and even the wires.

This chapter presents series circuits first, then parallel circuits, and finally complex circuits (which are combinations of series and parallel arrangements). You will see that there are several different methods for solving some of the problems, for example, straight series and parallel circuits. In other cases,

Vector Algebra must be relied on so that all facets of the problem are considered. Phase relations must be included in virtually all of the calculations if correct results are to be obtained. In this chapter vectors are used to represent magnitude and phase, and, therefore, they could be called phasors because of their usage.

In order to simplify calculations, it will be assumed that  $R$ ,  $X_L$ , and  $X_C$  are known. If  $L$  or  $C$  is given instead, the reactance can be calculated and then included with the problem. To further simplify the problems, small numbers are used for resistance, reactance, and voltage, and they are stated in basic units. The same ideas apply to larger numbers, but working with them is more cumbersome, offering a greater chance for error.

### SERIES CIRCUITS

There are certain characteristics of series circuits which apply for both DC and AC, and these form the basis for most of our calculations of series-circuit quantities.

1. The current is the same at any point in the circuit.
2. The voltage drop across each component is equal to the current times the resistance (or reactance) of that component.
3. The sum of the individual voltages is equal to the applied voltage, although for AC the vector sum is implied.
4. The total circuit opposition is the sum of all the separate oppositions, although for AC the vector sum is implied.
5. Ohm's law applies to both DC and AC circuits.

#### R and L in Series

As an example of an RL series circuit, Fig. 5-1A is considered.  $R$  is the resistance,  $X_L$  is the inductive reactance, and  $E_a$

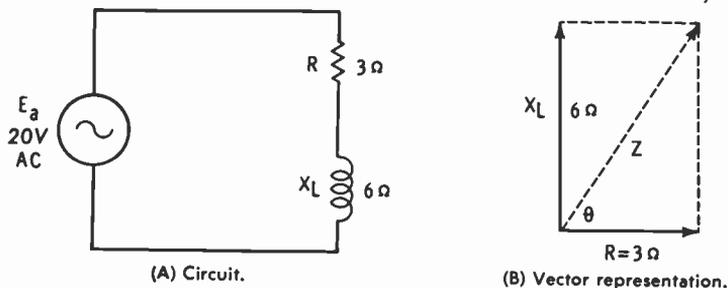


Fig. 5-1. R and L in series.

is the applied voltage. Current through the resistor is in phase with the voltage drop across it, but in the inductance the current lags the voltage drop by  $90^\circ$ . Current is the same through both components so that the separate voltage drops are  $90^\circ$  out of phase with each other. The circuit current lags the applied voltage by some angle between  $0^\circ$  and  $90^\circ$  because of the combined action of the two components.

A vector representation of the circuit oppositions is given in Fig. 5-1B. Resistance is plotted on the real axis (X), and reactance on the imaginary axis (Y). The reactance is inductive so it is plotted upward, giving a complex expression of  $3 + j6$  ohms. It is standard to plot  $X_L$  upward and  $X_C$  downward, the reason for which becomes more apparent from the vector representation of current and voltage (shown later).

The diagonal of the parallelogram in Fig. 5-1B represents the resultant of resistance and inductive reactance, which is impedance Z. To find the impedance, the resultant can be measured and calculated from the scale of the graph. A more accurate method of calculation, however, is the Pythagorean formula:

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{3^2 + 6^2} = \sqrt{45} = 6.71 \text{ ohms}$$

Calculations in this chapter are rounded off so as not to complicate the problems with long and tedious calculations involving many decimal places. Also, in the use of the trig tables no attempt at interpolation is made. Angles are read to the closest tenth of a degree. For these reasons there are slight discrepancies between similar answers obtained by different methods of calculation.

The phase angle ( $\theta$ ) must be calculated separately because it is not included in the Pythagorean formula:

$$\theta = \text{arc tan } \frac{6}{3} = \text{arc tan } 2 = 63.4^\circ$$

Therefore, impedance can be stated in two ways, as  $6.71/63.4^\circ$  ohms in polar form or as  $3 + j6$  ohms in rectangular form. Strict rules of mathematical writing may demand that parentheses be used with the rectangular form as  $(3 + j6)$  ohms since the unit applies to both terms. However, it is impossible to add ohms to any other quantity so it has become more or less conventional to omit the parentheses, as done previously.

Knowing the impedance of the circuit and the applied voltage, a simple Ohm's law calculation can be used to determine circuit current:

$$I = \frac{E_a}{Z} = \frac{20}{6.71} = 2.98 \text{ amperes}$$

Performing the same calculation in polar form gives this result:

$$I = \frac{E_a}{Z} = \frac{20/0^\circ}{6.71/63.4^\circ} = 2.98/-63.4^\circ \text{ amperes}$$

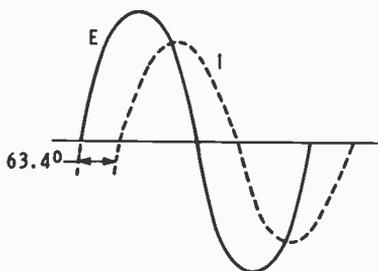
The negative angle is listed with respect to the applied voltage (assuming it to be at  $0^\circ$ ). This is logical because in an inductive circuit the current lags the voltage. Even so, it is customary to use current as the reference in series circuits because it is the same through all parts of the circuit. This means that a more standard representation is that of Fig. 5-2A, which is a graphic presentation of the phase conditions illustrated in Fig. 5-2B. And in the previous problem current would be assumed to be at  $0^\circ$  and applied voltage at  $+63.4^\circ$ .

Bear in mind though that these vectors represent phase and could be called phasors. The individual phases are continually changing because of the changing nature of the applied AC, but the phase difference is constant. Regardless of the part of the cycle that is considered, in every case the current is lagging the applied voltage by the phase angle. The vectors of Fig. 5-2A could have been drawn at any angles as long as voltage was shown to be  $63.4^\circ$  counterclockwise from the current. Considering one circuit parameter at  $0^\circ$  simplifies the calculations and has come to be conventional.

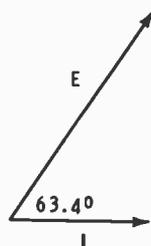
Voltage drop across a separate component can be calculated by Ohm's law, multiplying the current by the opposition offered by that component. In the same problem:

$$E_R = I \times R = 2.98 \times 3 = 8.94 \text{ volts}$$

$$E_L = I \times X_L = 2.98 \times 6 = 17.88 \text{ volts}$$



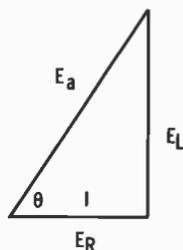
(A) Sine-wave displacement.



(B) Vector representation.

Fig. 5-2. Current and voltage with a phase difference of  $63.4^\circ$ .

Fig. 5-3. Vector representation of voltages in Fig. 5-1A.



Notice that the two voltages add up to an amount greater than the applied voltage because of the  $90^\circ$  phase difference between the two voltages. Fig. 5-3 is a vectorial representation of the voltages involved. Current and voltage across the resistor are plotted at  $0^\circ$ ; the reactive voltage is plotted at  $90^\circ$  because it leads the current (or current lags) by  $90^\circ$ . Applied voltage is the vector sum of the two individual voltages and is the diagonal as shown. The phase angle is that already calculated and is between  $0^\circ$  and  $90^\circ$ , depending on the relative values of resistance and reactance.

This triangle is similar to that of Fig. 5-1B. The sides are not the same lengths because the voltages are not the same as the circuit oppositions. But the relationships between the sides are the same, with  $\theta$  being  $63.4^\circ$ . From Fig. 5-3:

$$E_a = \sqrt{E_R^2 + E_L^2} = \sqrt{(8.94)^2 + (17.88)^2} = 20 \text{ volts}$$

And the phase angle could also have been determined by:

$$\theta = \arctan \frac{E_L}{E_R} = \arctan \frac{17.88}{8.94} = 63.4^\circ$$

Triangles are said to be similar when their angular distributions are the same, therefore, they have the same general shape. In both the impedance and the voltage triangles the angles are  $90^\circ$ ,  $63.4^\circ$ , and  $26.6^\circ$ , indicating that the comparable sides have the same length relationships in both cases.

Power in each of the components can be calculated from the current or from the voltage with the same results:

$$P_R = I^2 R = 2.98^2 \times 3 = 26.64 \text{ watts}$$

$$P_L = I^2 X_L = 2.98^2 \times 6 = 53.28 \text{ vars}$$

If voltage were used, the comparable formulas would be  $E^2/R$  and  $E^2/X_L$ . For reactive power the quantity can also be stated in watts, but it is conventional to state it in vars (volt-amperes reactive) to distinguish it from resistive or other types of power. That power in the resistance is called true

power; it is actually dissipated, or used. Reactive power is not used or dissipated because the voltage and current are  $90^\circ$  out of phase. This means that all of the power furnished to a circuit is not actually used, but some of it is returned to the supply line.

As previously shown, the true (resistive) power ( $P_T$ ) in the circuit under consideration is 26.64 watts. Line voltage times line current, however, gives a greater amount:

$$P_a = E_a \times I = 20(2.98) = 59.6 \text{ volt-amperes}$$

This is the apparent power ( $P_a$ ), the power that apparently is being used. The unit is volt-amperes and is used to distinguish it from true power which is expressed in watts. Power factor (PF) is the ratio of true to apparent power and expresses the fractional part of the total which is actually being used. In the same problem:

$$PF = \frac{P_t}{P_a} = \frac{26.64}{59.6} = .447$$

It is usually expressed as a decimal fraction, but it could also be expressed as a percentage, 44.7% in this example. This means that 44.7% of the apparent power is being dissipated. For a purely reactive circuit the power factor is zero; no power is used. For a purely resistive circuit the power factor is one (or 100%); all of the power that is supplied is actually used. For combination circuits the power factor is always between zero and one.

A power factor can also be expressed as the cosine of the phase angle, as is evident from reference to Fig. 5-7, which is described with reference to RCL circuits:

$$PF = \cos \theta = \cos 63.4 = .4478$$

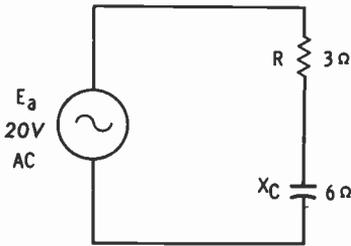
The slight difference is due to estimating arc tan 2 earlier in the discussion and in rounding off the results of the power calculation.

The power factor, for series circuits only, can also be calculated as follows:

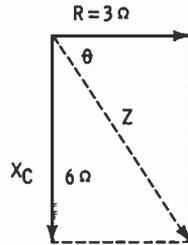
$$PF = \frac{R}{Z} = \frac{3}{6.71} = .4471$$

In summary, there are three methods of calculating the power factor of a series circuit:

$$PF = \frac{P_t}{P_a} = \cos \theta = \frac{R}{Z}$$



(A) Circuit.



(B) Vector representation.

Fig. 5-4. R and C in series.

The circuit under consideration is inductive so the current lags the applied voltage. Therefore the power factor can be referred to as being inductive, or lagging.

### R and C in Series

To simplify the coverage of RC series circuits the same values as used with the RL examples are used. The RC circuit is that of Fig. 5-4A and the impedance vectors are labeled Fig. 5-4B. In rectangular form the impedance is  $3 - j6$  ohms, but the absolute value of impedance is

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{3^2 + 6^2} = \sqrt{45} = 6.71 \text{ ohms}$$

The phase angle has the same magnitude as in the previous circuit, but it is in the opposite direction.

$$\theta = \arctan \frac{-6}{3} = \arctan (-2) = -63.4^\circ$$

An angle of  $-116.6^\circ$  is also the arc tangent of  $-2$ , but the phase angle is always  $90^\circ$  or less in these types of circuits—the obtuse angle is not considered. Impedance, then, can be expressed in rectangular form as  $3 - j6$  ohms or in polar form as  $6.71 / -63.4^\circ$  ohms. Current is calculated by  $E_a / Z$  just as with the RL circuit, and the values are the same. In fact all the calculations for the RC circuit are the same as for the RL circuit, except for the angle. However, the magnitudes of angles are the same; only the signs are different.

With current as the reference, the voltage of the RC circuit is at  $-63.4^\circ$ , lagging the current by that amount of angular displacement. So rather than repeat all the calculations, the various values for the circuit of Fig. 5-4A are as follows:

$$\begin{aligned} Z &= 6.71 / -63.4^\circ \text{ ohms} \\ I &= 2.98 / 63.4^\circ \text{ amperes} \\ E_R &= 8.94 \text{ volts} \end{aligned}$$

$$\begin{aligned}
 E_L &= 17.88 \text{ volts} \\
 P_R &= 26.64 \text{ watts} \\
 P_C &= 53.28 \text{ vars} \\
 P_a &= 59.6 \text{ volt-amperes} \\
 PF &= .447 \text{ (leading)}
 \end{aligned}$$

This means that series-circuit calculations can be simplified because the same relationships hold for both RL and RC circuits; i.e., as long as the direction of angle is considered whenever necessary.

