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THE MARCONI REVIEW

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Editor: L. E. Q. WALKER, A.R.C.S.

SOME DESIGN CONSIDERATIONS FOR LINKS CARRYING MULTICHANNEL TELEPHONY

PART II

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In Part I of this article a survey of some of the important factors affecting the design of links conveying multichannel telephony signals was given. This survey covered the amplitude and phase linearity of terminal modulators, demodulators and associated I.F. amplifiers, the phase linearity of the repeaters, the effect of weak echoes due to feeder mismatches or multipath propagation and some of the testing methods which have been found useful in the design of suitable equipment.

In this, the second part, the calculation of intermodulation distortion is treated.

INTERMODULATION DISTORTION DUE TO AMPLITUDE NON-LINEARITY

General

WE shall assume here that the non-linear system has a transfer characteristic of the form

$$V_0 = a_1 V_1 + a_2 V_1^2 + a_3 V_3^2 + \ldots . + a_n V_n^2 \qquad (1)$$

where $a_1, a_2, \ldots, a_n$ are constants which are independent of frequency.

The input signal into the system is made up of a large number of tones having random amplitudes and phases, fairly closely simulating random noise. The input spectrum need not necessarily be uniform. In fact, cases of non-uniform input spectra will be examined in some detail.

Symbols used

N = total number of channels carried by the multichannel system.
f_1, f_2 = lowest and highest frequencies in signal band.
t_n = n^{th} harmonic power in mW, produced by that sinusoidal input, which gives a fundamental output power of 1 mW.
T_n = total n^{th} order distortion power in mW.
P = total fundamental power in mW at the output of the system, measured at a point of zero level.

These symbols are the same as those used by Brockbank and Wass(3).

Harmonic Powers of Single Tones

If we assume the input voltage to be a single pure sine wave

$$V_1 = v_1 \sin x \tag{2}$$

then the output voltage is

$$V_0 = a_1 v_1 \sin x + a_2 v_1^2 \sin^2 x + \ldots + a_n v_1^n \sin^n x \tag{3}$$

The second and third harmonics produced by the second and third terms are given by the expansion

$$V_0 \fallingdotseq a_1 v_1 \sin x + \frac{a_2 v_1^2}{2} (1 - \cos 2x) + \frac{a_3 v_1^3}{4} (3 \sin x - \sin 3x) \tag{4}$$

If R is the load impedance across the output of the system, then the fundamental and the harmonic output powers are:

$$P_F = \frac{1}{2R} (a_1^2 v_1^2) \tag{5}$$

$$P_{H_2} = \frac{1}{2R} \left(\frac{a_2 v_1^2}{2} \right)^2 \tag{6}$$

$$P_{H_3} = \frac{1}{2R} \left(\frac{a_3 v_1^3}{4} \right)^2 \tag{7}$$

We can express the harmonic powers in terms of the fundamental output power by combining equations (5), (6) and (7), thus:

$$v_1^2 = \frac{2R}{a_1^2} P_F$$

$$P_{H_2} = \frac{a_2^2}{8R} \times \frac{4R^2 P_F^2}{a_1^4} = \frac{a_2^2 R}{2a_1^4} P_F^2 \tag{8}$$

$$P_{H_3} = \frac{a_3^2}{32R} \frac{8R^3 P_F^3}{a_1^6} = \frac{a_3^2 R^2}{4a_1^6} P_F^3 \tag{9}$$

If we take our unit of power as 1 mW, and make $P_F = 1$, we obtain from (8) and (9)

$$t_2 = \frac{a_2^2 R}{2 a_1^4} \tag{10}$$

$$t_3 = \frac{a_3^2 R^2}{4\, a_1^6} \tag{11}$$

In general the n^{th} harmonic is produced, almost exclusively, by the n^{th} term of the expansion (3), which contains $\sin^n x$. We have

$$\sin^n x = \frac{1}{2^{n-1}} \left[\begin{matrix} \pm \sin n x \\ \pm \cos n x \end{matrix} \; + \; \begin{matrix} \mp n \sin (n-2)x \\ \mp n \cos (n-2)x \end{matrix} \; + \ldots \right] \tag{12}$$

(see Dwight[11] p. 69) and consequently

$$P_{H_n} = \frac{1}{2R} \left(\frac{a_n\, v_1^n}{2^{n-1}} \right)^2 \tag{13}$$

$$P_{H_n} = \frac{a_n^2}{2^{2n-1}\, R} \times v_1^{2n} = \frac{a_n^2}{2^{2n-1} R} \; \frac{2^n\, R^n}{a_1^{2n}}\, P_F^n \tag{14}$$

$$P_{H_n} = \frac{a_n^2\, R^{n-1}}{2^{n-1}\, a_1^{2n}}\, P_F^n \tag{15}$$

and

$$t_n = \frac{a_n^2\, R^{n-1}}{2^{n-1}\, a_1^{2n}} \tag{16}$$

Equations (8), (9) and (15) can now be simplified and rewritten

$$P_{H_2} = t_2\, P_F^2 \tag{17}$$

$$P_{H_3} = t_3\, P_F^3 \tag{18}$$

$$\ldots \qquad \ldots\ldots$$

$$P_{H_n} = t_n\, P_F^n \tag{19}$$

Harmonic and Intermodulation Powers of Two Sinusoidal Tones

Let us now assume that two sinusoidal tones, $v_1 \sin x$ and $v_2 \sin y$, are simultaneously applied to the system.

The output voltage is

$$V_0 = a_1 (v_1 \sin x + v_2 \sin y) + a_2 (v_1 \sin x + v_2 \sin y)^2 + a_3 (v_1 \sin x + v_2 \sin y)^3 + \ldots . \tag{20}$$

This can be expanded to:

$$\begin{aligned} V_0 = {} & a_1 (v_1 \sin x + v_2 \sin y) \\ & + a_2 \left[\frac{v_1^2}{2} (1 - \cos 2x) + v_1 v_2 \cos (x - y) - v_1 v_2 \cos (x + y) + \frac{v_2^2}{2} (1 - \cos 2y) \right] \\ & + a_3 \left[\frac{v_1^3}{4} (3 \sin x - \sin 3 x) - \frac{3}{4} v_1^2 v_2 \sin (2x + y) + \frac{3}{4} v_1^2 v_2 \sin (2 x - y) \right. \\ & \left. - \frac{3}{4} v_1 v_2^2 \sin (2y + x) + \frac{3}{4} v_1 v_2^2 \sin (2y - x) + \frac{v_2^3}{4} (3 \sin y - \sin 3y) \right] + \ldots . \end{aligned} \tag{21}$$

In addition to harmonics, we have cross products or intermodulation terms. The fundamental, harmonic and intermodulation powers are as follows:

Fundamental: $P_{F_1} = \frac{1}{2} \frac{a_1^2\, v_1^2}{R}, \qquad P_{F_2} = \frac{1}{2} \frac{a_1^2\, v_2^2}{R}$

Harmonics: $$P_{H_{12}} = \frac{1}{2R}\left(\frac{a_2\, v_1^2}{2}\right)^2, \quad P_{H_{22}} = \frac{1}{2R}\left(\frac{a_2\, v_2^2}{2}\right)^2$$

$$P_{H_{13}} = \frac{1}{2R}\left(\frac{a_3\, v_1^3}{4}\right)^2, \quad P_{H_{23}} = \frac{1}{2R}\left(\frac{a_3\, v_2^3}{4}\right)^2$$

Intermodulation: $$P_2 = \frac{2}{2R}\,(a_2\, v_1\, v_2)^2$$

$$P_3 = \frac{2}{2R}\left(\frac{3}{4}\, a_3\, v_1^2\, v_2\right)^2 + \frac{2}{2R}\left(\frac{3}{4}\, a_3\, v_1\, v_2^2\right)^2$$

The harmonic and intermodulation powers can be expressed in terms of the distortion coefficients t_n, and the fundamental power, as follows:

$$P_{H_2} = t_2\,(P_{F_1}^{\,2} + P_{F_2}^{\,2}) \tag{22}$$

$$P_{H_3} = t_3\,(P_{F_1}^{\,3} + P_{F_2}^{\,3}) \tag{23}$$

$$P_2 = \frac{4\, a_2^2\, R}{a_1^4}\, P_{F_1}\, P_{F_2} = 8\, t_2\, P_{F_1} \cdot P_{F_2} \tag{24}$$

$$P_3 = \frac{9\, a_3^2\, R^2}{2\, a_1^6}\, P_{F_1}\, P_{F_2}\,(P_{F_1} + P_{F_2}) = 18\, t_3\, P_{F_1}\, P_{F_2}\,(P_{F_1} + P_{F_2}) \tag{25}$$

Let us now assume that the two tones are of equal amplitude,

i.e. $$P_{F_1} = P_{F_2} = \frac{P_F}{2}$$

We have then

$$P_{H_2} = \frac{1}{2}\, t_2\, P_F^{\,2} \tag{26}$$

$$P_{H_3} = \frac{1}{4}\, t_3\, P_F^{\,3} \tag{27}$$

$$P_2 = 2\, t_2\, P_F^{\,2} \tag{28}$$

$$P_3 = \frac{9}{2}\, t_3\, P_F^{\,3} \tag{29}$$

If we compare equations (28) and (29) with (17) and (18), we see that by replacing the single tone by two tones delivering the same fundamental power output, we have increased the amount of distortion (mainly intermodulation) by factors of 2·5 and 4·75 respectively. The harmonic powers also become relatively insignificant.

If the two input voltages v_1 and v_2 are each equal to half the original voltage v_1, of equation (2), we have

$$P_{F_1} = P_{F_2} = \frac{1}{4}\, P_F \tag{30}$$

$$\frac{P_{2+}}{P_{F_1}} = \frac{P_{2-}}{P_{F_1}} = t_2\, P_F \tag{31}$$

and from equation (17)

$$\frac{P_{H_2}}{P_F} = t_2\, P_F \tag{32}$$

The second order harmonic margin* is equal to the second order intermodulation margin* for the test conditions stated above.

Harmonics and Intermodulation Products for Large Numbers of Tones

If the number of simultaneous tones is very large, the amount of intermodulation power vastly exceeds the amount of harmonic power. Already for two tones only we can see from equations (26) to (29) that the harmonic powers are pretty well negligible.