### R, C, and L in Series

In a circuit containing all three circuit oppositions, R,  $X_L$ , and  $X_C$ , the two reactances effectively oppose each other. The circuit can act inductively or capacitively, depending on whichever reactance is larger. So the resultant circuit is composed of resistance and the difference between the two reactances. Such a circuit is shown in Fig. 5-5A where the rectangular form for impedance is:

$$\begin{aligned}
 Z &= R + jX_L - jX_C \\
 &= 5 + j7 - j3 = 5 + j4 \text{ ohms}
 \end{aligned}$$

The resultant circuit for calculation purposes is that of Fig. 5-5B. Impedance, current, phase angle, and power factor are calculated as though that were the circuit.

$$\begin{aligned}
 Z &= \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{5^2 + (7 - 3)^2} = \sqrt{5^2 + 4^2} \\
 &= \sqrt{25 + 16} = \sqrt{41} = 6.4 \text{ ohms}
 \end{aligned}$$

$$I = \frac{E_a}{Z} = \frac{20}{6.4} = 3.12 \text{ amperes}$$

$$\theta = \arctan \frac{4}{5} = \arctan .8 = 38.7^\circ$$

$$PF = \frac{R}{Z} = \frac{5}{6.4} = .7812$$

$$PF = \cos \theta = \cos 38.7^\circ = .7804$$

The small discrepancy occurred because, from the tangent table, the actual phase angle is somewhat less than  $38.7^\circ$ . But that is the closest value in the table. If the angle were slightly smaller than that shown, the cosine of that angle would be slightly larger.

Individual voltage drops and power in each of the components must be calculated from the actual circuit (Fig. 5-5A).

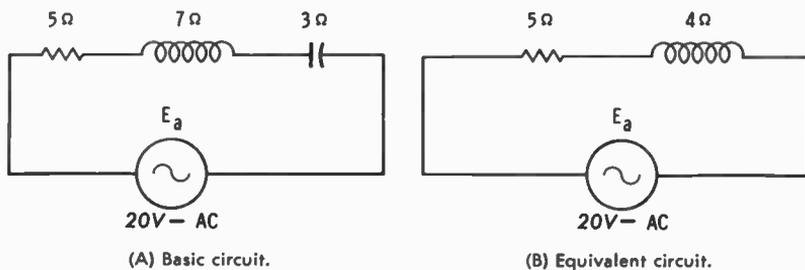


Fig. 5-5. An RCL circuit and its equivalent.

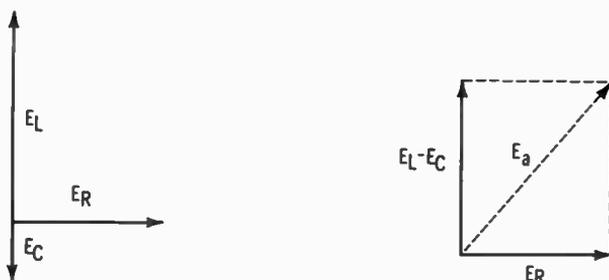
The combined voltages and powers, however, resolve to the simplified (Fig. 5-5B) because of the direct opposition of L and C in the circuit.

$$E_R = I \times R = 3.12(5) = 15.6 \text{ volts}$$

$$E_L = I \times X_L = 3.12(7) = 21.84 \text{ volts}$$

$$E_C = I \times X_C = 3.12(3) = 9.36 \text{ volts}$$

$$E_x = E_L - E_C = 21.84 - 9.36 = 12.48 \text{ volts}$$



(A) The three individual voltages.

(B) Resultant voltages.

Fig. 5-6. Voltages of Fig. 5-5 plotted as vectors.

The total reactive voltage is the difference between  $E_L$  and  $E_C$ , which is 12.48 volts. All three voltages are plotted as vectors (phasors) in Fig. 5-6A, but the resultant voltages are shown in Fig. 5-6B. The diagonal in this figure is the applied voltage, which can be determined from:

$$E_a = \sqrt{E_R^2 + E_x^2} = \sqrt{15.6^2 + 12.48^2} = \sqrt{243.36 + 155.75}$$

$$= \sqrt{399.11} = 19.98 \text{ volts}$$

which is very close to the 20-volt value given as applied voltage in the original problem.

Powers are solved similarly:

$$P_R = I^2R = (3.12)^2(5) = 9.73(5) = 48.65 \text{ watts}$$

$$P_L = I^2X_L = (3.12)^2(7) = 9.73(7) = 68.11 \text{ vars}$$

$$P_C = I^2X_C = (3.12)^2(3) = 9.73(3) = 29.19 \text{ vars}$$

$$P_X = P_L - P_C = 68.11 - 29.19 = 38.92 \text{ vars}$$

The total reactive power in the circuit is 38.92 vars, and a triangle can be set up from true power ( $P_R$ ), reactive power ( $P_X$ ), and apparent power ( $P_a$ ). Refer to the triangle in Fig. 5-7. This triangle is similar to the impedance triangle because the powers are obtained by multiplying the current squared times the number of ohms represented by each side of the impedance triangle. Therefore  $\theta$  is also the same value, as is also the power factor.

$$P_a = E_a \times I = 20 \times 3.12 = 62.4 \text{ volt-amperes}$$

$$PF = \frac{P_t}{P_a} = \frac{48.65}{62.4} = .78$$

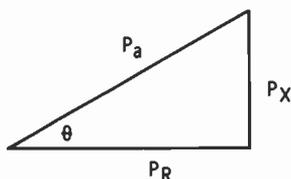


Fig. 5-7. Triangle showing  $P_R$ ,  $P_X$ , and  $P_a$ .

which is approximately the same as calculated by other methods.

Reactive power is not a major consideration in most requirements; therefore it will not be calculated in the remaining problems of this chapter. If required in a particular problem, however, it can be calculated as done in this example. And in all cases:

$$P_a = \sqrt{P_R^2 + P_X^2}$$

Three triangles are possible with series circuits—those representing impedance, voltage, and power. All are similar, although the side lengths are different in each triangle. If  $X_L$  is larger than  $X_C$  in a series circuit, then  $E_L$  is larger than  $E_C$ , and the circuit is inductive by an amount that depends on the relative circuit values. If  $X_C$  is larger than  $X_L$ , then  $E_C$  is larger than  $E_L$ , and the circuit is capacitive. Note that the straight series circuit can be calculated without using Vector Algebra, although it can be used as shown in some of the calculations. This is also true of straight-parallel arrangements which are described next. Vector Algebra finds its greatest application in combination or complex circuits, which are also

covered in this chapter. Its advantage is that the angles are carried with the problem along with the absolute quantities. This will become evident as complex circuit examples are presented.

### PARALLEL CIRCUITS

There are certain characteristics of parallel circuits that hold in all cases, and these should be considered in calculations involving parallel circuits.

1. Voltage is the same across each of the parallel branches.
2. Current through each branch is determined by voltage divided by the opposition (R or X) of that branch.
3. The sum of the currents of the individual branches is the total circuit current. For AC the vector sum is implied.
4. Ohm's law applies to each branch of the circuit, as well as to the total circuit.

#### RL in Parallel

In Fig. 5-8A a resistance of 4 ohms is connected in parallel with an inductive reactance of 6 ohms, with 24 volts applied. One of the easiest ways to calculate parallel circuit impedance is to use what is known as the current method. First the individual and then the total currents are determined, then the total current is divided into the applied voltage to calculate impedance. The advantage of this method is that only Ohm's law and the Pythagorean formula are needed. No special formulas are needed, although there are some that can be used.

$$I_R = \frac{E_a}{R} = \frac{24}{4} = 6 \text{ amperes}$$

$$I_L = \frac{E_a}{X_L} = \frac{24}{6} = 4 \text{ amperes}$$

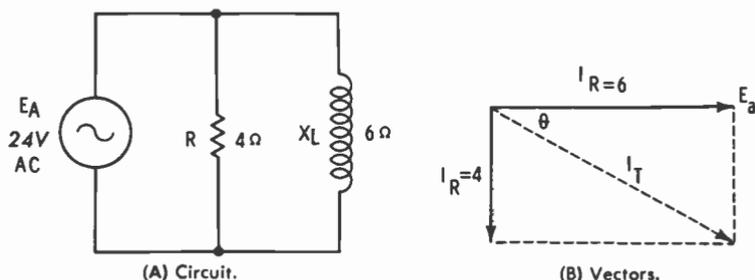


Fig. 5-8. Circuit and vectors of R and L in parallel.

The current through the resistance is in phase with the applied voltage; the current through the inductance lags the voltage by  $90^\circ$ , assuming pure inductance (no resistance). Vector representation of these currents is shown in Fig. 5-8B where  $I_T$  is the total current. The total current lags the applied voltage by phase angle  $\theta$ . This could be determined graphically, but an algebraic solution is more definite:

$$I_T = \sqrt{I_R^2 + I_L^2} = \sqrt{6^2 + 4^2} = \sqrt{36 + 16} = \sqrt{52} \\ = 7.21 \text{ amperes}$$

$$\theta = \text{arc tan } \frac{I_L}{I_R} = \text{arc tan } \frac{4}{6} = 33.7^\circ$$

The  $I_T$  vector is below the  $0^\circ$  reference so  $\theta$  is a negative angle. Total current, in polar form, is  $7.21 / -33.7^\circ$  amperes.

Impedance can be determined by Ohm's law, using polar notation and considering the voltage as the reference:

$$Z = \frac{E_a}{I_T} = \frac{24 / 0^\circ}{7.21 / -33.7^\circ} = 3.33 / 33.7^\circ \text{ ohms}$$

The positive impedance angle is indicative of an inductive circuit, which we already know in this problem. There is no capacitance. But the sign of the impedance angle is important for circuits containing both L and C and in which the resultant could be either inductive or capacitive, depending on relative values. This is especially true for complex circuits in which a visual examination cannot alone determine the type of reactance that predominates.

Note that the Pythagorean formula cannot be used for circuit oppositions connected in parallel. The arrangement  $Z = \sqrt{R^2 + X_L^2}$  holds only for series circuits. With parallel circuits the currents are added vectorially, not the circuit oppositions.

Phase angle has already been calculated so that the power factor can be calculated from it:

$$\text{PF} = \cos \theta = \cos 33.7^\circ = .8320$$

$\cos 33.7^\circ$  is the same value as  $\cos -33.7^\circ$  so either angle could have been used. We know, however, that it is a lagging power factor since the current lags the applied voltage. True power is that in the resistance, just as in series circuits, and can be calculated by any one of the three power formulas.

$$P_R = E_a \times I_R = 24 \times 6 = 144 \text{ watts}$$

$I^2R$  or  $E^2/R$  could have been used with the same results.

Apparent power is the applied voltage times the total current:

$$P_a = E_a \times I_T = 24 \times 7.21 = 173.04 \text{ volt-amperes}$$

$$PF = \frac{P_t}{P_a} = \frac{144}{173.04} = .8321$$

The results are very close to those obtained by determining the cosine of the phase angle. Also:

$$PF = \frac{Z}{R} = \frac{3.33}{4} = .8325$$

Note the difference here from the way that the power factor was calculated in series circuits. For series arrangements  $PF = R/Z$ , but  $\cos \theta$  and  $P_t/P_a$  hold for all types of circuits.

### RC in Parallel

Fig. 5-9A is used as an example of a parallel-connected RC circuit. This is the same circuit as Fig. 5-8A, except that  $X_L$  has been changed to  $X_C$  with the same numerical value. This simplifies the calculations for this part of the chapter since the absolute values for the various circuit characteristics are the same:

$$I_R = \frac{E_a}{R} = \frac{24}{4} = 6 \text{ amperes}$$

$$I_C = \frac{E_a}{X_C} = \frac{24}{6} = 4 \text{ amperes}$$

Current through R is in phase with the applied voltage, but the current through  $X_C$  leads the applied voltage by  $90^\circ$ , assuming pure capacitance (no resistance). Vector representation of these currents is shown in Fig. 5-9B.  $I_T$  is the vector sum of the

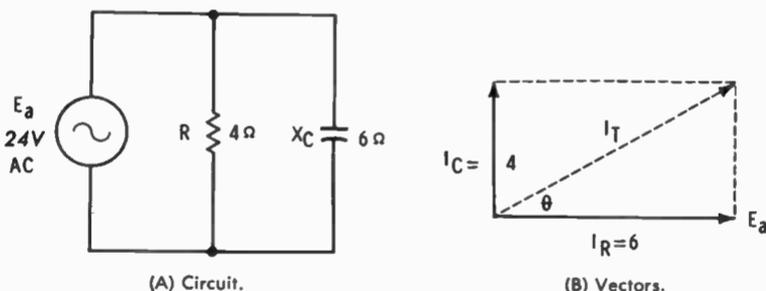


Fig. 5-9. Parallel-connected RC circuit.

individual currents; it leads the applied voltage by phase angle  $\theta$ .

$$I_T = \sqrt{I_R^2 + I_C^2} = \sqrt{6^2 + 4^2} = \sqrt{36 + 16} = \sqrt{52} \\ = 7.21 \text{ amperes}$$

$$\theta = \text{arc tan } \frac{I_C}{I_R} = \text{arc tan } \frac{4}{6} = 33.7^\circ$$

The  $I_T$  vector is above the  $0^\circ$  reference so that the phase angle is positive, and total current can be expressed as:

$$I_T = 7.21 / 33.7^\circ \text{ amperes}$$

$$Z = \frac{E_a}{I_T} = \frac{24 / 0^\circ}{7.21 / 33.7^\circ} = 3.33 / -33.7^\circ \text{ ohms}$$

The negative impedance angle is indicative of a capacitive circuit; conversion to rectangular form would show  $R - jX_C$ .

$$\text{PF} = \cos \theta = \cos 33.7^\circ = .8320$$

The result is the same as calculated for the comparable RL parallel circuit, but in the RC arrangement the power factor is leading. This is true because the current leads the applied voltage, as is normal in a capacitive circuit.

True and apparent powers are also calculated the same as in the previous circuit:

$$P_R = E_a \times I_R = 24 \times 6 = 144 \text{ watts}$$

$$P_a = E_a \times I_T = 24 \times 7.21 = 173.04 \text{ volt amperes}$$

$$\text{PF} = \frac{P_t}{P_a} = \frac{144}{173.04} = .8321$$

$\text{PF} = Z/R$  could also have been used with similar results. Note again that the absolute values of both circuits are the same, even the magnitude of the phase angle. The only difference is the sign of the angle, with total current lagging in the RL circuit and leading in the RC circuit.