To approach the case of a random noise input, with a continuous spectrum, take the input signal to be

$$V_1 = v_1 \sin x + v_2 \sin y + \ldots \tag{33}$$

made up of r terms (r is large). The n^{th} order products come from the term

$$a_n (v_1 \sin x + v_2 \sin y + \ldots)^n$$

where n is usually small and always much less than r. If this term is expanded, it gives r^n separate terms, and of these a number

$$m = r(r-1) \ldots (r-n+1) \tag{34}$$

involve n different frequencies. The ratio

$$\frac{m}{r^n} = \left(1 - \frac{1}{r}\right)\left(1 - \frac{2}{r}\right) \ldots \left(1 - \frac{n-1}{r}\right) \tag{35}$$

approaches unity as the number of terms r increases.

For example for $r = 100$ and $n = 4$, the ratio $m/r^4 = \cdot 99 \times \cdot 98 \times \cdot 97 = \cdot 941$. It follows that only the terms with n different frequencies need be considered when r is large, provided one, or more, of the input signals is not much greater than the rest. More n^{th} order terms will come from the $(n+2)^{th}$, $(n+4)^{th}$ power, but these will be terms for which two or more frequencies are equal, and hence a negligible fraction when r is large. Moreover, they are multiplied by a_{n+2} etc. which is generally much smaller than a_n.

A typical term of the expansion of (33) containing n different frequencies is

$$n!\, a_n v_{x_1} v_{x_2} \ldots \sin x_1 . \sin x_2 \ldots \quad (n \text{ terms}) \tag{36}$$

since there are $n!$ possible ways of selecting it.

For instance, for $r = 4$, $n = 3$ we have

$$\begin{aligned}(a+b+c+d)^3 = {} & a^3 + b^3 + c^3 + d^3 \\ & + 3(ab^2 + a^2 b + ac^2 + a^2 c + ad^2 + a^2 d + bc^2 + b^2 c \\ & + cd^2 + c^2 d + db^2 + d^2 b) \\ & + 6(abc + bcd + cda + dab)\end{aligned}$$

The total number of terms is $r^n = 64$. Of these $r(r-1)(r-2) = 24$ contain three different factors, which fall into groups of $n! = 6$, having identical factors. When r is large this type of product predominates.

The typical term (36) can now be expanded into sum and difference frequency terms as follows: For

$$n=2 \quad \sin x_1 \sin x_2 = \frac{1}{2}[\cos(x_1 - x_2) - \cos(x_1 + x_2)] \tag{36a}$$

* We define a harmonic margin as the ratio of harmonic power to fundamental power Similarly the intermodulation margin is defined in the same way. P_{2+} refers to the second order sum term, while P_{2-} refers to the difference term.

$$n=3 \quad \sin x_1 \sin x_2 \sin x_3 = \frac{1}{4}\{-\sin(x_1+x_2+x_3)+\sin(x_1+x_2-x_3)$$
$$+\sin(x_1-x_2+x_3)+\sin(-x_1+x_2+x_3)\} \quad (36b)$$

$$n=4 \quad \sin x_1 \sin x_2 \sin x_3 \sin x_4 = \frac{1}{8}\{\cos(x_1+x_2+x_3+x_4)-\cos(-x_1+x_2+x_3+x_4)$$
$$-\cos(x_1-x_2+x_3+x_4)-\cos(x_1+x_2-x_3+x_4)$$
$$-\cos(x_1+x_2+x_3-x_4)+\cos(x_1+x_2-x_3-x_4)$$
$$+\cos(x_1-x_2-x_3+x_4)+\cos(x_1-x_2+x_3-x_4)\} \quad (36c)$$

The general result, which can be proved by induction, is:
For n terms $\quad \sin x_1 \sin x_2 \sin x_3 \ldots . \sin x_n$

$$= \frac{1}{2^{n-1}}\left\{\pm(s/_c(x_1+..+x_n) \pm s/_c(\pm x_1 \pm .. \pm x_n) \pm s/_c(\pm x_1 \pm .. \pm x_n) \pm \ldots .\right\}$$

all +	one −	two −	(37)
1 term	n terms.	$\frac{n(n-1)}{2}$ terms.	

Here $s/_c$ means sine for n odd and cosine for n even. The total number of terms is 2^{n-1}.

The power in a typical n^{th} order term such as (36) is

$$p = \frac{1}{2R}\cdot\left(\frac{n!\,a_n}{2^{n-1}}\right)^2 (v_{x_1} v_{x_2} \ldots v_{x_n})^2 \cdot 2^{n-1} \quad (38)$$

$$p = \frac{n!^2\, a_n^2}{2^n R} v_{x_1}^2 v_{x_2}^2 v_{x_3}^2 \ldots . \quad (39)$$

and the total n^{th} order intermodulation power is

$$T_n = \frac{n!^2\, a_n^2}{2^n R} \sum v_{x_1}^2 v_{x_2}^2 v_{x_3}^2 \ldots \quad (40)$$

summed over all combinations of n different $x_1, x_2, \ldots x_n$, out of r.

Now $\quad (v_1^2 + v_2^2 + .. + v_2^2)^n \doteqdot n! \sum v_{x_1}^2 v_{x_2}^2 \ldots v_{x_n}^2 \quad (41)$

where the x's are all different, since the other components can be neglected for the reasons given at the beginning of this paragraph.

It follows that

$$T_n = \frac{n!\, a_n^2}{2^n R}[v_1^2 + v_2^2 + \ldots + v_r^2]^n \quad (42)$$

If P is the fundamental output power we have

$$P = \frac{a_1^2}{2R}(v_1^2 + v_2^2 + \ldots + v_r^2) \quad (43)$$

and

$$T_n = \frac{n!\, a_n^2}{2^n R}\left(\frac{2R}{a_1^2}\right)^n P^n \quad (44)$$

$$T_n = n! \frac{a_n^2 R^{n-1}}{a_1^{2n}} P^n \qquad (45)$$

and from (16)

$$T_n = 2^{n-1} n! t_n P^n \qquad (46)$$

Transition to Continuous Spectra

We shall assume that the case of a continuous signal spectrum is the limiting case investigated in the preceding section, where, without loss of generality we could take:

$$x_1 = f_1 + \Delta f, x_2 = f_1 + 2\Delta f, \ldots x_r = f_1 + r\Delta f = f_2.$$

The output spectrum in a small bandwidth df is:

$$dP = \Phi(f) \, df$$

while in the discrete case the output power due to a component $v_f \sin x_f$ is

$$\Delta P_f = \frac{a_1^2}{2R} V_f^2$$

The power per unit bandwidth is

$$\frac{\Delta P_f}{\Delta f} = \frac{a_1^2}{2R} \frac{V_f^2}{\Delta f}$$

which, in the limit, becomes the power spectrum $\Phi(f)$ and we have

$$\frac{dP}{df} = \Phi(f) = \frac{a_1^2}{2R} \text{ Limit of } \frac{V_f^2}{\Delta f}$$

and $V_f \doteqdot \sqrt{\frac{2R \, \Phi(f) \, \Delta f}{a_1^2}}$ for a very large number of terms.

The power in an individual n^{th} order term is, from equation (39)

$$p = \frac{n!^2 a_n^2}{2^n R} v_{x_1}^2 v_{x_2}^2 \ldots$$

which in the limit could be written as

$$dp = n!^2 \frac{a_n^2 R^{n-1}}{a_1^{2n}} dP_1 \cdot dP_2 \ldots dP_n \qquad (47)$$

The product in (47) contains each power element once only in whichever P it occurs, and hence it is $1/n!$ of the product with all elements occurring in each factor. Therefore we can write

$$T_n = 2^{n-1} n! t_n \int\int\int dP_1 \cdot dP_2 \ldots . dP_n$$

$$T_n = 2^{n-1} n! t_n \int_0^P dP_1 \int_0^P dP_2 \ldots \int_0^P dP_n = 2^{n-1} n! t_n P^n \qquad (48)$$

Intermodulation Spectra

If now we return to equation (47) and introduce the value of t_n given by equation (16)

$$t_n = \frac{a_n^2 R^{n-1}}{2^{n-1} a_1^{2n}}$$

we obtain

$$dp = n!^2 \, 2^{n-1} \, t_n \, dP_1 \cdot dP_2 \ldots dP_n \tag{49}$$

The n^{th} order intermodulation power dp, falling within a small frequency band at frequency f, say, is proportional to the product of the signal powers $dP_1, dP_2, .. dP_n$, falling within small bands at frequencies $f_1, f_2, .. f_n$, such that intermodulation products of the type $x \pm y \pm z$ . . , are at frequency f.

Take for example the case of second order intermodulation terms. They are of the type $x + y$ and $x - y$. The intermodulation power dp, at frequency f, is proportional to the product of the signal powers falling at frequencies u and $v = f - u$, for the sum terms, both u and v being signal frequencies. Thus, for every value of f, we can determine a value of dp (taking sum and difference terms) and we can consequently determine the frequency distribution of intermodulation powers (intermodulation spectra).

We see from equations (36a), (36b) and (36c) that intermodulation powers are weighted according to the type of term as follows:

2^{nd} order.	$x + y$	: $\frac{1}{2}$	Total power: 1
	$x - y$	: $\frac{1}{2}$	
3^{rd} order.	$x + y + z$	: $\frac{1}{4}$	Total power: 1
	$x + y - z$	: $\frac{3}{4}$	
4^{th} order.	$x + y + z + w$	: $\frac{1}{8}$	Total power: 1
	$x + y + z - w$	: $\frac{1}{2}$	
	$x + y - z - w$	: $\frac{3}{8}$	

Given that the input signal distribution is Φ_1, for the range $f_1 < f < f_2$ and zero outside this range, the second order $x + y$ distribution of intermodulation powers is

$$\Phi_2'(u) = \int \Phi_1(f) \, \Phi_1(u - f) \, df \tag{50}$$

taken between limits for which the integral is not zero.

i.e.

$$\begin{cases} f_1 < f < f_2 \\ f_1 < u - f < f_2 \end{cases} \quad \text{or} \quad \begin{cases} f_1 < f < f_2 \\ u - f_2 < f < u - f_1 \end{cases} \tag{51}$$

If $u < 2f_1$ or $u > 2f_2$, these two conditions cannot be satisfied simultaneously, and $\Phi_2' = 0$.
If $2f_1 < u < f_1 + f_2$, the limits for the integral are f_1 and $u - f_1$.
If $f_1 + f_2 < u < 2f_2$ the limits are $u - f_2$ and f_2.