### **RCL in Parallel**

In a circuit including all three types of circuit opposition the two reactances effectively oppose each other in determining total current. The total current can be either inductive or capacitive, depending on the relative values of the two reactances. Fig. 5-10 is used as an illustration of such a circuit. The applied voltage is common to all three branches so that it is used as the  $0^\circ$  reference in the calculations:

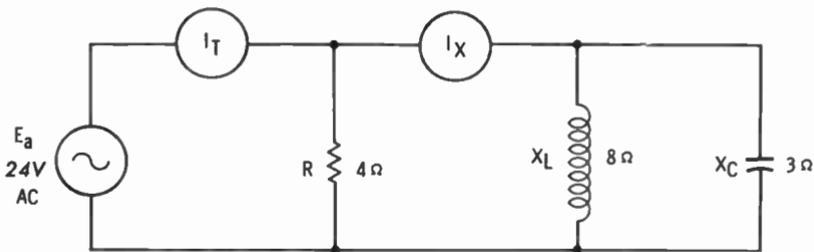


Fig. 5-10. Parallel-connected RCL circuit.

$$I_R = \frac{E_a}{R} = \frac{24}{4} = 6 \text{ amperes} = (6/0^\circ) = 6$$

$$I_L = \frac{E_a}{X_L} = \frac{24}{8} = 3 \text{ amperes} = (3/-90^\circ) = -j3$$

$$I_C = \frac{E_a}{X_C} = \frac{24}{3} = 8 \text{ amperes} = (8/+90^\circ) = +j8$$

Vector representation of these currents is illustrated in Fig. 5-11A with the resultant of these currents in Fig. 5-11B.  $I_C$  leads  $E_a$  by  $90^\circ$ , and  $I_L$  lags by  $90^\circ$ . Thus, the vector sum of the reactive currents is  $8 - 3$ , or 5 amperes of capacitive current. This means that the  $I_X$  meter in Fig. 5-10 reads 5 amperes and that current leads the voltage by  $90^\circ$ ; total circuit current is  $6 + j5$  amperes. As the two currents are at  $90^\circ$  with respect to each other, they can be added by the Pythagorean formula:

$$I_T = \sqrt{6^2 + 5^2} = \sqrt{36 + 25} = \sqrt{61} = 7.81 \text{ amperes}$$

$$\theta = \arctan \frac{5}{6} = \arctan .8333 = 39.8^\circ$$

$$I_T = 7.81 / 39.8^\circ \text{ amperes}$$

$$Z = \frac{E_a}{I_T} = \frac{24/0^\circ}{7.81/39.8^\circ} = 3.07 / -39.8^\circ \text{ ohms}$$

The impedance angle is negative, which is indicative of a capacitive circuit and which agrees with the previous calculations that showed capacitive current to be greater than inductive current.

Impedance could be solved directly by using Vector Algebra, but it is a longer process with more probable chance of error. The current method is probably the easier of the two, and therefore it is generally used. However, a vector solution is shown here to illustrate the ideas involved.

First the equivalent of  $X_L$  and  $X_C$  in parallel is determined by the product-over-the-sum method:

$$\begin{aligned} X &= \frac{(jX_L)(jX_C)}{jX_L + jX_C} = \frac{(j8)(-j3)}{j8 - j3} = \frac{-j^2 24}{j5} \\ &= \frac{24}{j5} \cdot \frac{j}{j} = \frac{j24}{-5} = -j4.8 \text{ ohms} \end{aligned}$$

Then this  $X$  value is combined with  $R$  in another product-over-the-sum calculation, the result of which is the impedance:

$$\begin{aligned} Z &= \frac{(R)(jX)}{R + jX} = \frac{(4)(-j4.8)}{4 - j4.8} = \frac{-j19.2}{4 - j4.8} \\ &= \frac{19.2/\underline{-90^\circ}}{6.25/\underline{50.2^\circ}} = 3.07/\underline{50.2} \text{ ohms} \end{aligned}$$

Results are the same as obtained using the current method.

True power is that amount dissipated in the resistance, just

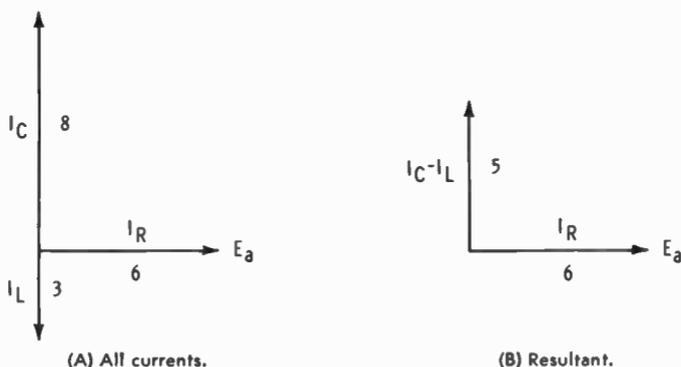


Fig. 5-11. Vector representation of the currents in Fig. 5-10.

as in other circuits. Apparent power is also calculated in the same manner—applied voltage times total current.

$$P_t = E_a \times I_R = 24 \times 6 = 144 \text{ watts}$$

$$P_a = E_a \times I_T = 24 \times 7.81 = 187.44 \text{ volt amperes}$$

$$PF = \frac{P_t}{P_a} = \frac{144}{187.44} = .7682$$

From the trig table:

$$PF = \cos \theta = \cos 39.8^\circ = .7683$$

$PF = Z/R$  could also have been used.

It should be remembered that in almost every problem the results vary slightly because of rounding off in the various methods used, and due to approximating the angles from the trig tables. By carrying the various computations to a larger number of decimal places and interpolating with the trig tables, almost exact results can be obtained, regardless of the method and assuming, of course, that a correct method is being used.

### COMPLEX CIRCUITS

As was previously pointed out, simple series and parallel AC circuits can be handled by Ohm's law and the Pythagorean relationship for almost all the usual types of calculations. Complex numbers are not required, although they can be used if

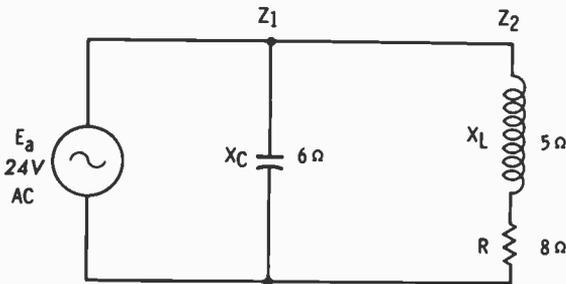


Fig. 5-12. A combination series-parallel circuit.

desired. For complex circuits (combinations of series and parallel) Vector Algebra can be used to the greatest advantage. The circuit of Fig. 5-12 is an example. For convenience of notation,  $Z_1$  indicates the impedance of the capacitive branch;  $Z_2$  the combined impedance of the resistance and inductance. Solution is by the current method, and as will be seen, vector notation is required because of the relationship of the angles.

$$Z_1 = 0 - j6 \text{ ohms} = 6 / \underline{-90^\circ} \text{ ohms}$$

$$I_1 = 0 + j4 \text{ amperes} = 4 / \underline{90^\circ} \text{ amperes}$$

$$Z_2 = 8 + j5 \text{ ohms} = 9.43 / \underline{32^\circ} \text{ ohms}$$

$$I_2 = 2.15 - j1.35 \text{ amperes} = 2.54 / \underline{-32^\circ} \text{ amperes}$$

Total current ( $I_T$ ) is:

$$\begin{array}{r} 0 \quad + j4 \\ \hline 2.15 - j1.35 \\ \hline 2.15 + j2.65 \text{ amperes} \end{array}$$

Converting to polar form :

$$I_T = 3.41 / \underline{50.9^\circ} \text{ amperes}$$

$$Z = \frac{E_a}{I_T} = \frac{24 / 0^\circ}{3.41 / \underline{50.9^\circ}} = 7.04 / \underline{-50.9} \text{ ohms}$$

Capacitive current is greater than that through the other branch, so the total circuit is capacitive. Notice that the vector notation was necessary because the phase angle of branch 2 was not  $0^\circ$  nor  $90^\circ$ , but a value between those two limits. Quantities at  $0^\circ$  or  $180^\circ$  can be combined by simple addition or subtraction; at  $90^\circ$  they can be combined by the Pythagorean formula. For any other angle, however, a graphic or an algebraic solution is necessary, including the angles as definite parts of the problem. This was pointed out in Chapter 3.

Impedance of the same circuit can also be determined by the product-over-the-sum method :

$$\begin{aligned} Z_T &= \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(0 - j6)(8 + j5)}{0 - j6 + 8 + j5} = \frac{-j48 - j^2 30}{8 - j} \\ &= \frac{30 - j48}{8 - j} \cdot \frac{8 + j}{8 + j} = \frac{240 - j384 + j30 - j^2 48}{64 - j^2} \\ &= \frac{288 - j354}{65} = 4.43 - j5.44 \text{ ohms} \end{aligned}$$

Conversion to polar form gives :

$$Z_T = 7.02 / \underline{-50.8^\circ} \text{ ohms}$$

This is very close to the quantity obtained by use of the current method. Power-factor calculations are also close :

$$P_t = I_2^2 \times R = 2.54^2 \times 8 = 51.61 \text{ watts}$$

$$P_a = E_a \times I_T = 24 \times 3.41 = 81.84 \text{ volt-amperes}$$

$$PF = \frac{P_t}{P_a} = \frac{51.61}{81.84} = .6306$$

And from the trig table :

$$PF = \cos \theta = \cos 50.9^\circ = .6307$$

Neither  $R/Z$  nor  $Z/R$  can be used to calculate the power factor because of the complex circuit arrangement. These relationships hold only for straight series and straight parallel circuits, as illustrated previously.

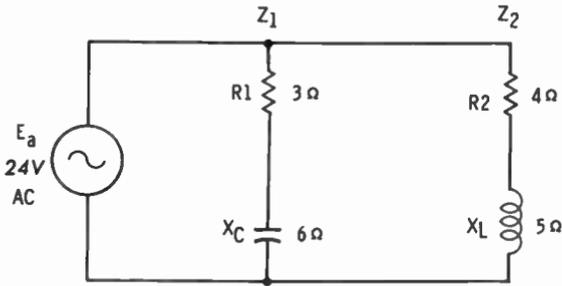


Fig. 5-13. A more complex series-parallel circuit.

A slightly more complex circuit arrangement is that of Fig. 5-13. The methods of calculation are similar to those used for the previous circuit:

$$Z_1 = 3 - j6 \text{ ohms} = 6.71 / -63.4^\circ \text{ ohms}$$

$$I_1 = \frac{E_a}{Z_1} = \frac{24 / 0^\circ}{6.71 / -63.4^\circ} = 3.58 / 63.4^\circ \text{ amperes}$$

$$Z_2 = 4 + j5 \text{ ohms} = 6.4 / 51.3^\circ \text{ ohms}$$

$$I_2 = \frac{E_a}{Z_2} = \frac{24 / 0^\circ}{6.4 / 51.3^\circ} = 3.75 / -51.3^\circ \text{ amperes}$$

Total current is  $(3.58 / 63.4^\circ + 3.75 / -51.3^\circ)$  amperes. Converting to rectangular form and adding:

$$I_1 = 3.58 / 63.4^\circ = 1.60 + j3.20$$

$$I_2 = 3.75 / -51.3^\circ = 2.34 - j2.93$$

$$I_T = 3.94 + j.27 = 3.95 / 3.9^\circ \text{ amperes}$$

$$Z = \frac{E_a}{I_T} = \frac{24 / 0^\circ}{3.95 / 3.9^\circ} = 6.08 / -3.9^\circ \text{ ohms}$$

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(3 - j6)(4 + j5)}{3 - j6 + 4 + j5} = \frac{12 + j15 - j24 - j^2 30}{7 - j}$$

$$= \frac{42 - j9}{7 - j} \cdot \frac{7 + j}{7 + j} = \frac{294 + j42 - j63 - j^2 29}{49 - j^2}$$

$$= \frac{303 - j21}{50} = 6.06 - j.42 \text{ ohms}$$

Conversion to polar form gives:

$$Z_t = 6.08 / -4^\circ \text{ ohms}$$

The negative phase angle for impedance is indicative of a capacitive circuit. This is correct even though the current through the inductive branch is greater. Relative values of oppositions and angles combine to determine the total result, which cannot be determined solely by magnitudes.

$$I_T = \frac{E_a}{Z_T} = \frac{24/0^\circ}{6.08/-4^\circ} = 3.95/4^\circ \text{ amperes}$$

$$PF = \cos \theta = \cos 4^\circ = .9976$$

Current is leading the voltage so the power factor is leading.

$$\begin{aligned} P_T &= (I_1^2 \times R_1) + (I_2^2 \times R_2) \\ &= (3.58^2 \times 3) + (3.75^2 \times 4) \\ &= 38.45 + 56.25 = 94.7 \text{ watts} \end{aligned}$$

$$P_a = E_a \times I_t = 24 \times 3.95 = 94.8 \text{ watts}$$

$$PF = \frac{P_t}{P_a} = \frac{94.7}{94.8} = .9989$$

From  $\cos \theta$ ,  $PF = .9976$ , a slight difference resulting from rounding off the results of calculations and in not interpolating the trig ratios. The more involved the circuit, the more chances there are for accumulating inaccuracies. The purpose here is to

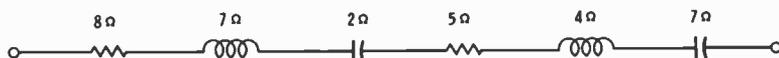


Fig. 5-14. RCL series circuit.

illustrate methods rather than extreme accuracy because the latter requires more time without increasing the understanding of the subject.