Therefore for

$$2f_1 < u < f_1 + f_2 \qquad \Phi_2'(u) = \int_{f_1}^{u-f_1} \Phi_1(f) \quad \Phi_1(u-f)\, df \tag{52}$$

$$f_1 + f_2 < u < 2f_2 \qquad \Phi_2'(u) = \int_{u-f_2}^{f_2} \Phi_1(f) \quad \Phi_1(u-f)\, df \tag{53}$$

Similarly the $(x - y)$ distribution is

$$\Phi_2''(u) = \int \Phi_1(f)\, \Phi_1(u+f)\, df \tag{54}$$

with limits given by

$$\begin{cases} f_1 < f < f_2 \\ f_1 < u + f < f_2 \end{cases} \qquad \text{or} \qquad \begin{cases} f_1 < f < f_2 \\ f_1 - u < f < f_2 - u \end{cases} \tag{55}$$

These only overlap if $f_1 - f_2 < u < f_2 - f_1$ (56)

Therefore for

$$f_1 - f_2 < u < 0 \qquad \Phi_2''(u) = \int_{f_1-u}^{f_2} \Phi_1(f) \quad \Phi_1(f+u)\, df \tag{57}$$

$$0 < u < f_2 - f_1 \qquad \Phi_2''(u) = \int_{f_1}^{f_2-u} \Phi_1(f) \quad \Phi_1(f+u)\, df \tag{58}$$

Further distributions are built up step by step in a similar way. If u is negative, as it is in equation (57), no distinction is made and the spectra for equal positive and negative u's are added together.

By way of example we shall apply equations (52), (53), (57) and (58) to the case of a uniform signal spectrum.

Here

$$\Phi_1(f) = \frac{1}{f_2 - f_1} \tag{59}$$

the signal power in a small band, at frequency f, is

$$dP = \Phi_1(f)\, df = \frac{df}{f_2 - f_1}$$

so that the total signal power is

$$P = \int_{f_1}^{f_2} \frac{df}{f_2 - f_1} = 1$$

We have

$$2f_1 < u < f_1 + f_2 \qquad \Phi_2'(u) = \int_{f_1}^{u-f_1} \frac{1}{f_2 - f_1} \cdot \frac{1}{f_2 - f_1}\, df$$

$$= \frac{1}{(f_2 - f_1)^2} \cdot (u - 2f_1) \tag{60}$$

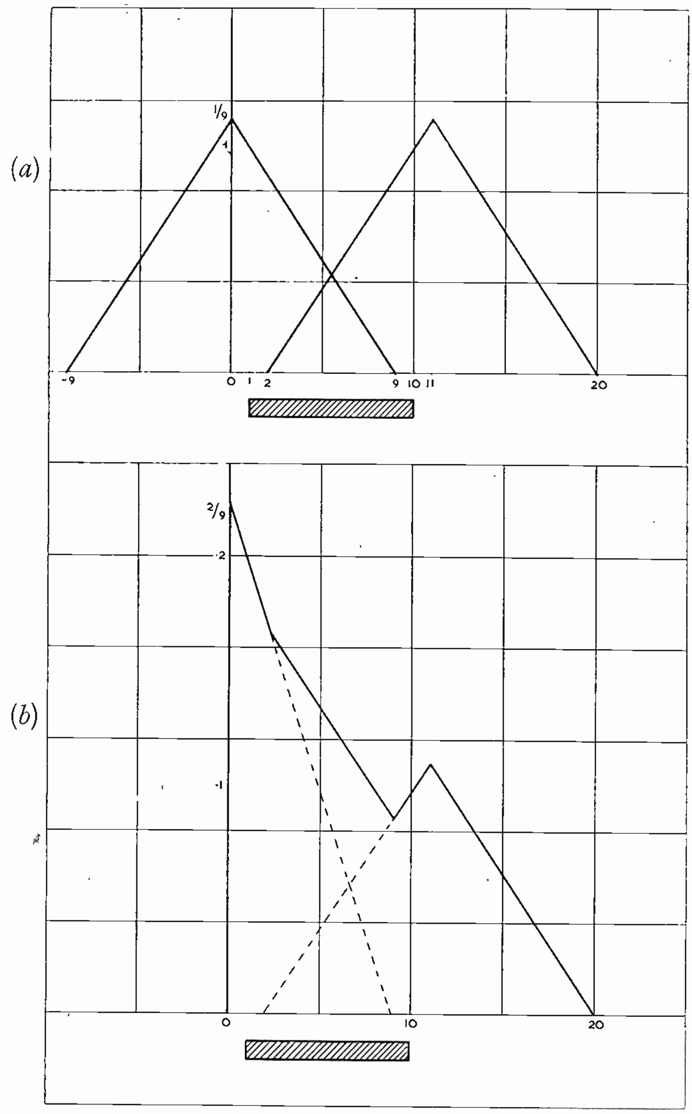

FIG. 1

2nd order intermodulation distribution for $f_2 = 10, f_1 = 1$.

$$f_1 + f_2 < u < 2f_2$$

$$\Phi_2'(u) = \int_{u-f_2}^{f_2} \frac{1}{(f_2 - f_1)^2}\, df = \frac{2f_2 - u}{(f_2 - f_1)^2} \qquad (61)$$

$$f_1 - f_2 < u < 0$$

$$\Phi_2''(u) = \int_{f_1-u}^{f_2} \frac{1}{(f_2 - f_1)^2}\, df = \frac{f_2 - f_1 + u}{(f_2 - f_1)^2} \qquad (62)$$

$$0 < u < f_2 - f_1$$

$$\Phi_2''(u) = \int_{f_1}^{f_2-u} \frac{df}{(f_2 - f_1)^2} = \frac{f_2 - f_1 - u}{(f_2 - f_1)^2} \qquad (63)$$

The complete spectrum, as given by equations (60)-(63), is plotted in Fig. 1b for $f_1 = 1, f_2 = 10$.

We have, for the above values of f_1 and f_2

$$\Phi_2'(u) = \frac{u - 2}{81} \qquad 2 < u < 11 \qquad\qquad \Phi_2''(u) = \frac{9 + u}{81} \qquad -9 < u < 0$$

$$\Phi_2'(u) = \frac{20 - u}{81} \qquad 11 < u < 20 \qquad\qquad \Phi_2''(u) = \frac{9 - u}{81} \qquad 0 < u < 9$$

The spectrum for $-9 < u < 0$ is folded back into the positive frequency region and added to the spectrum for $0 < u < 9$.

The amount of second order intermodulation power, falling within a small band b, at frequency u is

$$d_2 = \Phi_2(u) \times b \times 4\, t_2\, P^2.$$

If for example $b = 0{\cdot}1$ and $u = 0$, then

$$d_2 = 4\, t_2\, P^2 \times \frac{2}{9} \times 0{\cdot}1$$

$$= \frac{\cdot 8}{9}\, t_2\, P^2$$

where P is the total signal power.

Intermodulation Spectra for a "White Noise" Input

If the input spectrum is flat between the limits f_1, f_2, the intermodulation spectra can be simply derived from the $x + y + z + w + \ldots.$ spectrum (sum frequencies only) for the band 0, 1.

Take the n^{th} order intermodulation distribution for $x + y \ldots - z - w - \ldots$ (i.e. for $n - t$ positive terms and t negative terms), $[\Phi_n]_{n-t}$.

$$[\Phi_n]_{n-t}(u)\, du = \Phi_1(x)\, \Phi_1(y) \ldots \Phi_1(z)\, dx\, dy \ldots dz$$

$$u = x + y + \ldots - z - w - \ldots$$

where $\Phi_1(u) = \dfrac{1}{f_2 - f_1}$ in the range f_1 to f_2, and zero outside.

Change variables x, y, z etc. to X, Y, Z etc. as follows:

$$\begin{aligned} X &= (x - f_1)/(f_2 - f_1) \\ Y &= (y - f_1)/(f_2 - f_1) \text{ etc.} \\ &\ldots\ldots \\ Z &= (f_2 - z)/(f_2 - f_1) \text{ etc.} \end{aligned} \qquad (64)$$

and

$$U = X + Y + \ldots + Z + W + \ldots$$

X, Y, .. range from 0 to 1, and U from 0 to n.

We also have:

$$\begin{aligned} x &= f_1 + X\,(f_2 - f_1) \\ y &= f_1 + Y\,(f_2 - f_1) \\ &\ldots\ldots \end{aligned} \qquad (65)$$

$$\begin{aligned} z &= f_2 - Z\,(f_2 - f_1) \\ w &= f_2 - W\,(f_2 - f_1) \end{aligned}$$

and

$$u = (n - t)\, f_1 - t\, f_2 + U\,(f_2 - f_1) \qquad (66)$$

The intermodulation spectrum in terms of the new variables is

$$F_n(U)\, dU = F_1(X)\, F_1(Y) \ldots F_1(Z) \cdot dX \cdot dY \ldots dZ$$

and we have $F_n(U)\, dU = [\Phi_n]_{n-t}(u)\, du$ (67)

Since u and U are linearly related (equation 66) and Φ_1 is uniform, the reversal of sign between Z and z in (64) does not change the form of the distribution and the $\Phi(u)$ and $F(U)$ spectra are similar in shape and only differ by a shift of the frequency scale, as specified by equation (66).

From (66) $$du = (f_2 - f_1)\, dU \tag{68}$$

therefore $$\Phi_n(u) = F_n(U)/(f_2 - f_1) \tag{69}$$

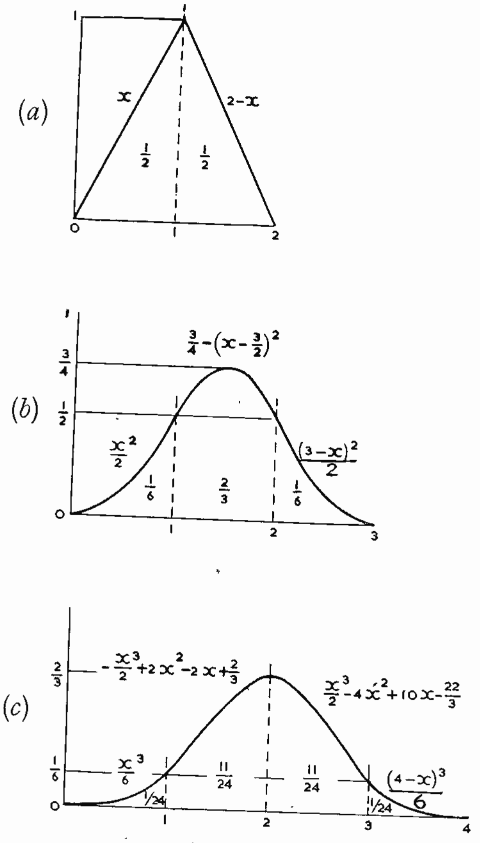

FIG. 2
Basic intermodulation spectrum $F_n(U)$. *Input signal band* 0 *to* 1, *sum terms only.*

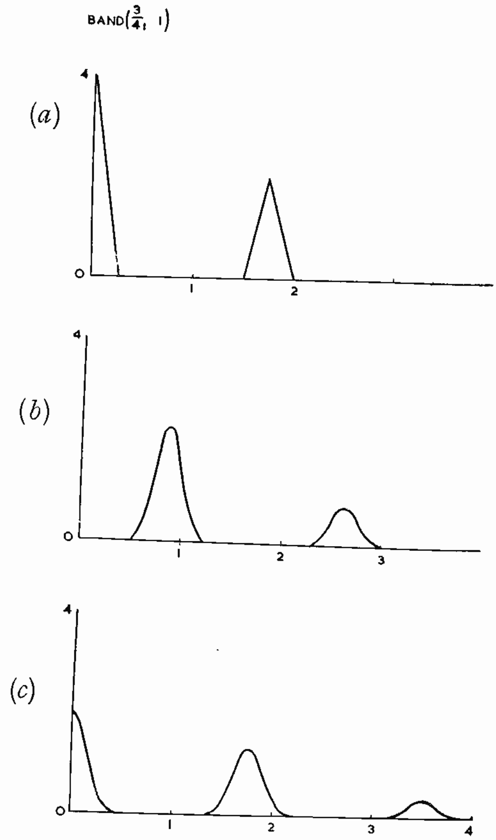

FIG. 3
2nd, 3rd and 4th order intermodulation spectra for a level signal covering the band 3/4 to 1.

$F_n(U)$ is the n^{th} order spectrum for the band (0,1), with all terms positive, and it is the only distribution which needs to be worked out.

For instance, $\Phi_5(u)$ for $n - t = 3$ and $t = 2$ in the band $f_2 = 7$ to $f_1 = 5$ is related to $F_5\,(U)$ for the band 0, 1 and all positive terms. The lowest and highest u are, according to (66), 1 and 11 respectively, which is correct since the lowest u comes from three fives minus two sevens, while the highest comes from three sevens minus two fives.

Taking the F distributions themselves $F_{n+1}\,(U_{n+1})$ may be obtained by taking all cases where the first n frequencies give U_n $(0 < U_n < n)$, and the last gives x $(0 < x < 1)$.

Therefore

$$F_{n+1}(U_{n+1}) = \int F_1(x)\, F_n(U_n)\, dx \text{ and } U_n + x = U_{n+1}$$

$$F_{n+1}(U_{n+1}) = \int F_1(x)\, F_n(U_{n+1} - x)\, dx \tag{70}$$

We have $F_1 = 1$, $0 < x < 1$ and $F_1 = 0$ outside this range of x so that

$$F_{n+1}(U_{n+1}) = \int_0^1 F_n(U_{n+1} - x)\, dx \tag{71}$$

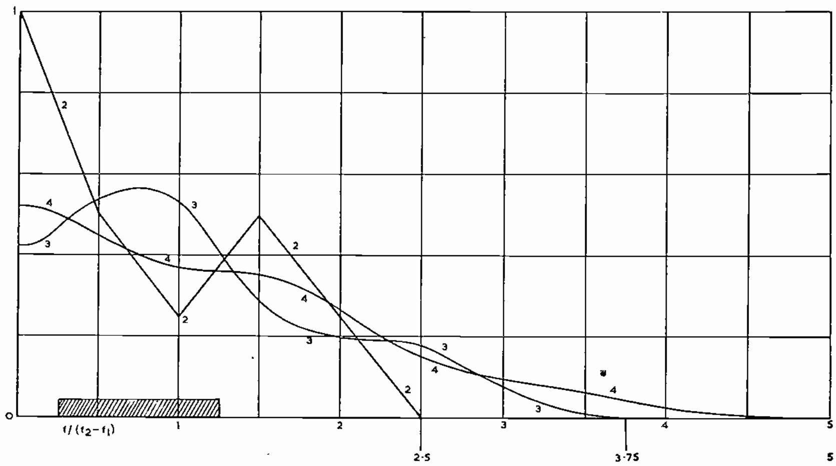

FIG. 4

Amplitude intermodulation spectra for $f_2/f_1 = 5$ (60 channel system).

If we now change variables as follows:

$$U_{n+1} - x = y \qquad \text{for } x = 0,\ y = U_{n+1}$$

$$-dx = dy \qquad x = 1,\ y = U_{n+1} - 1$$

we have

$$F_{n+1}(U_{n+1}) = \int_{U_{n+1}-1}^{U_{n+1}} F_n(y)\, dy \tag{72}$$

Thus the spectrum of order $(n+1)^{\text{th}}$ can be obtained simply from the n^{th} order spectrum according to equation (72).

$$F_1(U_1) = 1 \qquad 0 < U_1 < 1$$

$$= 0 \qquad \text{elsewhere}$$

$$F_2(U_2) = \int_0^{U_2} 1\, dy = U_2 \qquad 0 < U_2 < 1 \tag{73}$$

$$= \int_{U_2-1}^{1} 1\, dy = 2 - U_2 \qquad 1 < U_2 < 2 \tag{74}$$

$$F_3(U_3) = \int_0^{U_3} y\,dy = \frac{1}{2} U_3^2 \qquad 0 < U_3 < 1 \qquad (75)$$

$$= \int_{U_3-1}^{1} y\,dy + \int_1^{U_3} (2-y)\,dy = \frac{3}{4} - \left(U_3 - \frac{3}{2}\right)^2 \qquad 1 < U_3 < 2 \qquad (76)$$

$$= \int_{U_3-1}^{2} (2-y)\,dy = \frac{1}{2}(3-U_3)^2 \qquad 2 < U_3 < 3 \qquad (77)$$

$$F_4(U_4) = \int_0^{U_4} \frac{1}{2} y^2\,dy = \frac{1}{6} U_4^3 \qquad 0 < U_4 < 1 \qquad (78)$$

$$= \int_{U_4-1}^{1} \frac{1}{2} y^2\,dy + \int_1^{U_4} \left\{\frac{3}{4} - \left(y - \frac{3}{2}\right)^2\right\} dy$$

$$= -\frac{1}{2} U_4^3 + 2U_4^2 - 2U_4 + \frac{2}{3} \qquad 1 < U_4 < 2 \qquad (79)$$

$$= \int_{U_4-1}^{2} \left\{\frac{3}{4} - \left(y - \frac{3}{2}\right)^2\right\} dy + \int_2^{U_4} \frac{1}{2}(3-y)^2\,dy = \frac{1}{2} U_4^3 - 4U_4^2 + 10U_4 - \frac{22}{3} \qquad 2 < U_4 < 3 \qquad (80)$$

$$= \int_3^{U_4} \frac{1}{2}(3-y)^2\,dy = \frac{1}{6}(4-U_4)^3 \qquad 3 < U_4 < 4 \qquad (81)$$

Given the basic spectra $F_n(\omega)$, the procedure for determining the intermodulation spectra for a given input spectrum, extending from f_1 to f_2, is illustrated in the following example. Let $f_1 = \frac{3}{4}$ and $f_2 = 1$. The second order terms are as $x + y$ and $x - y$. Taking the sum term first the relation between u and U is, from (66)

$$u = 2 \times \frac{3}{4} + \frac{1}{4} U$$

$$u = 1{\cdot}5 + 0{\cdot}25\, U.$$

We also have from equation (69)

$$\Phi_n(u) = \frac{F_n(U)}{\frac{1}{4}} = 4\, F_n(U) \qquad (82)$$

For $U = 0$ $u = 1{\cdot}5$, $U = 2$ $u = 2$. Therefore the triangle of Fig. 2a is shifted to the band 1·5 to 2 and its ordinates multiplied by 2.

In the case of the difference term $x - y$, we have

$$u = \frac{3}{4} - 1 + \frac{1}{4} U$$

$$u = -\frac{1}{4} + \frac{1}{4} U$$

For $U = 0$ $u = -\frac{1}{4}$, $U = 2$ $u = +\frac{1}{4}$. The triangle of Fig. 2a is now shifted to the band $-\frac{1}{4}$ to $+\frac{1}{4}$ and the negative part folded back to double the positive part. The complete 2nd order spectrum is thus as shown in Fig. 3a. Higher order spectra

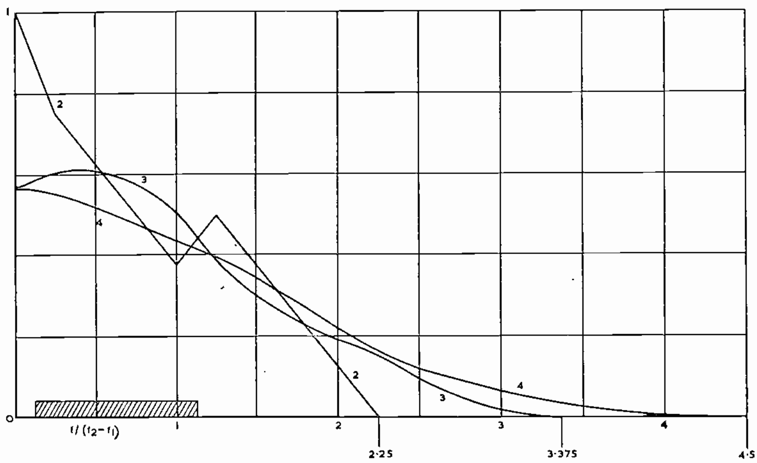

FIG. 5

Amplitude intermodulation spectra for $f_2/f_1 = 9$ (120 *channel system*).

are obtained in the same way. Third and fourth order spectra are given in Figs. 3b and 3c.

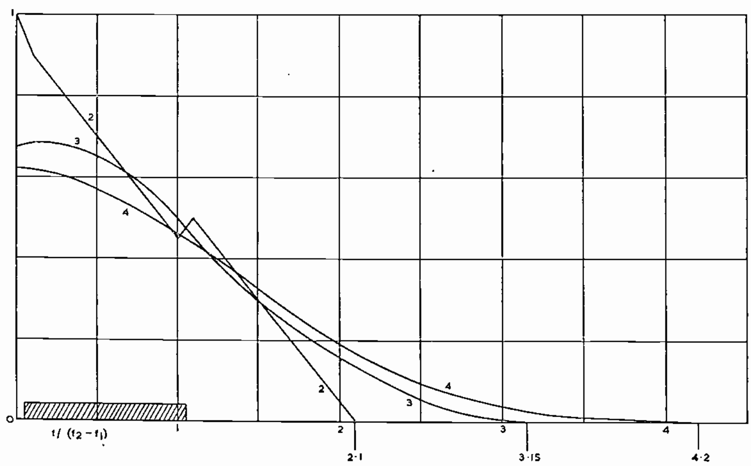

FIG. 6

Amplitude intermodulation spectra for $f_2/f_1 = 21$ (300 *channel system*).