The ideas presented here can also be used for circuits that are even more complex than those illustrated. In series circuits the resistances and reactances should be combined so that the impedance can be expressed in a single complex number. For example, in Fig. 5-14 the total resistance is 13 ohms, inductive reactance is 11 ohms, and capacitive reactance is 9 ohms. Consequently, the total circuit impedance is  $13 + j2$  ohms.

In a circuit consisting of several parallel branches this type of simplification should be performed in each branch before proceeding with the parallel calculations. When more than two branches are involved, any two can be combined, as illustrated previously, and the resultant so obtained combined with other parts of the circuit. For example, if three branches

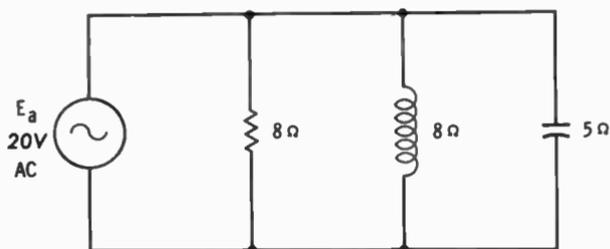


Fig. 5-15. Circuit for Problem 13.

were involved, any two could be solved for a resultant. Then that resultant could be calculated with the third branch to give a total resultant circuit. If four branches were being considered, numbers one and two could be calculated for a resultant, numbers three and four treated likewise, then the two resultants combined for the total arrangement.

### PRACTICE PROBLEMS

1. A series RL circuit includes 4 ohms of resistance and 5 ohms of inductive reactance. Calculate impedance, phase angle, and power factor of the circuit.
2. If 20 volts are applied to this circuit, what is the current? Is it leading or is it lagging the applied voltage?
3. A series RL circuit has 8 volts across the resistance and 12 volts across the inductance. What is the applied voltage? What is the phase angle?
4. A series RC circuit includes 6 ohms resistance and 9 ohms capacitive reactance. Calculate impedance, phase angle, and power factor of the circuit.
5. If 12 volts are applied to this circuit, what is the current? Is it leading, or is it lagging the applied voltage?
6. Calculate the true power and the apparent power of this same circuit.

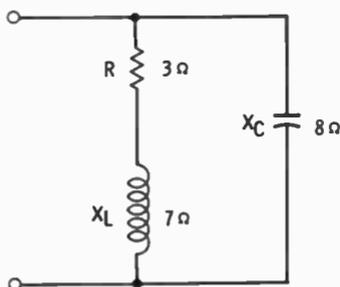


Fig. 5-16. Circuit for Problem 15.

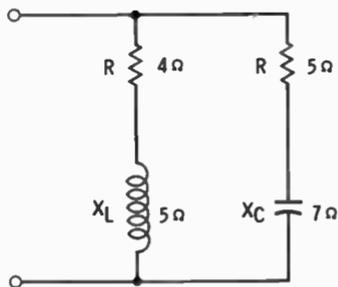


Fig. 5-17. Circuit for Problem 16.

7. A series RC circuit has 24 volts applied, and there are 12 volts across the resistor. What is the voltage drop across the capacitor?
8. A series RCL circuit includes 6 ohms of resistance, 8 ohms of capacitive reactance, and 5 ohms of inductive reactance. Express the impedance in both rectangular and polar forms.
9. If 24 volts are applied to this circuit, what is the current and the phase angle? Is the current leading or is it lagging the applied voltage?
10. There are 8 ohms of inductive reactance connected in parallel with 5 ohms of resistance. Calculate the impedance and phase angle of this circuit.
11. If 20 volts are applied to this circuit, calculate the reactive power and the power factor.
12. A parallel RC circuit has an impedance of 3 ohms and a resistance of 5 ohms. Calculate the capacitive reactance.
13. What is the impedance of the circuit of Fig. 5-15?
14. What is the true power and the phase angle of the same circuit? Is the current leading or is it lagging the applied voltage?
15. Calculate the impedance of the circuit of Fig. 5-16. Is this circuit inductive, or is it capacitive?
16. Calculate the impedance of the circuit of Fig. 5-17. What is the phase angle?

## CHAPTER 6

# Miscellaneous Vector Applications

In the previous chapter vectors and vector ideas were used in the solutions of various types of networks involving resistance, inductance, and capacitance. But vectors can be used in many other applications, either as a tool for calculations, or in analyzing circuit operation for many different types of circuits. A conventional amplifier shifts the phase of an input signal by  $180^\circ$  between grid and plate. Oscillators must have in-phase feedback to sustain oscillations. High-quality audio amplifiers use degenerative feedback ( $180^\circ$  out of phase) to improve the frequency response. Then, on the other hand, receivers in two-way radio systems may use regenerative (in-phase) feedback to reinforce the signal and thus increase audio output. There are many other examples in which degenerative or regenerative actions play a part in circuit operation.

In-between angles also enter into electricity and electronics to a large degree. Some circuits use phase as an analog type of information, representing some particular quantity. Frequency modulation and demodulation are examples; changes of frequency or phase represent changes of audio amplitude. This chapter explains several different electrical and electronic circuits or systems in which phase plays an important part in the operation. The choice of circuits follows no particular pattern; those described were chosen simply because they are relatively

simple analyses involving phase. The idea here is not so much to teach circuitry and operation, but to illustrate how vectors and phase can be used to secure a better understanding of circuit operation.

## RESONANCE

The LCR circuits solved previously used more or less random values of circuit parameters. Resonance, however, is a special condition of an LCR circuit, such as that in Fig. 6-1. Such a



Fig. 6-1. An LCR series circuit.

series circuit is resonant at the frequency at which inductive and capacitive reactances are equal. The resonant frequency formula,

$$f = \frac{1}{2\pi\sqrt{LC}}$$

has been derived from that relationship of reactances.

Currents through all the components of a series circuit are the same so that with equal reactances the voltage drop across the inductance equals the voltage drop across the capacitance. This is shown in Fig. 6-2A, in which circuit current is used as reference. The voltage across the resistance is in phase with the current, the inductive voltage leads the current by  $90^\circ$ , and the capacitive voltage lags current by  $90^\circ$ . A vector addition of the reactive voltages shows that they effectively cancel each other, leaving a vector resultant as shown in Fig. 6-2B. The circuit at resonance is purely resistive, and as the reactances cancel the effects of each other, the only effective circuit opposition is the resistance. So impedance is minimum and current

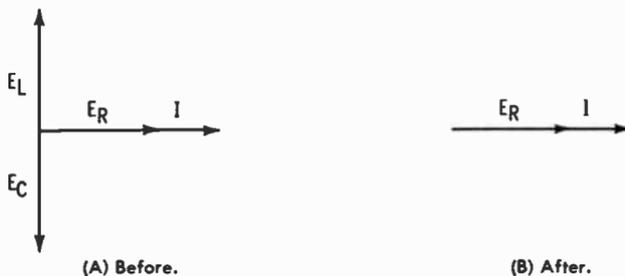


Fig. 6-2. Voltage and current vectors of Fig. 6-1 before and after vector addition.

is maximum in the series LCR circuit when operated at its resonant frequency.

The ideal parallel resonant circuit would include no resistance at all, only a perfect capacitor connected in parallel with a perfect inductor across an AC source. In such a circuit the capacitive current leads the applied voltage by  $90^\circ$ , as shown in Fig. 6-3A. The inductive current lags the applied voltage by  $90^\circ$ . As the currents are equal and opposite, the total circuit current is zero (Fig. 6-3B), meaning that the parallel LC circuit represents an infinite impedance at the resonant frequency.

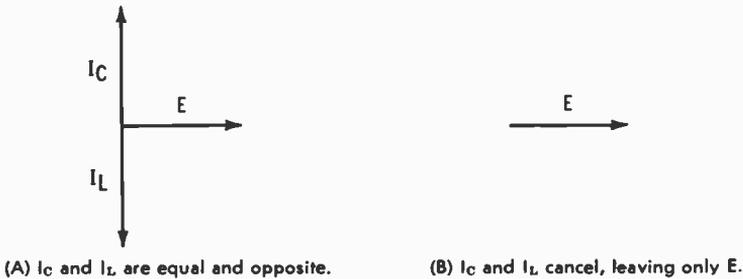


Fig. 6-3. Vectors of an ideal parallel resonant circuit.

However, a practical parallel LC circuit includes resistance, most of which is in the inductive branch and which consists primarily of the resistance of the coil. Assuming a small amount of resistance in the inductive branch and zero resistance in the capacitive branch, a vector representation is shown in Fig. 6-4. As in any parallel circuit, the same voltage appears across both branches, and it is labeled  $E_a$  as in Fig. 6-4. Current through the capacitor ( $I_c$ ) leads the applied voltage by  $90^\circ$ . Current through the inductive branch (including L and R) lags the applied voltage by something less than  $90^\circ$  and is labeled  $I_L$ . Voltage  $E_L$  leads the current through it by  $90^\circ$  and leads the applied voltage by a smaller angle.

The voltage across the resistance ( $E_R$ ) is in phase with current  $I_L$ , bearing in mind that the same current flows through R and L. Adding  $E_L$  and  $E_R$  at  $90^\circ$  gives the applied voltage  $E_a$ , shown as the diagonal of the parallelogram. This part of the analogy would hold for the series  $R_L$  branch regardless of whether or not the capacitor were connected in parallel. The total circuit current cannot be zero because the currents in the two branches are not equal. Total current ( $I_T$ ) is the diagonal of the parallelogram that includes  $I_c$  and  $I_L$  as adjacent sides. This arrangement acts resistively because total current is in

phase with the applied voltage, but the total is smaller than that through either of the separate branches.

From this analogy it is seen that true parallel resonance does not occur at the frequency at which the reactances are equal. For true resonance the inductive current must be larger than the capacitive current by an amount that causes  $I_T$  to be in phase with the applied voltage. In order to have this condition occur the inductive reactance must be less than the capacitive reactance. The conditions illustrated in Fig. 6-4 are those of a

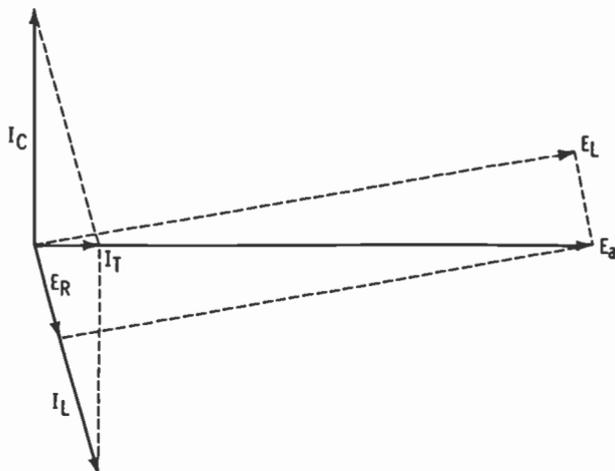


Fig. 6-4. Vector representation of a practical parallel LC circuit.

unity power factor condition—the true parallel resonance situation. Since  $X_L$  is smaller than  $X_C$ , the frequency for the unity power factor is lower than that at which the reactances are equal.

If  $R$  were made larger, the  $E_R$  vector would swing upward toward  $E_a$ , and the circuit current would increase. If  $R$  were made smaller, the  $E_R$  vector would swing downward, approaching the  $X_L = X_C$  resonant condition. Calculating the resistance of the capacitive branch into the problem, the  $I_C$  vector would swing downward, meaning that in order to maintain unity power factor it would be necessary for the  $I_L$  vector to also swing downward.

## TRANSFORMERS

Vectors (phasors) can be used to show the current-voltage relationships in power or signal transformers, as in Fig. 6-5.

Primary voltage ( $E_p$ ) is used as the reference because it is the voltage that is applied and thus initiates circuit activity. Current through the primary winding lags the primary voltage by  $90^\circ$ , assuming pure inductance in the primary winding. However, inductive reactance is usually considerably larger than resistance so the phase angle approaches  $90^\circ$ .

Voltage is induced into the secondary winding because the changes of current in the primary set up magnetic lines of force that cross the secondary. This induced voltage is maximum when the rate of change of primary current is maximum. This occurs whenever the primary current cycle passes through zero in either direction. As a result of this inductive action the

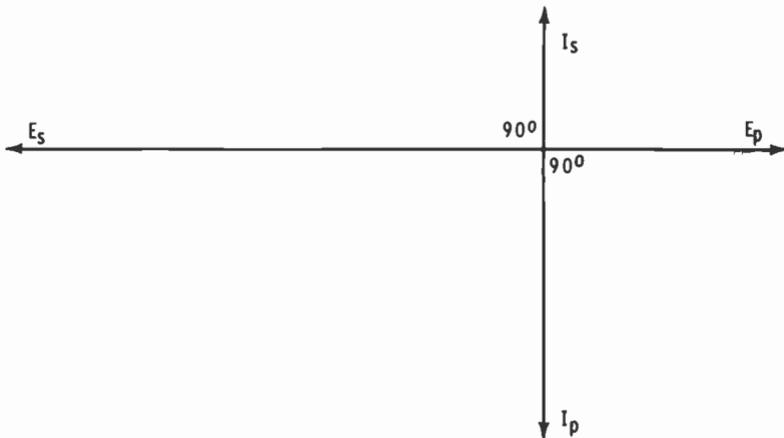


Fig. 6-5. Current-voltage relationships in transformers.

voltage induced into the secondary lags the primary current by  $90^\circ$ , as shown in Fig. 6-5. It could also be said that the secondary voltage has the same phase as the counter emf induced into the primary winding. The secondary winding is also inductive so that the secondary current lags the voltage by  $90^\circ$ , as shown, indicating that primary and secondary currents are  $180^\circ$  out of phase with each other.