It may be of interest to note that there is no distortion at the points $\frac{1}{2}$, $1\frac{1}{4}$, $2\frac{1}{4}$, 3 for all orders up to and including the fourth.

Typical intermodulation spectra are given in Figs. 4 to 7 for various values of f_2/f_1. The frequency scale is normalised to the signal bandwidths.

n^{th} Order Intermodulation Powers in a channel

The n^{th} order intermodulation power d_n, falling within a small band b, centred at frequency f, can be deduced directly from the above figures.

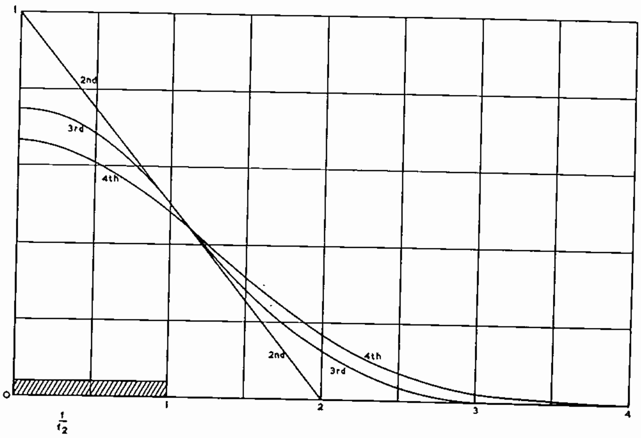

FIG. 7

Amplitude intermodulation spectra for $f_2/f_1 = \infty$.

If B is the signal bandwidth we have

$$d_n = T_n \frac{b}{B} F_n(\omega) \qquad (83)$$

where $F_n(\omega)$ is the amplitude of the intermodulation spectrum at the frequency of the selected channel. In general $\frac{b}{B} = \frac{1}{N}$, where N is the number of channels and we have then

$$d_n = \frac{T_n F_n(\omega)}{N} \qquad (84)$$

$F_n(\omega)$ may be obtained from Figs. 4 to 7, depending on the system used.

INTERMODULATION DISTORTION DUE TO PHASE NON-LINEARITY

General

The distortion of f.m. signals in passive networks has been the subject of several articles. A number of approximate formulae for the calculation of distortion have been derived, and one which has been found extremely useful in problems demanding solutions in general terms, is the quasi-stationary approximation of Van der Pol[4].

According to this approximation the output signal $\Delta\omega\, g_0(t)$ obtained when an f.m. signal $\Delta\omega\, g(t)$ is applied to a passive network is given by

$$g_0(t) = g(t) + \frac{d}{dt} g(t) \left[\varphi'(\omega) + \Delta\omega\, g(t)\, \varphi''(\omega) + \frac{\Delta\omega^2\, g(t)^2}{2} \varphi'''(\omega) + \ldots \right] \qquad (85)$$

where $\varphi'(\omega), \ldots \varphi^{(n)}(\omega)$ are the derivatives of the phase response $\varphi(\omega)$, with respect to the angular frequency ω.

Let

$$\varphi'(\omega) = \tau_1 + \tau_2 \left(\frac{\omega - \omega_0}{\frac{B}{2}} \right) + \tau_3 \left(\frac{\omega - \omega_0}{\frac{B}{2}} \right)^2 + \ldots \qquad (86)$$

where B is the bandwidth of the network (to 3 db say)

ω_0 the angular centre frequency.

τ_1, τ_2, etc. group delay coefficients of the network

We have, from equations (85) and (86)

$$g_0(t) = g(t) + \frac{d\,g(t)}{dt}\left[\tau_1 + \tau_2 \frac{\Delta\omega\, g(t)}{\frac{B}{2}} + \tau_3\left(\frac{\Delta\omega\, g(t)}{\frac{B}{2}}\right)^2 + \cdot\cdot\right] \tag{87}$$

That the quasi-stationary approximation is not quite correct is clear enough from the above formula. A system with a flat amplitude and linear phase response ($\tau_2 = \tau_3 = \ldots = 0$) should give an output signal equal to the input signal except for a time delay.

According to equation (87) however we obtain

$$g_0(t) = g(t) + \tau_1 \frac{d\,g(t)}{dt} \tag{88}$$

and it appears that a quadrature component has been added to the input signal. This of course cannot be true.[12]

Harmonics of a Pure Tone

If the input signal is $\Delta f \sin \omega_f t$, the output signal is $\Delta f\, g_0(t)$ and we have

$$g_0(t) = \sin \omega_f t + \omega_f \cos \omega_f t \left[\tau_1 + \tau_2\left(\frac{\Delta f}{\frac{B}{2}}\right) \sin \omega_f t + \tau_3\left(\frac{\Delta f}{\frac{B}{2}}\right)^2 \sin^2 \omega_f t + \ldots\right] \tag{89}$$

The amplitudes of the second and third harmonics produced are:

$$H_2 = \omega_f \cdot \frac{\Delta f}{B} \tau_2 \cdot \Delta f \cdot \sin 2\omega_f t \tag{90}$$

$$H_3 = \omega_f \cdot \left(\frac{\Delta f}{B}\right)^2 \tau_3 \cdot \Delta f \cdot \cos 3\omega_f t \tag{91}$$

It is clear, from equation (4) and equations (90) and (91), that the harmonics produced by amplitude and phase distortion, respectively, are in phase quadrature to one another.

The harmonic margins (ratio of amplitudes), are, in terms of the delay coefficients:

$$\frac{H_2}{F} = \omega_f \cdot \left(\frac{\Delta f}{B}\right) \tau_2 = \frac{1}{2}\,\omega_2 \left(\frac{\Delta f}{B}\right) \tau_2 \tag{92}$$

$$\frac{H_3}{F} = \omega_f \cdot \left(\frac{\Delta f}{B}\right)^2 \tau_3 = \frac{1}{3}\,\omega_3 \left(\frac{\Delta f}{B}\right)^2 \tau_3 \tag{93}$$

where ω_2 and ω_3 are the frequencies of the harmonic terms.

The fundamental output power P_F is, if d and R are constants of the discriminator and line amplifier

$$P_F = \frac{d^2}{2R} \Delta f^2 \tag{94}$$

while the harmonic powers are:

$$P'_{H_2} = \frac{d^2}{2R}\ \omega_f^{\,2}\ \tau_2^{\,2}\ \Delta f^2 \left(\frac{\Delta f}{B}\right)^2 \tag{95}$$

$$P'_{H_3} = \frac{d^2}{2R}\ \omega_f^{\,2}\ \tau_3^{\,2}\ \Delta f^2 \left(\frac{\Delta f}{B}\right)^4 \tag{96}$$

If now we combine equations (94), (95) and (96) we obtain

$$P'_{H_2} = \frac{2R}{d^2\ B^2}\ \omega_f^{\,2}\ \tau_2^{\,2}\ P_F^{\,2} \tag{97}$$

$$P'_{H_3} = \left(\frac{2R}{d^2\ B^2}\right)^2\ \omega_f^{\,2}\ \tau_3^{\,2}\ P_F^{\,3} \tag{98}$$

These two formulae may be generalised by writing

$$n^2\, t'_n = \left(\frac{2R}{d^2\ B^2}\right)^{n-1}\ \tau_n^{\,2} \tag{99}$$

$$P'_{H_n} = n^2\, t'_n\ \omega_f^{\,2}\ P_F^{\,n} = t'_n\ (n\ \omega_f)^2\ P_F^{\,n} \tag{100}$$

Here the coefficient $n^2\, t'_n$ may be defined as the n^{th} harmonic power in mW, produced by that sinusoidal input, which gives a fundamental output power of 1 mW, at an angular frequency ω_f of unity. Thus $n^2 t'_n$ corresponds to the coefficient t_n for the case of amplitude non-linearity.

Intermodulation Terms Produced by Two Sinusoidal Tones

Let $g(t) = \sin\ \omega_1 t + \sin\ \omega_2 t$ and assume that each tone has an amplitude of $\frac{1}{2}\,\Delta f$, we have

$$g_0(t) = \sin\ \omega_1 t + \sin\ \omega_2 t$$

$$+ (\omega_1 \cos\ \omega_1 t + \omega_2 \cos\ \omega_2 t) \left[\tau_1 + \tau_2\, \frac{1}{2}\left(\frac{\Delta f}{\frac{B}{2}}\right)(\sin\ \omega_1 t + \sin\ \omega_2 t)\right.$$

$$\left. + \tau_3\, \frac{1}{4}\left(\frac{\Delta f}{\frac{B}{2}}\right)^2 (\sin\ \omega_1 t + \sin\ \omega_2 t)^2 + \ldots\right] \tag{101}$$

There are four second and six third order terms, with the following amplitudes, normalised to the amplitudes of each of the fundamental tones.

Second order terms

Harmonics
$$\frac{1}{4}\,(2\omega_1)\ \tau_2\left(\frac{\Delta f}{B}\right),\ \frac{1}{4}\,(2\omega_2)\ \tau_2\left(\frac{\Delta f}{B}\right) \tag{102}$$

Cross products
$$\frac{1}{2}\,(\omega_1 + \omega_2)\ \tau_2\left(\frac{\Delta f}{B}\right),\ \frac{1}{2}\,(\omega_1 - \omega_2)\ \tau_2\left(\frac{\Delta f}{B}\right) \tag{103}$$

Third order terms

Harmonics $\frac{1}{12}(3\omega_1)\,\tau_3\left(\frac{\Delta f}{B}\right)^2,\ \frac{1}{12}(3\,\omega_2)\,\tau_3\left(\frac{\Delta f}{B}\right)^2$ (104)

Cross products

$$\frac{1}{4}(2\,\omega_1+\omega_2)\,\tau_3\left(\frac{\Delta f}{B}\right)^2$$

$$\frac{1}{4}(2\,\omega_2+\omega_1)\,\tau_3\left(\frac{\Delta f}{B}\right)^2 \qquad (105)$$

$$\frac{1}{4}(2\,\omega_1-\omega_2)\,\tau_3\left(\frac{\Delta f}{B}\right)^2$$

$$\frac{1}{4}(2\,\omega_2-\omega_1)\,\tau_3\left(\frac{\Delta f}{B}\right)^2$$

The total fundamental output power is

$$P_F=\frac{d^2}{2R}\left[\frac{1}{4}\,\Delta f^2+\frac{1}{4}\,\Delta f^2\right]=\frac{d^2}{4R}\,\Delta f^2$$

while the total second and third order harmonic and intermodulation powers are:

$$P'_{H_2}=t'_2\,(\omega_1^2+\omega_2^2)\,P_F^2 \qquad (106)$$

$$P'_{i_2}=t'_2\left[(\omega_1+\omega_2)^2+(\omega_1-\omega_2)^2\right]P_F^2$$

$$P'_{H_3}=\frac{9}{8}\,t'_3\left[(\omega_1^2+\omega_2^2)\right]P_F^3 \qquad (107)$$

$$P'_{i_3}=\frac{9}{8}\,t'_3\Big[(2\,\omega_1+\omega_2)^2+(2\,\omega_2+\omega_1)^2+(2\,\omega_1-\omega_2)^2+(2\,\omega_2-\omega_1)^2\Big]P_F^3$$

It is clear, from this example, that the intermodulation power is larger than the harmonic power and, that for large numbers of simultaneous tones, the latter is negligible as compared with the former. Furthermore the amplitude of a given harmonic, or intermodulation term, is proportional to its frequency.

Comparison of Amplitude and Phase Distortion

The power of each individual harmonic and intermodulation term, for amplitude distortion, may be obtained, from equations (26) to (29), in terms of the power of each fundamental term P_{F_1}. Thus

$$P_{i_2}=4t_2\,(P_F)_1^2 \qquad (108)$$

$$P_{i_3}=9t_3\,(P_F)_1^3 \qquad (109)$$

$$P_{H_2}=t_2\,(P_F)_1^2 \qquad (110)$$

$$P_{H_3}=t_3\,(P_F)_1^3 \qquad (111)$$

In the case of phase distortion, we have from equations (106) and (107)

$$P'_{i_2} = 4\ \omega^2 t'_2\ (P_F)_1^2 \qquad (112)$$

$$P'_{i_3} = 9\ \omega^2 t'_3\ (P_F)_1^3 \qquad (113)$$

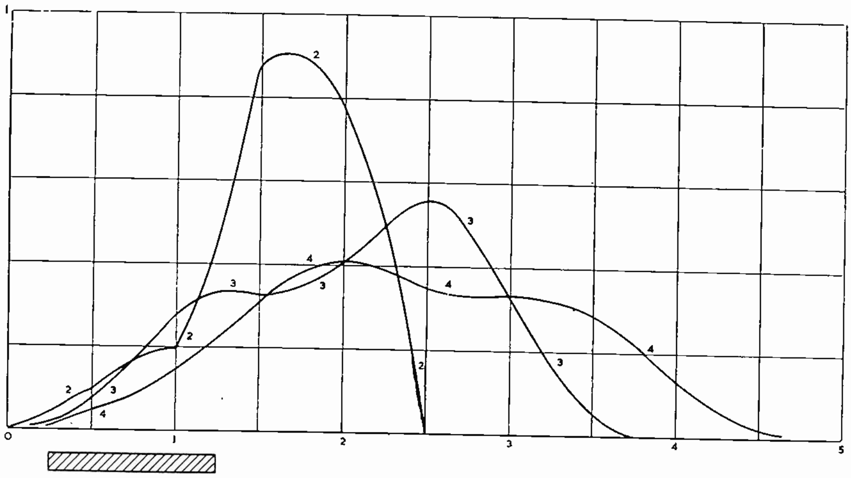

FIG. 8
Intermodulation spectra due to phase distortion $f_2/f_1 = 5$.

In general

$$P'_{i_n} = n^2\ \omega^2 t'_n\ P_F^n \qquad (114)$$

$$P_{H_2} = \omega^2 t'_2\ (P_F)_1^2 \qquad (115)$$

$$P_{H_3} = \omega^2 t'_3\ (P_F)_1^3 \qquad (116)$$

Here ω is the frequency of the actual harmonic or intermodulation term.

It is useful to collect equations (108) to (116) in the following table.

POWERS OF INDIVIDUAL HARMONIC AND INTERMODULATION TERMS ARISING OUT OF r EQUAL TONES

	Second order powers		Third order powers	
	Harmonics	Intermodulation	Harmonics	Intermodulation
Amplitude distortion	$t_2\ (P_F)_1^2$	$4t_2\ (P_F)_1^2$	$t_3\ (P_F)_1^3$	$9t_3\ (P_F)_1^3$
Phase distortion	$\omega^2 t'_2\ (P_F)_1^2$	$4\,\omega^2 t'_2\ (P_F)_1^2$	$\omega^2 t'_3\ (P_F)_1^3$	$9\,\omega^2 t'_3\ (P_F)_1^3$

ω is the frequency of the harmonic or intermodulation term and $(P_F)_1$ is the power of each of the individual r tones.

It is clear from the above table that it is possible to deduce the total distortion power and the distortion spectra, for phase non-linearity, from the corresponding distortion power and spectrum distributions already obtained for amplitude non-linearity, simply by substituting t_n by t'_n and multiplying all terms by ω^2.

Intermodulation Power due to Phase Distortion

The intermodulation spectrum distribution for phase distortion may be obtained from the corresponding amplitude distortion case, by multiplying all ordinates by ω^2. The intermodulation spectra for signal bands extending from 0·25 to 1·25 (60 channel systems) and 0 to 1 (systems with very large numbers of channels) obtained in this way, and normalised to an area of unity, are shown in Figs. 8 and 9.

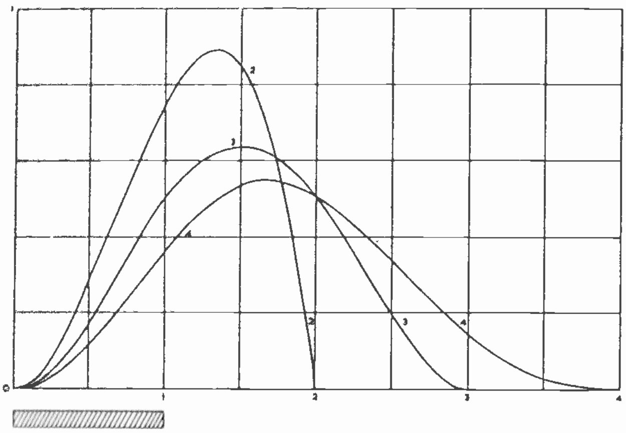

FIG. 9

Intermodulation spectra due to phase distortion $f_2/f_1 = \infty$.

The total intermodulation power for amplitude distortion was shown to be

$$T_n = 2^{n-1}\, n!\, t_n\, P^n \qquad (46)$$

Two corrections are necessary in order to deduce, from this, the total intermodulation power T'_n, due to phase distortion. The one consists in replacing t_n by t'_n and the second in making allowance for the frequency weighting. To an n^{th} order amplitude intermodulation spectrum Φ_n there corresponds an n^{th} order phase intermodulation spectrum Φ'_n where

$$\int \Phi'_n(\omega)\, d\omega = \int \omega^2\, \Phi_n(\omega)\, d\omega. \qquad (117)$$

Let $\int \omega^2\, \Phi_1(\omega)\, d\omega = I$, where $\Phi_1(\omega)$ is the normalised input spectrum.

The n^{th} order normalised spectrum Φ_n, is the distribution of the sum of n independent elements from Φ_1. These n elements are taken all positive, $(n-1)$ positive and one negative, and so on, with various weights proportional to the binomial coefficients.

The picture is simplified if we think of Φ_1 as made of two equal parts, one positive and one negative, symmetrical about zero. Φ_n is then the distribution of n independent elements from this Φ_1 and it is also symmetrical about zero. The weights attached to the different combinations of positive and negative terms are taken into account automatically. For instance in the case of Φ_3, the three independent elements are chosen from the positive and negative halves of Φ_1, with equal likelihood as $+++$, $++-$, $+-+$, $-++$, $+--$, $-+-$, $--+$, $---$ (Fig. 10). These make up the sub-distributions of Φ_3, and when the negative frequencies are treated as positive, the all + and the one −, sub-distributions are seen to have the weights $\frac{1}{4}$ and $\frac{3}{4}$.

The second and third moments about the mean of the distribution obtained by summing n independent members of any distribution are n times the moments of the first distribution. $\int x^2\, \Phi(x)\, dx$ is the second moment.

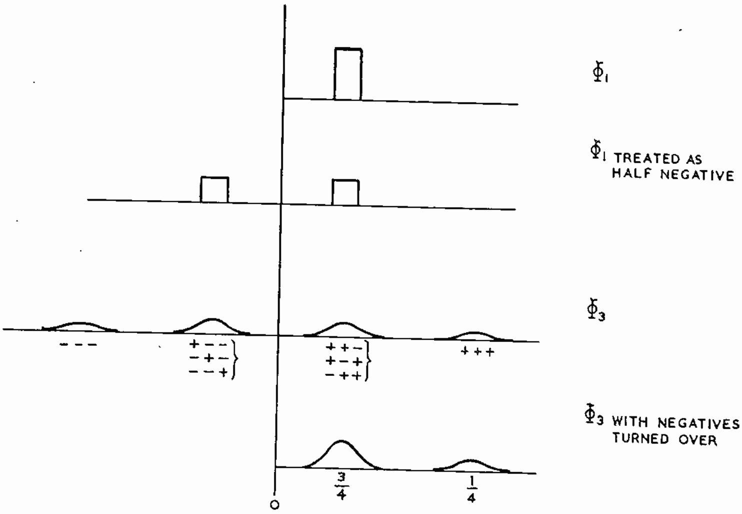

FIG. 10

Derivation of third order intermodulation spectrum by considering positive and negative distributions of input signal.

Therefore.
$$\int \omega^2\, \Phi_n(\omega)\, d\omega = n \int \omega^2\, \Phi_1(\omega)\, d\omega \tag{118}$$

Since ω is squared in the integrals, it makes no difference whether the Φ distribution is all positive, or split up into symmetrical positive and negative components.