As shown in Fig. 6-5, the transformer is connected step-up so that secondary voltage is greater than primary voltage. Secondary current then is smaller than primary current because, neglecting losses, the product of primary voltage and current should equal the product of secondary voltage and current. However, the vectors shown do not apply in all cases, especially if the primary or secondary, or both, are tuned or if the transformer is loaded to any degree.

As to loading, it is assumed with the vectors in Fig. 6-5 that only a very small current flows through the secondary winding, a condition of very light loading. If the load resistance across the secondary is decreased, the secondary current is increased, causing the secondary to appear more resistive. One condition of this is illustrated in Fig. 6-6. Voltage and current in the secondary approach an in-phase condition, assuming a parallel-circuit analogy of source and load. Because of reflected impedance, the effective reactance of the primary is decreased, causing that circuit to also appear more resistive. This is true because  $X_L$  and  $R$  act as a series RL circuit. As secondary loading

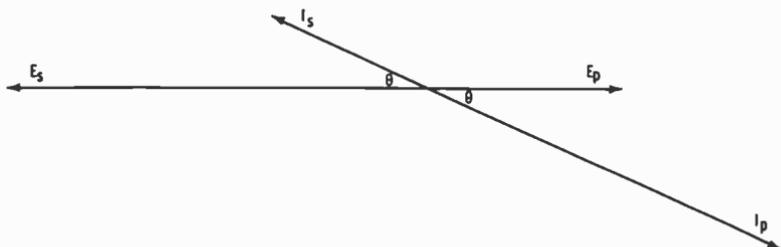


Fig. 6-6. Voltage and current approaching an in-phase condition.

(current) is increased, angle  $\theta$  in Fig. 6-6 becomes smaller, causing voltage and current to more closely approximate an in-phase condition. An ideal transformer would have voltage and current in phase in each winding. The source would then work into a perfect resistance with maximum transfer of energy.

### THREE-PHASE POWER

Three-phase systems used primarily in power work include three separate voltages (or currents), each separated from the other by 120 electrical degrees of phase difference. These voltages are produced from three separate coils (or windings) spaced  $120^\circ$  apart on the alternator. One method of connecting three-phase coils is shown in Fig. 6-7. This is a Y-connection (also called star) in which 0 is the common connection point, and one end of each of the coils is connected to this common point.

Vectors (phasors) representing these voltages are shown in Fig. 6-8. All the vectors are the same length, and they are spaced  $120^\circ$  from each other. The vector lengths are not drawn in a time relationship, because to do so, each voltage would have a different value, and each would be continuously chang-

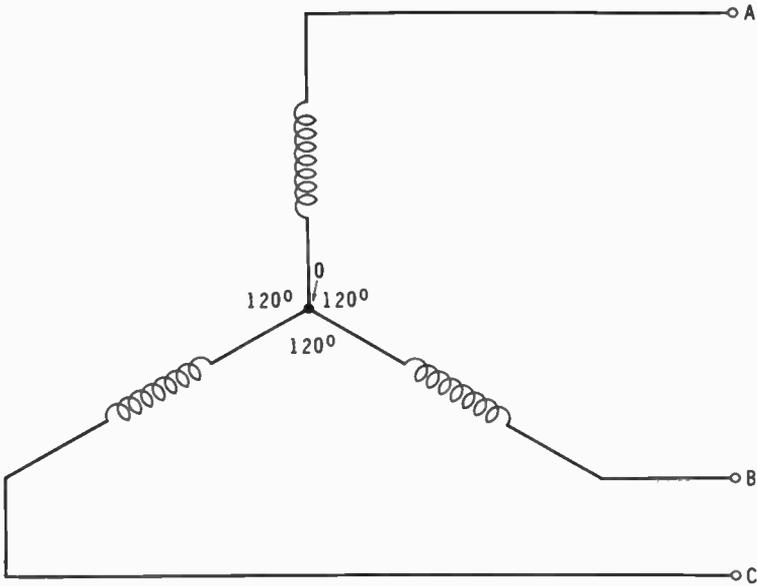


Fig. 6-7. One method of connecting three-phase coils.

ing with respect to the others. So these vectors can be assumed to represent some specific voltage, for example the rms value, of each of the separate phases. It could be added, however, that at any instant of time the sum of the three instantaneous values is zero, adding positive and negative values algebraically.

From the connection diagram (Fig. 6-7) it can be seen that the voltage between any two terminals is actually the voltage across two coils. Voltage AB, for example, is the voltage difference of coil-A and coil-B voltages. But these separate coil

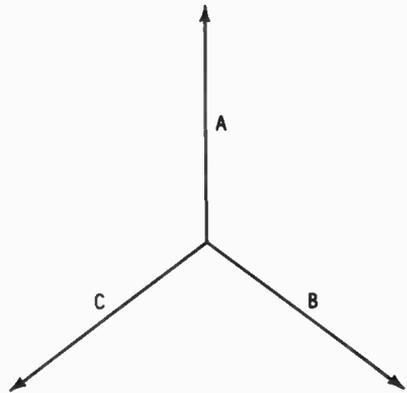


Fig. 6-8. Vectors representing Fig. 6-7.

voltages are at  $120^\circ$  with respect to each other, and therefore cannot be added (or subtracted) directly. Vectors can be used to determine the multiplying factor for calculating terminal voltage. In Fig. 6-9 the voltage across coils A and B is solved by vectors. Other terminal voltages would be solved similarly with the same length of resultant.

A in the diagram has been reversed by  $180^\circ$  and is labeled  $-A'$ . The vector resultant is the diagonal (R) of the parallelogram formed from sides  $-A'$  and B. A was reversed because the terminal voltage is the difference (not the sum) of the separate coil voltages. Perhaps Fig. 6-10 may clarify this.

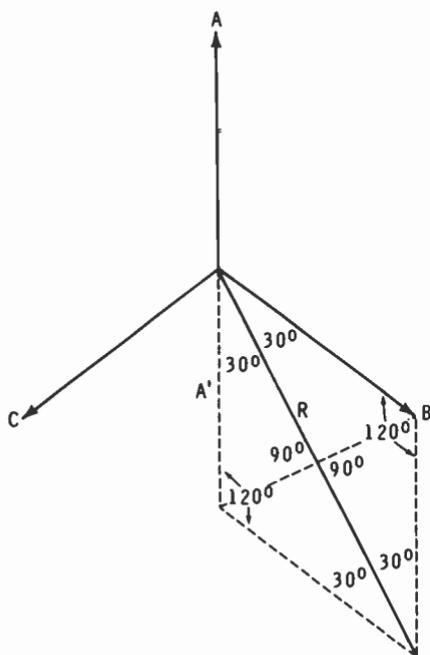


Fig. 6-9. Solving the voltage across coils A and B of Fig. 6-7.

Two batteries are shown in Fig. 6-10A connected series-aiding. With respect to 0 the upper terminal is  $-3$  volts; and the lower terminal is  $+5$  volts. With respect to the lower terminal the upper terminal is  $8$  volts ( $-3 - 5$ ) negative. But compare that to Fig. 6-10B. Here the upper terminal is  $-2$  volts ( $3 - 5$ ). In each case the voltages were added by changing the sign of one of them, which means that actually they were subtracted. The voltages could have been stated with respect to the upper terminal, but no matter which way they are stated the potential difference in Fig. 6-10A is  $8$  volts and that in B is  $2$  volts.

Referring back to Fig. 6-9 we see that a line has been drawn perpendicular to the resultant  $R$ . This line is the opposite diagonal of the same parallelogram, and with  $R$  it divides the parallelogram into four identical triangles. Vector  $B$  is the hypotenuse of one of them, and half of  $R$  is the side adjacent to the  $30^\circ$  angle.  $\cos 30^\circ = .8660$  so one-half of the length of  $R$  is  $.866$  of the  $B$  voltage. So the entire length of  $R$  is  $2 \times .866$ , or 1.732 times, the length of vector  $B$ .

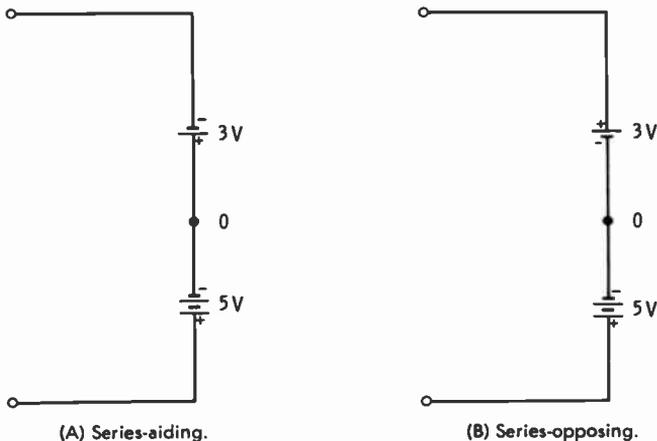


Fig. 6-10. Connection of batteries.

We can conclude from this that the rms voltage across terminals A and B is 1.73 times the rms voltage of either A or B. Resultant phase is a combination of the two separate phases, but it is not normally an important factor in power circuits. It is the voltage that is of prime concern; however, in normal operation all three phases are used with the result that coil B is not only in the AB circuit, but it is in the BC circuit as well. And similarly, each coil is part of two different current paths.

### TUNED-PLATE, TUNED-GRID OSCILLATOR

In the tuned-plate, tuned-grid (TPTG) oscillator feedback is accomplished through the plate-to-grid capacitance of the tube. And, as in any oscillator, the feedback must reinforce the oscillatory voltage and replace power losses in the tank circuit. A typical TPTG circuit is that of Fig. 6-11 in which  $L1C1$  is the grid tank and the oscillating section of the circuit. On the positive peaks of oscillation the tube is driven into conduction, and this, in turn, causes the  $L2C2$  plate tank to also oscillate.

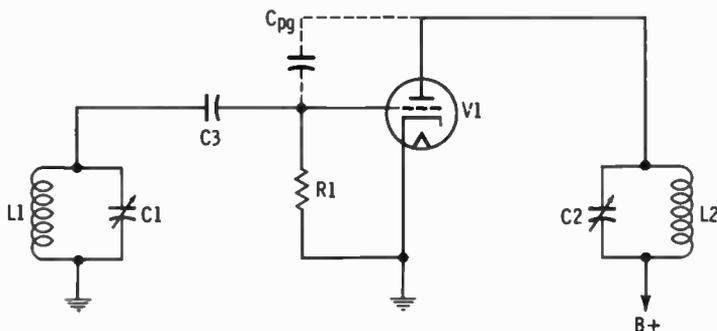
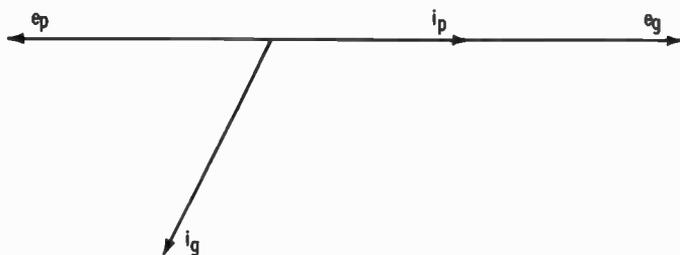


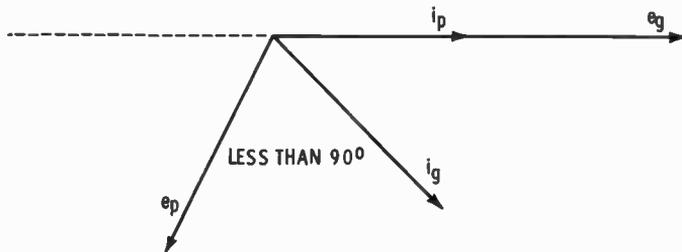
Fig. 6-11. A typical TPTG circuit.

To sustain oscillations, energy from the plate circuit is fed back to the grid in proper phase to replace the grid losses.

When the grid and plate tanks are both operated at the resonant frequency, they act resistively to the oscillation voltage on the grid. The plate current of the tube is therefore in phase with the grid signal, as indicated in Fig. 6-12A. Because of the normal phase inversion of the stage, plate signal ( $e_p$ ) and grid signal ( $e_g$ ) are  $180^\circ$  out of phase with respect to each other. The plate signal also appears across the plate-to-grid capaci-



(A) Plate current in phase with grid signal.



(B)  $e_p$  leads  $i_p$ .

Fig. 6-12. Vectors of the circuit in Fig. 6-11.

tance and the grid tank as the feedback signal. But the reactance of the tube capacitance is much larger than the impedance of the grid tank (which is resistive) so the grid current ( $i_g$ ) leads  $e_p$  by nearly  $90^\circ$ . The voltage across the L1C1 tank circuit is in phase with  $i_g$ , but the oscillations cease because the grid current is too far out of phase with respect to the grid signal.

When the plate tank is tuned to a higher frequency than the grid tank, the plate circuit acts inductively, so  $e_p$  leads  $i_g$  as indicated in Fig. 6-12B. Plate voltage decreases when plate current increases so that the additional lead of  $e_p$  places it below the horizontal line as shown. The leading condition occurs because in the inductive plate circuit the voltage leads the current. The total lead is the normal  $180^\circ$  difference plus that caused by the inductive condition.

Again the reactance of the tube capacitance is much larger than the impedance of the grid tank, therefore grid current  $i_g$  leads plate voltage  $e_p$  by nearly  $90^\circ$ . The voltage across L1C1 leads  $i_g$  by nearly  $90^\circ$ , and the feedback voltage across the input is in phase with the original grid drive signal. In operation, the L1C1 tank operates as an inductive circuit because its frequency is above that of the oscillation signal. The plate tank, however, is tuned to a higher frequency than the grid tank, hence it acts even more inductively. In this type of circuit the feedback need not be exactly in phase for oscillations to occur. But the phase relationship must be close enough so that the grid losses are replaced, which allows oscillations to continue. These latter statements can be shown to be true by the fact that there is a range of plate-tank settings, not just one particular setting, over which oscillations can occur.