It follows that we may write
$$T'_n = 2^{n-1}\, n!\, t'_n\, P^n\, nI \tag{119}$$

$$T'_n = 2^{n-1}\, (n-1)!\, t'_n\, P^n\, In^2 \tag{120}$$

If the input spectrum extends from zero to unity
$$\int_0^1 \omega^2\, \Phi_1(\omega)\, d\omega = I = \frac{1}{3} \tag{121}$$

and for this case
$$T'_n = \frac{2^{n-1}\, (n-1)!\, n^2\, t'_n\, P^n}{3} \tag{122}$$

It may be verified that equation (118) holds for a continuous spectrum from 0 to unity.

$$\text{If } \Phi_1 = \frac{1}{\lambda} \text{ with } 0 < \omega < \lambda$$

$$\int_0^{\lambda} \frac{d\omega}{\lambda} = 1. \qquad I = \int_0^{\lambda} \frac{\omega^2\, d\omega}{\lambda} = \frac{\lambda^2}{3}$$

We have seen that

$$\Phi_2 = \frac{1}{\lambda}\left(1 - \frac{\omega}{2\lambda}\right) \qquad 0 < \omega < 2\lambda$$

Then

$$\int_0^{2\lambda} \Phi_2(\omega)\, d\omega = 1.$$

$$\int_0^{2\lambda} \frac{\omega^2}{\lambda}\left(1 - \frac{\omega}{2\lambda}\right) d\omega = \frac{2}{3}\lambda^2 = 2I.$$

The amount of intermodulation power falling within a channel bandwidth b is

$$d'_{\text{n}} = \frac{b}{B} T'_{\text{n}} F'_{\text{n}}(\omega) \tag{123}$$

$$d'_{\text{n}} = \frac{T'_{\text{n}} F'_{\text{n}}(\omega)}{N} \tag{124}$$

where N is the number of channels and $F'(\omega)$ is the spectrum amplitude at the frequency of the selected channel, which may be obtained from Figs. 8 and 9 for the particular system used.

INTERMODULATION DISTORTION DUE TO WEAK ECHOES

General

The distortion of frequency modulated waves, caused by mismatched feeders, has been investigated by Lewin [7,8]. He considers first the case of a single tone frequency modulated carrier, which is transmitted along a feeder, mismatched at both ends, the reflection coefficients of the generator and load being r_1 and r_2 and the feeder attenuation α.

Let the modulation frequency be ω_{f} and the carrier frequency ω_{c} and consider the case of a feeder, where in the matched condition, the group velocity is independent of frequency, i.e. the feeder is not dispersive. This of course excludes the case of waveguide feeders, which are generally dispersive.

The output wave is considered to be made up of the incident wave, and a secondary wave, which has suffered two reflections, one at the aerial end and the other at the generator end. All other waves, caused by multiple reflections, are sufficiently small to be neglected.

The incident modulated carrier is

$$\cos\left(\omega_{\text{c}} t - \frac{\Delta'\omega}{\omega_{\text{f}}} \cos \omega_{\text{f}} t\right)$$

which is written as

$$V_{\text{i}} = \cos\left[\omega_{\text{c}} t + s(t)\right]$$

The secondary wave is delayed by $2\tau = 2l/v$, with respect to the primary wave, so that the signal, effectively transmitted is

$$S(t) = \cos\left[\omega_c t + s(t)\right] + r_1 r_2 e^{-2\alpha l} \cos\left[\omega_c (t - 2\tau) + s(t - 2\tau) + \theta_1 + \theta_2\right]$$

where θ_1 and θ_2 are the phase changes caused by reflection.

If $s(t - 2\tau) = s(t)$, there is no distortion, as the two interfering waves have the same instantaneous frequency, and the effect of the secondary reflection is only to alter the magnitude of the carrier by a small amount depending on r_1, r_2, θ_1 and θ_2. In general this does not apply, and if we write $u = r_1 r_2 e^{-2\alpha l}$ we have

$$S = \cos\left[\omega_c t + s(t)\right] + u \cos\left[\omega_c (t - 2\tau) + s(t - 2\tau) + \theta_1 + \theta_2\right] \tag{125}$$

$$= \cos\left[\omega_c t + s(t)\right] + u \cos\left[\left(\omega_c t + s(t)\right) - \left(2\omega_c\tau + s(t) - s(t - 2\tau) - \theta_1 - \theta_2\right)\right]$$

$$= \cos\left[\omega_c t + s(t)\right] + u \cos\left[\omega_c t + s(t)\right] \cos\left[2\omega_c\tau + s(t) - s(t - 2\tau) - \theta_1 - \theta_2\right]$$
$$+ u \sin\left[\omega_c t + s(t)\right] \sin\left[2\omega_c\tau + s(t) - s(t - 2\tau) - \theta_1 - \theta_2\right]$$

$$= \cos\left[\omega_c t + s(t)\right] \left[1 + u \cos\left(2\omega_c\tau + s(t) - s(t - 2\tau) - \theta_1 - \theta_2\right)\right]$$
$$+ \sin\left[\omega_c t + s(t)\right] u \sin\left(2\omega_c\tau + s(t) - s(t - 2\tau) - \theta_1 - \theta_2\right)$$

$$= A \cos\left[\omega_c t + s(t)\right] + B \sin\left[\omega_c t + s(t)\right] \tag{126}$$

where

$$A = 1 + u \cos\left[2\omega_c\tau + s(t) - s(t - 2\tau) - \theta_1 - \theta_2\right] \tag{127}$$

$$B = u \sin\left[2\omega_c\tau + s(t) - s(t - 2\tau) - \theta_1 - \theta_2 .\right] \tag{128}$$

S may now be rewritten as follows:

$$S = \sqrt{A^2 + B^2} \cos\left[\omega_c t + s(t) - \phi\right] \text{ where } \phi = \tan^{-1}\frac{B}{A} \tag{129}$$

If u is very small then $\phi \fallingdotseq B$.

The amplitude variation of the carrier signal, as given by $\sqrt{A^2 + B^2}$ is removed by the limiters and the recovered signal is proportional to the instantaneous frequency of S.

$$\text{Thus } V_0 = \frac{d}{dt}\left[\omega_c t + s(t) - \phi\right] = \omega_c + s'(t) - \frac{d\phi}{dt} \tag{130}$$

$s'(t)$ is the recovered input signal, while $\frac{d\phi}{dt}$ represents its distortion.

The output modulation is thus

$$g_0(t) = \Delta'\omega \sin \omega_f t - \frac{d}{dt}\left[u \sin (2\omega_c\tau + s(t) - s(t-2\tau) - \theta_1 - \theta_2)\right] \quad (131)$$

We may write

$$s(t) - s(t-2\tau) = -\frac{\Delta'\omega}{\omega_f}\left[\cos \omega_f t - \cos (\omega_f t - 2\omega_f \tau)\right] \quad (132)$$

$$= \frac{2\Delta'\omega}{\omega_f} \sin \omega_f (t-\tau) \sin \omega_f \tau$$

$$= 2\Delta'\omega\tau \left(\frac{\sin \omega_f \tau}{\omega_f \tau}\right) \sin \omega_f (t-\tau) \quad (133)$$

The quantity $2\Delta'\omega\tau \dfrac{\sin \omega_f\tau}{\omega_f\tau}$ is termed y.

Thus the full output signal of the system is

$$g_0(t) = \Delta'\omega \sin \omega_f t - \frac{d}{dt}\left[u \sin \left(2\omega_c\tau - \theta_1 - \theta_2 + y \sin \omega_f (t-\tau)\right)\right] \quad (134)$$

The quantity in brackets may be expanded into

$$\sin\left(2\omega_c\tau - \theta_1 - \theta_2 + y \sin \omega_f (t-\tau)\right) = \sin\left(2\omega_c\tau - \theta_1 - \theta_2\right)\cos\left[y \sin \omega_f (t-\tau)\right]$$

$$+ \cos\left(2\omega_c\tau - \theta_1 - \theta_2\right) \sin\left[y \sin \omega_f (t-\tau)\right]$$

The right hand side may be expanded in a Fourier/Bessel series thus:

$$\cos\left[y \sin \omega_f (t-\tau)\right] = J_0(y) + 2J_2(y) \cos 2\omega_f (t-\tau) + 2J_4(y) \cos 4\omega_f (t-\tau) \ \ldots$$

$$\sin\left[y \sin \omega_f (t-\tau)\right] = 2J_1(y) \sin \omega_f (t-\tau) + 2J_3(y) \sin 3\omega_f (t-\tau) + \ldots$$

It follows that, if we put $2\omega_c\tau - \theta_1 - \theta_2 = \theta$, the output signal is

$$g_0(t) = \Delta'\omega \sin \omega_f t + u \sin \theta \frac{d}{dt}\left[J_0(y) + 2J_2(y) \cos 2\omega_f(t-\tau) + 2J_4(y) \cos 4\,\omega_f(t-\tau) + \ldots\right]$$

$$\ldots + u \cos \theta \frac{d}{dt}\left[2J_1(y) \sin \omega_f(t-\tau) + 2J_3(y) \sin 3\omega_f(t-\tau) + \ldots\right] \quad (135)$$

After differentiating we find

$$g_0(t) = \Delta'\omega \sin \omega_f t - 2u \sin \theta \left[2\omega_f J_2(y) \sin 2\omega_f(t-\tau) + 4\omega_f J_4(y) \sin 4\omega_f(t-\tau) \ldots\right.$$

$$+ 2u \cos \theta \left[\omega_f J_1(y) \cos \omega_f(t-\tau) + 3\omega_f J_3(y) \cos 3\omega_f(t-\tau)\right.$$

$$\left. + 5\omega_f J_5(y) \cos 5\omega_f(t-\tau) + \ldots\right]$$

$$g_0(t) = \Delta'\omega \sin \omega_f t + \sum_{n=2}^{\infty} 2u(n\,\omega_f)\, J_n(y) \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \begin{bmatrix} \sin \\ \cos \end{bmatrix} n\,\omega_f(t-\tau) \quad (136)$$

The fundamental output power is, using the notation introduced in the case of phase distortion

$$P_F = \frac{d^2}{8\,\pi^2\,R}\,\Delta'\omega^2 \quad (137)$$

The harmonic powers are

$$P''_{H_n} = \frac{d^2}{8\,\pi^2\,R}\,4\,u^2\,n^2\,\omega_f^2\,J_n^2(y) \begin{bmatrix} \sin^2\theta \\ \cos^2\theta \end{bmatrix} \quad (138)$$

and the harmonic margin of n^{th} order, for a peak deviation $\Delta'\omega$ is

$$\frac{P''_{H_n}}{P_F} = \frac{4\,u^2\,n^2\,\omega_f^2\,J_n^2(y)}{\Delta'\omega^2} \begin{bmatrix} \sin^2\theta \\ \cos^2\theta \end{bmatrix} \quad (139)$$

If we assume here that y is small, we may use the approximation

$$J_n(y) = \frac{y^n}{2^n\,n!}$$

so that

$$\frac{y^n}{2^n} = (\Delta'\omega\tau)^n \text{ and } J_n(y) = \frac{(\Delta'\omega\tau)^n}{n!} \quad (140)$$