## PHASE MODULATION

A system of phase modulation, such as that of Fig. 6-13, can also be explained by use of vector relationships. In this circuit the RF and audio are applied across R1, R2, and R4 in series. The RF input is labeled  $e_r$  and appears across R1. C1 has low reactance to the radio frequencies, and the lower end of R1 is effectively at RF ground. The tube is cathode biased, but additional bias is provided by the audio voltage across R4. However, this audio signal is continually reversing polarity so that the R4 voltage alternately adds to and subtracts from the cathode bias. This means that total bias is changing with the audio signal, and the gain of the stage varies with the changes of bias.

The instantaneous plate load voltage is the resultant of two RF voltages. One of these is that coupled through the grid-to-plate capacitance of the tube; this is labeled  $e_1$  in the vector diagram (Fig. 6-14A). It is not in phase with  $e_g$ , but it leads because the circuit is capacitive. The other plate signal is that supplied through normal amplifier action, and it is labeled  $e_2$  in the diagram. It is  $180^\circ$  out of phase with  $e_g$ , but the plate tank is at resonance and thus acts resistively. The resultant of these two voltages ( $e_1$  and  $e_2$ ) is shown as  $e_o$  in Fig. 6-14A.

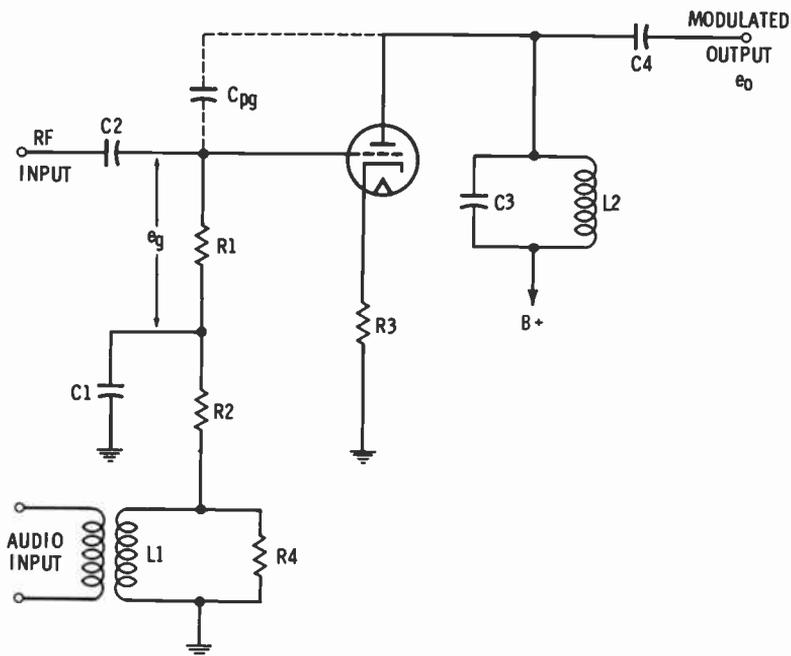


Fig. 6-13. A phase modulator circuit.

The amplitude of the amplified signal ( $e_2$ ) is relatively low because of the degenerative feedback that is produced as a result of not bypassing the cathode resistor.

When the audio swing is maximum, the gain of the stage is minimum, and  $e_2$  has relatively small amplitude. Then  $e_o$  leads  $e_g$  by phase angle  $\theta$ . In the second vector diagram (Fig. 6-14B) the audio-signal swing has caused the bias to decrease and the gain of the stage to increase. The increased gain causes  $e_2$  to increase in amplitude; it is labeled  $e_2'$  in the second diagram. This shifts the phase of the resultant  $e_o$  to that shown as  $e_o'$ , changing  $\theta$  to the value shown as  $\theta'$ .

In this way the audio signal causes the output to be phase-shifted by an amount proportional to the audio amplitude. The greater the amplitude, the greater is the degree of phase shift that occurs. When phase-shifted,  $e_o$  is applied to an oscillator tuned circuit, the time interval between signal peaks is shifted, thus phase modulating the carrier frequency. It is an indirect form of FM, achieving frequency modulation through altering

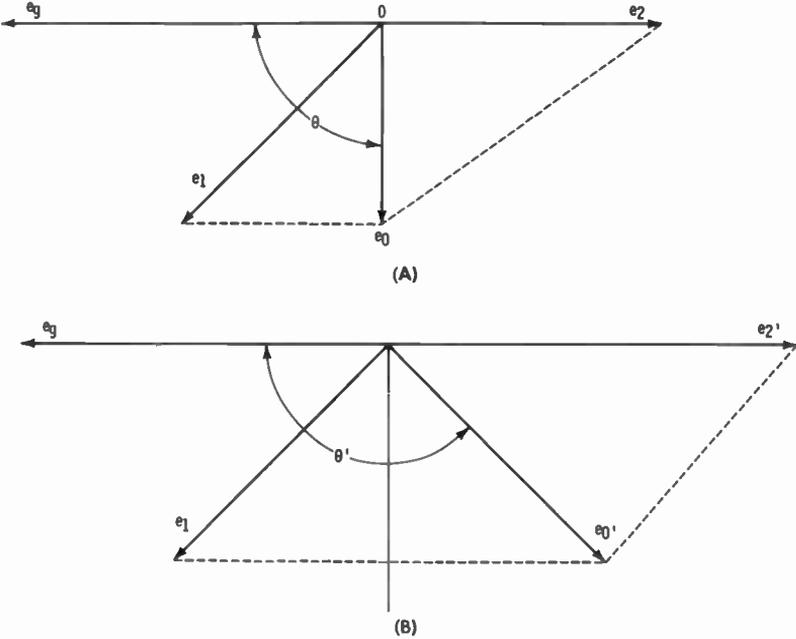


Fig. 6-14. Vectors of the circuit in Fig. 6-13.

the phase. Only one change is shown, but a continuous audio signal causes  $e_o$  to shift back and forth in step with the audio variations. The degree of shift is determined by the audio amplitude, and the rate of shift (number of changes per second) is determined by the frequency of the audio modulating signal.

### DISCRIMINATOR

A discriminator can be used in an FM receiver to change a frequency-modulated signal input to audio voltage variations at the output. Fig. 6-15 shows an example circuit in which  $L1C1$  is tuned to the intermediate frequency (IF) of the receiver.  $L1$  is inductively coupled to  $L2L3$ , the center-tapped

secondary, and L2L3C2 is also tuned to the intermediate frequency. There is capacitive coupling from the transformer primary to the center tap of the secondary through  $C_c$ , the coupling capacitor. This voltage is developed across L4, with the center tap of the output circuit used as a reference point. Each of the diodes has two sources of voltage acting in series. For V1 the applied signals are those across L2 and L4; for V2 the applied signals are those across L3 and L4. Basic operation depends on the phase relations and the comparative conductions of the two diode circuits. By using vectors we can obtain a much clearer picture of the overall operation.

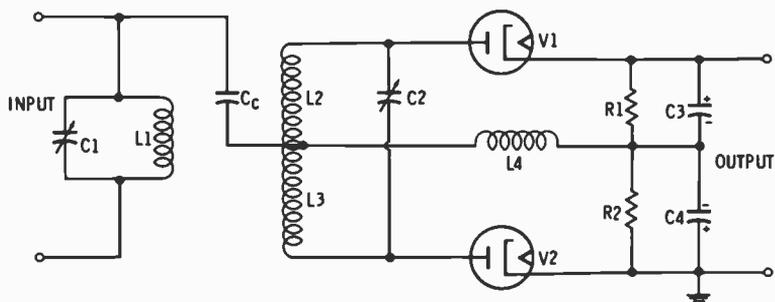
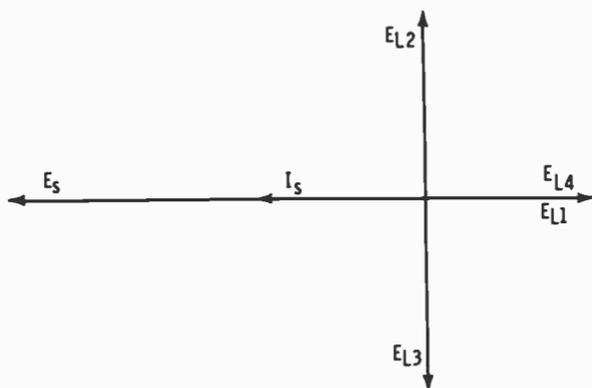


Fig. 6-15. A circuit in which L1C1 is tuned to the IF frequency.

$E_{L1}$  is the input signal voltage that has the same phase as  $E_{L4}$  (Fig. 6-16A). The secondary voltage ( $E_s$ ) is  $180^\circ$  out of phase because of the  $180^\circ$  phase reversal of the transformer. At the resonant frequency (IF) the secondary tank acts resistive so  $I_s$  is in phase with  $E_s$ , the voltage causing it. The voltage across the secondary inductance leads the current through it by  $90^\circ$ , and with the center tap serving as a reference point,  $E_{L2}$  and  $E_{L3}$  have opposite polarities, as shown. The reactance of L4 is high compared to that of coil L1 so that almost all the signal at the input appears across L4 in phase with  $E_{L1}$ .

Fig. 6-16B shows the phase conditions existing when the resonant frequency (IF) is applied.  $E_{L2}$  and  $E_{L4}$  are  $90^\circ$  out of phase with respect to each other; the voltage applied to the V1 circuit is the diagonal  $E_{V1}$ .  $E_{L3}$  and  $E_{L4}$  are  $90^\circ$  out of phase so that the voltage applied to the V2 circuit is the diagonal  $E_{V2}$ . Equal magnitudes of voltage are thus applied across the diode circuits, the conductions are equal, and the voltage drops across R1 and R2 are also equal. But the currents through them are in opposite directions so the same potential exists at both output terminals. This means that when the unmodulated IF is applied the output voltage is zero.

When the frequency of the input signal goes above resonance,  $X_L$  increases,  $X_C$  decreases, and the L2L3C2 tuned circuit acts inductive. As shown in Fig. 6-17A, the secondary current ( $I_s$ ) lags  $E_s$  by angle  $\theta$ . The magnitude of  $\theta$  is determined by how much above resonance the frequency changed. Secondary voltages across L2L3 are always  $90^\circ$  out of phase with respect to  $I_s$ ; therefore they shift as shown.  $E_{L2}$  and  $E_{L3}$ , however, are still



(A)  $E_{L1}$  has same phase as  $E_{L4}$ .

(B) When the resonant frequency is applied.

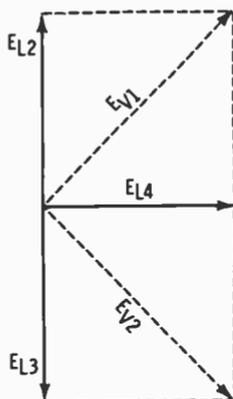


Fig. 6-16. Vectors of the circuit in Fig. 6-15.

$180^\circ$  out of phase with respect to each other. The  $E_{L4}$  reference remains constant and does not shift.

Fig. 6-17B shows the resultant voltages applied to the two diodes, and since  $E_{V1}$  is greater than  $E_{V2}$ , V1 conduction is greater than that of V2. This increases the voltage drop across R1 and decreases that across R2, resulting in an output that is positive with respect to ground. Below resonance the circuit

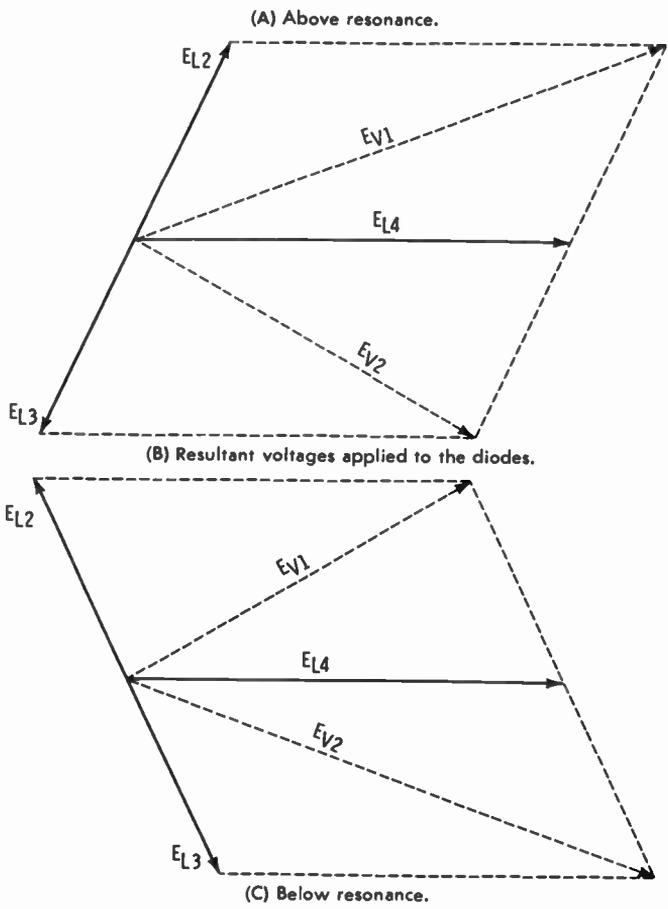
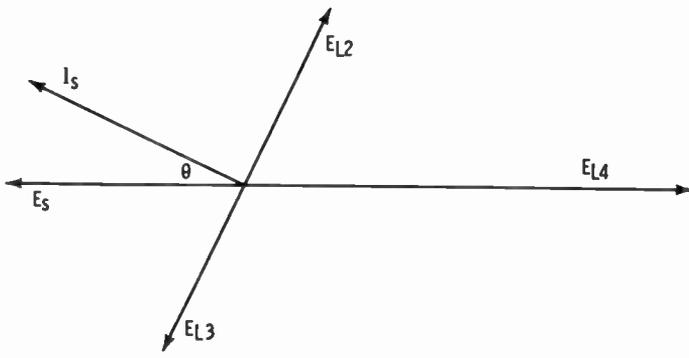


Fig. 6-17. Vector representation of Fig. 6-15 when the frequency is above and below resonance.

acts capacitively so the shift is in the opposite direction, as shown in Fig. 6-17C. Then the voltage applied to V2 is greater than that applied to V1, and the voltage across R2 is greater than that across R1. This results in an output that is negative with respect to ground. In every case the amplitude of the output signal is proportional to the change of frequency at the input, effectively changing frequency variations into corresponding voltage variations.

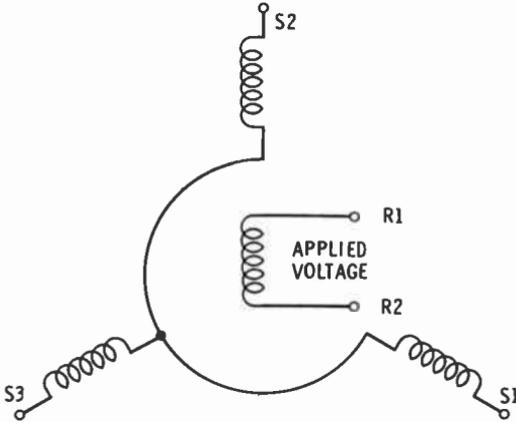


Fig. 6-18. Circuit of a synchro.

### SYNCHROS

Synchros are motor-type devices, each having three stator coils and one rotor coil. The magnetic axis of each of these coils is separated from the other by 120°. They are Y-connected, but are not three-phase units. AC is applied to the rotor, as indicated in Fig. 6-18, and a voltage is induced into each of the stator coils. Magnitudes of these induced voltages are determined by the angular position of the rotor and the maximum voltage that can be induced. Stator voltages may be calculated by:

$$E_s = E_m \times \cos \theta$$

where,

$E_s$  is the stator voltage,

$E_m$  is the maximum stator coil voltage,

$\theta$  is the angle between R1 end of rotor and the stator coil.

Maximum voltage is induced when the rotor and a stator coil are in line with each other, for example, S2 in Fig. 6-18. At this

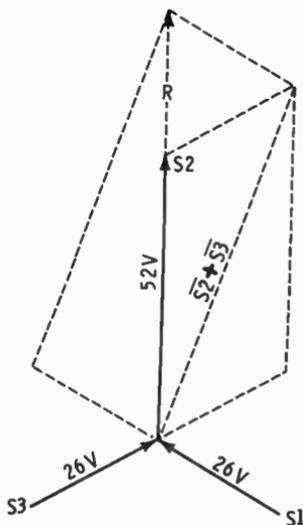


Fig. 6-19. Vectors representing the voltages in Fig. 6-18.

rotor setting the synchro is assumed to be set at  $0^\circ$ . Minimum voltage is induced into a coil when the rotor is positioned at right angles to it. Assuming that the maximum stator coil voltage is 52, then :

$$E_{s1} = 52 \times \cos 120^\circ = 52 (-.5) = -26 \text{ volts}$$

$$E_{s2} = 52 \times \cos 0^\circ = 52 (1) = 52 \text{ volts}$$

$$E_{s3} = 52 \times \cos 120^\circ = 52 (-.5) = -26 \text{ volts}$$

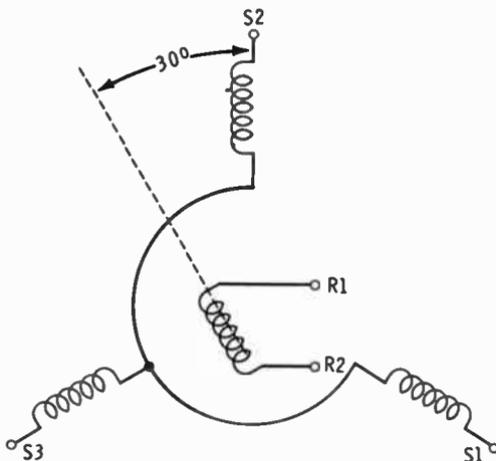


Fig. 6-20. Rotor has been rotated to  $30^\circ$ .

Here the positive and negative signs do not indicate polarity; they indicate phase. If the voltage is positive, then it is in phase with the voltage at the R1 (reference) end of the rotor coil. If negative, they are  $180^\circ$  out of phase with respect to each other. There are no in-between phases, they are either in phase or  $180^\circ$  out of phase.

The three stator voltages combine to produce a resultant magnetic flux in the direction the rotor is turned. Vectors representing these voltages are in Fig. 6-19, each vector pointing in the direction of the positive end of the rotor coil. The resultant (R) is as shown, and it is in the same direction that the R1

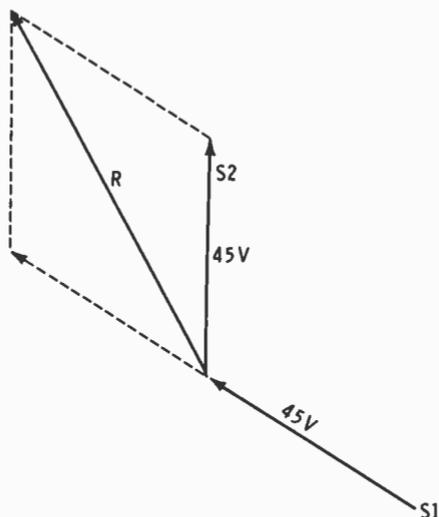


Fig. 6-21. Vector representation of Fig. 6-20.

end of the rotor is pointing. The length of the line is not important. It is the direction in which we are interested. However, at any angle the length of the resultant is always the same for a given synchro.

In Fig. 6-20 the rotor has been rotated to  $30^\circ$ . Remembering that for synchros,  $0^\circ$  is considered as being the direction of the S2 stator coil, the induced voltages are:

$$E_{s1} = 52 \times \cos 150^\circ = 52(-.866) = -45 \text{ volts}$$

$$E_{s2} = 52 \times \cos 30^\circ = 52(.866) = 45 \text{ volts}$$

$$E_{s3} = 52 \times \cos 90^\circ = 52(0) = 0 \text{ volts}$$

Fig. 6-21 shows the vector representation with the resultant (R) at  $30^\circ$ , the same angle at which the rotor was set. This

same process can be used for any angle, and in every case it gives the direction of the resultant magnetic field. When two synchros are connected together with corresponding terminals interconnected, any angular position set with one of the rotors is automatically followed by the rotor of the other unit.

There are many other circuits and systems which can be similarly analyzed by vectorial methods. Some of these are motors, generators, amplitude modulation, frequency modulation, phase-shift oscillators, coupled circuits, transmission lines, amplifier feedback, color-TV circuits, and multiplexing.

# Appendix

**Natural Trigonometric Functions**

**Answers To Practice Problems**

### SINES 0° to 45°

Deg.	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Comp	
<b>0°</b>	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	<b>89°</b>
<b>1</b>	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	<b>88</b>
<b>2</b>	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	0523	<b>87</b>
<b>3</b>	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	0698	<b>86</b>
<b>4</b>	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	0872	<b>85</b>
<b>5°</b>	0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	1045	<b>84°</b>
<b>6</b>	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	1219	<b>83</b>
<b>7</b>	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	1392	<b>82</b>
<b>8</b>	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	1564	<b>81</b>
<b>9</b>	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	1736	<b>80</b>
<b>10°</b>	1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	1908	<b>79°</b>
<b>11</b>	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	2079	<b>78</b>
<b>12</b>	2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	2250	<b>77</b>
<b>13</b>	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	2419	<b>76</b>
<b>14</b>	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	2588	<b>75</b>
<b>15°</b>	2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	2756	<b>74°</b>
<b>16</b>	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	2924	<b>73</b>
<b>17</b>	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3090	<b>72</b>
<b>18</b>	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3256	<b>71</b>
<b>19</b>	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3420	<b>70</b>
<b>20°</b>	3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3584	<b>69°</b>
<b>21</b>	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3746	<b>68</b>
<b>22</b>	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3907	<b>67</b>
<b>23</b>	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	4067	<b>66</b>
<b>24</b>	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	4226	<b>65</b>
<b>25°</b>	4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	4384	<b>64°</b>
<b>26</b>	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	4540	<b>63</b>
<b>27</b>	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	4695	<b>62</b>
<b>28</b>	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	4848	<b>61</b>
<b>29</b>	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	5000	<b>60</b>
<b>30°</b>	5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	5150	<b>59°</b>
<b>31</b>	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	5299	<b>58</b>
<b>32</b>	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	5446	<b>57</b>
<b>33</b>	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	5592	<b>56</b>
<b>34</b>	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	5736	<b>55</b>
<b>35°</b>	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	5878	<b>54°</b>
<b>36</b>	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	6018	<b>53</b>
<b>37</b>	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	6157	<b>52</b>
<b>38</b>	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	6293	<b>51</b>
<b>39</b>	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	6428	<b>50</b>
<b>40°</b>	6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	6561	<b>49°</b>
<b>41</b>	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	6691	<b>48</b>
<b>42</b>	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	6820	<b>47</b>
<b>43</b>	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	6947	<b>46</b>
<b>44</b>	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	7071	<b>45</b>
<b>Comp</b>	.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	<b>Deg.</b>	

### COSINES—45° to 90°

### SINES—45° to 90°

Deg.	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Comp	
<b>45°</b>	7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	7193	<b>44°</b>
<b>46</b>	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	7314	<b>43</b>
<b>47</b>	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	7431	<b>42</b>
<b>48</b>	7431	7443	7455	7466	7478	7490	7501	7515	7524	7536	7547	<b>41</b>
<b>49</b>	7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	7660	<b>40</b>
<b>50°</b>	7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	7771	<b>39°</b>
<b>51</b>	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	7880	<b>38</b>
<b>52</b>	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	7986	<b>37</b>
<b>53</b>	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	8090	<b>36</b>
<b>54</b>	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	8192	<b>35</b>
<b>55°</b>	8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	8290	<b>34°</b>
<b>56</b>	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	8387	<b>33</b>
<b>57</b>	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	8480	<b>32</b>
<b>58</b>	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	8572	<b>31</b>
<b>59</b>	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	8660	<b>30</b>
<b>60°</b>	8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	8746	<b>29°</b>
<b>61</b>	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	8829	<b>28</b>
<b>62</b>	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	8910	<b>27</b>
<b>63</b>	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	8988	<b>26</b>
<b>64</b>	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	9063	<b>25</b>
<b>65°</b>	9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	9135	<b>24°</b>
<b>66</b>	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	9205	<b>23</b>
<b>67</b>	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	9272	<b>22</b>
<b>68</b>	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	9336	<b>21</b>
<b>69</b>	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	9397	<b>20</b>
<b>70°</b>	9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	9455	<b>19°</b>
<b>71</b>	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	9511	<b>18</b>
<b>72</b>	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	9563	<b>17</b>
<b>73</b>	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	9613	<b>16</b>
<b>74</b>	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	9659	<b>15</b>
<b>75°</b>	9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	9703	<b>14°</b>
<b>76</b>	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	9744	<b>13</b>
<b>77</b>	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	9781	<b>12</b>
<b>78</b>	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	9816	<b>11</b>
<b>79</b>	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	9848	<b>10</b>
<b>80°</b>	9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	9877	<b>9°</b>
<b>81</b>	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	9903	<b>8</b>
<b>82</b>	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	9925	<b>7</b>
<b>83</b>	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	9945	<b>6</b>
<b>84</b>	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	9962	<b>5</b>
<b>85°</b>	9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	9976	<b>4°</b>
<b>86</b>	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	9986	<b>3</b>
<b>87</b>	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	9994	<b>2</b>
<b>88</b>	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	9998	<b>1</b>
<b>89</b>	9998	9999	9999	9999	9999	1.00	1.00	1.00	1.00	1.00	1.00	<b>0</b>
<b>Comp</b>	<b>.9</b>	<b>.8</b>	<b>.7</b>	<b>.6</b>	<b>.5</b>	<b>.4</b>	<b>.3</b>	<b>.2</b>	<b>.1</b>	<b>.0</b>	<b>Deg.</b>	

### COSINES—0° to 45°

### TANGENTS—0° to 45°

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Comp	
<b>0°</b>	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	<b>89°</b>
<b>1</b>	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	<b>88</b>
<b>2</b>	0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	0524	<b>87</b>
<b>3</b>	0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	0699	<b>86</b>
<b>4</b>	0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	0875	<b>85</b>
<b>5°</b>	0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	1051	<b>84°</b>
<b>6</b>	1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	1228	<b>83</b>
<b>7</b>	1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	1405	<b>82</b>
<b>8</b>	1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	1584	<b>81</b>
<b>9</b>	1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	1763	<b>80</b>
<b>10°</b>	1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	1944	<b>79°</b>
<b>11</b>	1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	2126	<b>78</b>
<b>12</b>	2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	2309	<b>77</b>
<b>13</b>	2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	2493	<b>76</b>
<b>14</b>	2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	2679	<b>75</b>
<b>15°</b>	2679	2698	2717	2736	2754	2774	2792	2811	2830	2849	2867	<b>74°</b>
<b>16</b>	2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3057	<b>73</b>
<b>17</b>	3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3249	<b>72</b>
<b>18</b>	3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3443	<b>71</b>
<b>19</b>	3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3640	<b>70</b>
<b>20°</b>	3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3839	<b>69°</b>
<b>21</b>	3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	4040	<b>68</b>
<b>22</b>	4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	4245	<b>67</b>
<b>23</b>	4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	4452	<b>66</b>
<b>24</b>	4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4663	<b>65</b>
<b>25°</b>	4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4877	<b>64°</b>
<b>26</b>	4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	5095	<b>63</b>
<b>27</b>	5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	5317	<b>62</b>
<b>28</b>	5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	5543	<b>61</b>
<b>29</b>	5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	5774	<b>60</b>
<b>30°</b>	5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	6009	<b>59°</b>
<b>31</b>	6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	6249	<b>58</b>
<b>32</b>	6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	6494	<b>57</b>
<b>33</b>	6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	6745	<b>56</b>
<b>34</b>	6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	7002	<b>55</b>
<b>35°</b>	7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	7265	<b>54°</b>
<b>36</b>	7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	7536	<b>53</b>
<b>37</b>	7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	7813	<b>52</b>
<b>38</b>	7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	8098	<b>51</b>
<b>39</b>	8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	8391	<b>50</b>
<b>40°</b>	8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	8693	<b>49°</b>
<b>41</b>	8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	9004	<b>48</b>
<b>42</b>	9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	9325	<b>47</b>
<b>43</b>	9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	9657	<b>46</b>
<b>44</b>	9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	1.00	<b>45</b>
<b>Comp.</b>	.9	.8	.7	.6	.5	.4	.3	.2	.1	.0		

### COTANGENTS—45° to 90°

## TANGENTS—45° to 90°

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Comp	
<b>45°</b>	1.0000	1.0035	1.0070	1.0105	1.0141	1.0176	1.0212	1.0247	1.0283	1.0319	1.0355	<b>44°</b>
<b>46</b>	1.0355	1.0392	1.0428	1.0464	1.0501	1.0538	1.0575	1.0612	1.0649	1.0686	1.0724	<b>43</b>
<b>47</b>	1.0724	1.0761	1.0799	1.0837	1.0875	1.0913	1.0951	1.0990	1.1028	1.1067	1.1106	<b>42</b>
<b>48</b>	1.1106	1.1145	1.1184	1.1224	1.1263	1.1303	1.1343	1.1383	1.1423	1.1463	1.1504	<b>41</b>
<b>49</b>	1.1504	1.1544	1.1585	1.1626	1.1667	1.1708	1.1750	1.1792	1.1833	1.1875	1.1918	<b>40</b>
<b>50°</b>	1.1918	1.1960	1.2002	1.2045	1.2088	1.2131	1.2174	1.2218	1.2261	1.2305	1.2349	<b>39°</b>
<b>51</b>	1.2349	1.2393	1.2437	1.2482	1.2527	1.2572	1.2617	1.2662	1.2708	1.2753	1.2799	<b>38</b>
<b>52</b>	1.2799	1.2846	1.2892	1.2938	1.2985	1.3032	1.3079	1.3127	1.3175	1.3222	1.3270	<b>37</b>
<b>53</b>	1.3270	1.3319	1.3367	1.3416	1.3465	1.3514	1.3564	1.3613	1.3663	1.3713	1.3764	<b>36</b>
<b>54</b>	1.3764	1.3814	1.3865	1.3916	1.3968	1.4019	1.4071	1.4124	1.4176	1.4229	1.4281	<b>35</b>
<b>55°</b>	1.4281	1.4335	1.4388	1.4442	1.4496	1.4550	1.4605	1.4659	1.4715	1.4770	1.4826	<b>34°</b>
<b>56</b>	1.4826	1.4882	1.4938	1.4994	1.5051	1.5108	1.5166	1.5224	1.5282	1.5340	1.5399	<b>33</b>
<b>57</b>	1.5399	1.5458	1.5517	1.5577	1.5637	1.5697	1.5757	1.5818	1.5880	1.5941	1.6003	<b>32</b>
<b>58</b>	1.6003	1.6066	1.6128	1.6191	1.6255	1.6319	1.6383	1.6447	1.6512	1.6577	1.6643	<b>31</b>
<b>59</b>	1.6643	1.6709	1.6775	1.6842	1.6909	1.6977	1.7045	1.7113	1.7182	1.7251	1.7321	<b>30</b>
<b>60°</b>	1.7321	1.7391	1.7461	1.7532	1.7603	1.7675	1.7747	1.7820	1.7893	1.7966	1.8040	<b>29°</b>
<b>61</b>	1.8040	1.8115	1.8190	1.8265	1.8341	1.8418	1.8495	1.8572	1.8650	1.8728	1.8807	<b>28</b>
<b>62</b>	1.8807	1.8887	1.8967	1.9047	1.9128	1.9210	1.9292	1.9375	1.9458	1.9542	1.9626	<b>27</b>
<b>63</b>	1.9626	1.9711	1.9797	1.9883	1.9970	2.0057	2.0145	2.0233	2.0323	2.0413	2.0503	<b>26</b>
<b>64</b>	2.0503	2.0594	2.0686	2.0778	2.0872	2.0965	2.1060	2.1155	2.1251	2.1348	2.1445	<b>25</b>
<b>65°</b>	2.1445	2.154	2.164	2.174	2.184	2.194	2.204	2.215	2.225	2.236	2.246	<b>24°</b>
<b>66</b>	2.246	2.257	2.267	2.278	2.289	2.300	2.311	2.322	2.333	2.344	2.356	<b>23</b>
<b>67</b>	2.356	2.367	2.379	2.391	2.402	2.414	2.426	2.438	2.450	2.463	2.475	<b>22</b>
<b>68</b>	2.475	2.488	2.500	2.513	2.526	2.539	2.552	2.565	2.578	2.592	2.605	<b>21</b>
<b>69</b>	2.605	2.619	2.633	2.646	2.660	2.675	2.689	2.703	2.718	2.733	2.747	<b>20</b>
<b>70°</b>	2.747	2.762	2.778	2.793	2.808	2.824	2.840	2.856	2.872	2.888	2.904	<b>19°</b>
<b>71</b>	2.904	2.921	2.937	2.954	2.971	2.989	3.006	3.024	3.042	3.060	3.078	<b>18</b>
<b>72</b>	3.078	3.096	3.115	3.133	3.152	3.172	3.191	3.211	3.230	3.250	3.271	<b>17</b>
<b>73</b>	3.271	3.291	3.312	3.333	3.354	3.376	3.398	3.420	3.442	3.465	3.487	<b>16</b>
<b>74</b>	3.487	3.511	3.534	3.558	3.582	3.606	3.630	3.655	3.681	3.700	3.732	<b>15</b>
<b>75°</b>	3.732	3.758	3.785	3.812	3.839	3.867	3.895	3.923	3.952	3.981	4.011	<b>14°</b>
<b>76</b>	4.011	4.041	4.071	4.120	4.134	4.165	4.198	4.230	4.264	4.297	4.331	<b>13</b>
<b>77</b>	4.331	4.366	4.402	4.437	4.474	4.511	4.548	4.586	4.625	4.665	4.705	<b>12</b>
<b>78</b>	4.705	4.745	4.787	4.829	4.872	4.915	4.959	5.005	5.050	5.097	5.145	<b>11</b>
<b>79</b>	5.145	5.193	5.242	5.292	5.343	5.396	5.449	5.503	5.558	5.614	5.67	<b>10</b>
<b>80°</b>	5.67	5.73	5.79	5.85	5.91	5.98	6.04	6.11	6.17	6.24	6.31	<b>9°</b>
<b>81</b>	6.31	6.39	6.46	6.54	6.61	6.69	6.77	6.85	6.94	7.03	7.12	<b>8</b>
<b>82</b>	7.12	7.21	7.30	7.40	7.49	7.60	7.70	7.81	7.92	8.03	8.14	<b>7</b>
<b>83</b>	8.14	8.26	8.39	8.51	8.64	8.78	8.92	9.06	9.21	9.36	9.51	<b>6</b>
<b>84</b>	9.51	9.68	9.84	10.0	10.2	10.4	10.6	10.8	11.0	11.2	11.4	<b>5</b>
<b>85°</b>	11.4	11.7	11.9	12.2	12.4	12.7	13.0	13.3	13.6	14.0	14.3	<b>4°</b>
<b>86</b>	14.3	14.7	15.1	15.5	15.9	16.3	16.8	17.3	17.9	18.5	19.1	<b>3</b>
<b>87</b>	19.1	19.7	20.4	21.2	22.0	22.9	23.9	24.9	26.0	27.3	28.6	<b>2</b>
<b>88</b>	28.6	30.1	31.8	33.7	35.8	38.2	40.9	44.1	47.7	52.1	57.0	<b>1</b>
<b>89</b>	57.0	64.0	72.0	82.0	95.0	115.0	143.0	191.0	286.0	573.0	—	<b>0</b>
<b>Comp</b>	.9	.8	.7	.6	.5	.4	.3	.2	.1	.0		

## COTANGENTS—0° to 45°

## ANSWERS TO PRACTICE PROBLEMS

### Chapter 1

1. 360
2. 810
3. .002 second
4. 20 cps
5. 141 volts  
282 volts
6. 176.75 volts  
159 volts
7. .3535
8. .025 second
9. 1.9 amperes
10. open-circuited
11. .03 second
12. 126.4 volts

### Chapter 2

1.  $78^\circ$
2.  $105^\circ$
3.  $250^\circ$
4.  $32.3^\circ$
5.  $43^\circ 42'$
6.  $15.178^\circ$
7.  $270^\circ$
8.  $\frac{4\pi}{3}$
9. .5270  
.9627  
.8185
10. .9527  
.3649  
4.0108
11.  $40.2^\circ$   
 $27.4^\circ$   
 $12.3^\circ$

12.  $60.9^\circ$   
 $83.0^\circ$   
 $70.8^\circ$
13. 6.71 feet
14.  $26.6^\circ$   
 $63.4^\circ$
15. 15.36  
19.49
16. 4.47 feet  
6.63 feet
17.  $29.1^\circ$   
 $60.9^\circ$   
 $90.0^\circ$
18.  $-.6428$   
.6428  
 $-1.4281$
19. .9659  
 $-.5299$   
11.43
20.  $70^\circ$   
 $-70^\circ$

### Chapter 3

1.  $X = 9.83$   
 $Y = 6.88$
2.  $7.21/33.7^\circ$
3.  $10.41/40.8^\circ$
4.  $11.17/-3.8^\circ$
5.  $3 + j9$
6.  $12.27/99.6^\circ$
7.  $-12 - j1$
8.  $18/35^\circ$
9.  $40/-6^\circ$
10.  $2/30^\circ$

#### Chapter 4

1.  $-\sqrt{10}$
2.  $-2\sqrt{3}$
3.  $j3\sqrt{5}$
4.  $j2$
5.  $j4$
6.  $j13$
7.  $-j3$
8. 2
9.  $-j\sqrt{2}$
10.  $j2$
11.  $7.81/50.2^\circ$
12.  $7.62/-66.8^\circ$
13.  $4.728 + j3.694$
14.  $-3.76 + j1.37$
15.  $7 - j$
16.  $2 - j5$
17.  $4 + j5$
18.  $4 + j8$
19.  $21 + j$
20. 10
21.  $6/54^\circ$
22.  $12/140^\circ$
23.  $2/42^\circ$
24.  $5/18^\circ$
25.  $\frac{3 - j2}{2}$
26.  $\frac{3 - j}{2}$
27.  $\frac{-3 - j15}{13}$
28.  $9/30^\circ$

29.  $8/60^\circ$

30.  $5/41^\circ$

#### Chapter 5

1. 6.4 ohms  
51.3°  
.6250
2. 3.125 amps; lagging
3. 14.42 volts  
56.3°
4. 10.81 ohms  
56.3°  
.5548
5. 1.11 amps; leading
6. 7.39 watts  
13.32 volt-amperes
7. 20.78 volts
8.  $6 - j3$  ohms  
 $6.71/-26.6^\circ$  ohms
9. 3.58 amps  
26.6°  
leading
10. 4.24 ohms  
32°
11. 50 vars  
.8480
12. 3.75 ohms
13. 6.86 ohms
14. 50 watts  
31°  
leading
15.  $19.28/-4.8^\circ$  ohms  
capacitive
16. 5.97 ohms  
9.4°

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*essentials of*  
**VECTOR & PHASE  
ANALYSIS**

by *ALAN ANDREWS*

In order to have a thorough understanding of AC circuit operation and calculations, it is necessary to consider the phase relationships of currents and voltages. The basic tools for this understanding are trigonometry and vectors. If the vector approach is not used, AC circuit calculations can become long and tedious.

How vectors are used to simplify AC calculations is thoroughly presented in **Essentials of Vector & Phase Analysis**. The author, who has an extensive electronics background and has taught mathematics at one of the larger electronic technical institutes, covers the subject in a clear and concise manner. The first two chapters offer a comprehensive review of AC fundamentals, with emphasis on phase. These chapters review sine waves, inductive and capacitive circuits, angles, units of measurements, trig functions, solutions of triangles, and the co-ordinate system.

Vectors can be added, subtracted, multiplied, and divided. How to perform these mathematical operations is covered in Chapter 3. A discussion on imaginary numbers, conversion of forms, and vector algebra follows.

The last two chapters cover AC network calculations (series, parallel, and complex circuits) and vector applications.

To clarify the various topics, the author includes numerous examples as well as practice problems. The answers to the practice problems are contained in an appendix, along with a table of Natural Trigonometric Functions, enabling the reader to use this book as a self-teaching medium. It is also particularly suited for use as a supplementary textbook in any school or training program in electronics.

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