From equations (15) and (16) we obtain

$$\frac{P''_{H_n}}{P_F} = \frac{4\,u^2\,n^2\,\omega_f^2}{\Delta'\omega^2}\;\frac{(\Delta'\omega\tau)^{2n}}{(n!)^2} \begin{bmatrix} \sin^2\theta \\ \cos^2\theta \end{bmatrix} \quad (141)$$

$$\frac{P''_{H_n}}{P_F} = \frac{4\,u^2\,n^2\,\omega_f^2\,\tau^{2n}}{(n!)^2}\;\Delta'\omega^{2n-2} \begin{bmatrix} \sin^2\theta \\ \cos^2\theta \end{bmatrix} \quad (142)$$

Using equation (137) we find

$$\Delta'\omega^{2n-2} = \left(\frac{8\,\pi^2\,R}{d^2}\right)^{n-1} P_F^{\,n-1}$$

and consequently

$$P''_{H_n} = \frac{4\,u^2\,\tau^{2n}}{(n!)^2}\left(\frac{8\,\pi^2\,R}{d^2}\right)^{n-1} \begin{bmatrix} \sin^2\theta \\ \cos^2\theta \end{bmatrix} n^2\,\omega_f^2\,P_F^{\,n} \quad (143)$$

We may write

$$t''_n = \frac{4\,u^2\,\tau^{2n}}{(n!)^2}\left(\frac{8\,\pi^2\,R}{d^2}\right)^{n-1} \begin{bmatrix} \sin^2\theta \\ \cos^2\theta \end{bmatrix} \quad (144)$$

and consequently

$$P''_{H_n} = n^2\,t''_n\,\omega_f^2\,P_F^{\,n} = t''_n\,\omega^2\,P_F^{\,n} \quad (145)$$

where ω is the frequency of the harmonic term.

It is clear that, provided the approximation of equation (140) is valid, distortion arising on account of weak echoes, with short delays, is very similar to that caused by group delay variations (Cf. equation (100)).

Before proceeding further we shall summarise below the results obtained.

Harmonic margins of single tones

$$\frac{P''_{H_n}}{P_F} = \frac{4\, u^2\, n^2\, \omega_f^{\,2}\, J_n^{\,2}(y)}{\Delta'\omega^2} \begin{bmatrix}\sin^2\theta\\ \cos^2\theta\end{bmatrix} \tag{146}$$

where $s(t)$ = input signal $= \Delta'\omega \sin \omega_f t$

u = echo amplitude $= r_1\, r_2\, e^{-2\alpha l} << 1$

$y = 2\Delta'\omega\tau \dfrac{\sin \omega_f\tau}{\omega_f\tau}$

$\theta = 2\,\omega_c\tau - \theta_1 - \theta_2$

Harmonic powers of single tones (approximate values)

$$P''_{H_n} = n^2\, t''_n\, \omega_f^{\,2}\, P_F^{\,n} = \omega^2\, t''_n\, P_F^{\,n} \tag{147}$$

where

$$t''_n = \frac{4\, u^2\, \tau^{2n}}{(n!)^2} \left(\frac{8\,\pi^2\, R}{d^2}\right)^{n-1} \begin{bmatrix}\sin^2\theta\\ \cos^2\theta\end{bmatrix} \tag{148}$$

The ratio of the second to the first term in $J_n(y)$ is $y^2/4(n+1)$. If this ratio is to be less than 0·1 (so that an error of no more than 10% in the harmonic amplitudes is incurred) we must have

$$y^2 < 0{\cdot}4\ (n+1)$$
$$y < 0{\cdot}63\ \sqrt{n+1}.$$

For second and third order distortion we should then have $2\Delta'\omega\tau \leqslant 1{\cdot}09$ and $2\Delta'\omega\tau \leqslant 1{\cdot}26$ respectively. It follows that for peak deviations of 1 Mc/s, say, the approximate formulae (24 and (25)) are valid for echo delays of up to 100 m μsecs.

If we assume that modulation frequencies of up to 2·5 Mc/s are used, the ratio $\sin \omega_f\tau/\omega_f\tau$ is always between 1 and 0·62, with the result that the approximation, used above, holds better still, especially for the high modulation frequencies.

Intermodulation Terms for Two Tones

We assume that the input carrier is

$$V_1 = \cos\left[\omega_c t - \frac{a}{\omega_a}\cos \omega_a t - \frac{b}{\omega_b}\cos \omega_b t\right]$$

and $\quad s(t) = a \sin \omega_a t + b \sin \omega_b t$

The output signal is, from equation (131)

$$g_0^{\cdot}(t) = a \sin \omega_a t + b \sin \omega_b t - \frac{d}{dt}\left[u \sin\left(\theta + s(t) - s(t-2\tau)\right)\right] \tag{149}$$

where $\theta = 2\omega_c\tau - \theta_1 - \theta_2$

As in equation (132) we may write:

$$s(t) - s(t-2\tau) = y_a \sin \omega_a(t-\tau) + y_b \sin \omega_b\,(t-\tau) \tag{150}$$

where

$$y_a = 2a\tau \frac{\sin \omega_a\tau}{\omega_a\tau}, \quad y_b = 2b\tau \frac{\sin \omega_b\tau}{\omega_b\tau}$$

It follows that

$$g_0(t) = a \sin \omega_a t + b \sin \omega_b t - \frac{d}{dt}\left[u \sin \left(\theta + y_a \sin \omega_a(t-\tau) + y_b \sin \omega_b (t-\tau)\right)\right]$$

The distortion terms may be expanded as follows:

$$- d_0(t) = u \frac{d}{dt}\Big[\sin \theta \cos \left(y_a \sin \omega_a(t-\tau)\right) \cos \left(y_b \sin \omega_b(t-\tau)\right)$$
$$+ \cos \theta \sin \left(y_a \sin \omega_a(t-\tau)\right) \cos \left(y_b \sin \omega_b(t-\tau)\right)$$
$$+ \cos \theta \cos \left(y_a \sin \omega_a(t-\tau)\right) \sin \left(y_b \sin \omega_b(t-\tau)\right)$$
$$- \sin \theta \sin \left(y_a \sin \omega_a(t-\tau)\right) \sin \left(y_b \sin \omega_b(t-\tau)\right)\Big]$$

Each of the above terms may be expanded as the product of two series. Thus the first term expands into:

$$u \frac{d}{dt} \sin \theta \left[J_0(y_a) + 2J_2(y_a) \cos 2\omega_a(t-\tau) + 2J_4(y_a) \cos 4\omega_a(t-\tau) + \ldots\right]$$
$$\times \left[J_0(y_b) + 2J_2(y_b) \cos 2\omega_b (t-\tau) + 2J_4(y_a) \cos 4\omega_b (t-\tau) + \ldots\right]$$

Write, $\omega_a(t-\tau) = \alpha$, $\omega_b(t-\tau) = \beta$, $J_n(y_a) = J_{na}$, $J_n(y_b) = J_{nb}$.

The distortion terms are then:

$$-d_0(t) = u \sin \theta \frac{d}{dt}\Big[\left(J_{oa} + 2J_{2a} \cos 2\alpha + 2J_{4a} \cos 4\alpha + \ldots\right)$$
$$\left(J_{ob} + 2J_{2b} \cos 2\beta + 2J_{4b} \cos 4\beta + \ldots\right)\Big]$$
$$+ u \cos \theta \frac{d}{dt}\Big[\left(2J_{1a} \sin \alpha + 2J_{3a} \sin 3\alpha + \ldots\right)$$
$$\left(J_{ob} + 2J_{2b} \cos 2\beta + 2J_{4b} \cos 4\beta + \ldots\right)\Big]$$
$$+ u \cos \theta \frac{d}{dt}\Big[\left(J_{oa} + 2J_{2a} \cos 2\alpha + 2J_{4a} \cos 4\alpha + \ldots\right)$$
$$\left(2J_{1b} \sin \beta + 2J_{3b} \sin 3\beta + \ldots\right)\Big]$$
$$- u \sin \theta \frac{d}{dt}\Big[\left(2J_{1a} \sin \alpha + 2J_{3a} \sin 3\alpha + \ldots\right)$$
$$\left(2J_{1b} \sin \beta + 2J_{3b} \sin 3\beta + \ldots\right)\Big] \qquad (151)$$

If now we expand this expression and neglect the small d.c. fundamental terms and all terms of order higher than the fourth, we obtain

$$
\begin{aligned}
-d_0(t) = u \sin\theta \frac{d}{dt}\Big[& 2J_{oa}J_{2b}\cos 2\beta + 2J_{oa}J_{4b}\cos 4\beta + 2J_{2a}J_{ob}\cos 2\alpha \\
& + 2J_{2a}J_{2b}\cos(2\alpha - 2\beta) \\
& + 2J_{2a}J_{2b}\cos(2\alpha + 2\beta) \\
& + 2J_{4a}J_{ob}\cos 4\alpha \Big] \\
+ u\cos\theta \frac{d}{dt}\Big[& 2J_{ob}J_{3a}\sin 3\alpha + 2J_{1a}J_{2b}\sin(\alpha + 2\beta) \\
& + 2J_{1a}J_{2b}\sin(\alpha - 2\beta)\Big] \\
+ u\cos\theta \frac{d}{dt}\Big[& 2J_{oa}J_{3b}\sin 3\beta + 2J_{2a}J_{1b}\sin(2\alpha + \beta) \\
& + 2J_{2a}J_{1b}\sin(\beta - 2\alpha)\Big] \\
- u\sin\theta \frac{d}{dt}\Big[& 2J_{1a}J_{1b}\cos(\alpha - \beta) - 2J_{1a}J_{1b}\cos(\alpha + \beta) \\
& + 2J_{1a}J_{3b}\cos(3\beta - \alpha) \\
& - 2J_{1a}J_{3b}\cos(3\beta + \alpha) \\
& + 2J_{3a}J_{1b}\cos(3\alpha - \beta) \\
& - 2J_{3a}J_{1b}\cos(3\alpha + \beta)\Big]
\end{aligned} \qquad (152)
$$

The individual distortion terms may now be tabulated as shown below

Harmonics			*Crossmodulation terms*
2nd *Order*			
$-u\sin\theta$	$2\omega_a$	$2J_{2a}J_{ob}\sin 2\alpha$	$+u\sin\theta\,(\omega_a - \omega_b)\,2J_{1a}J_{1b}\sin(\alpha - \beta)$
$-u\sin\theta$	$2\omega_b$	$2J_{oa}J_{2b}\sin 2\beta$	$-u\sin\theta\,(\omega_a + \omega_b)\,2J_{1a}J_{1b}\sin(\alpha + \beta)$
3rd *Order*			
$u\cos\theta$	$3\omega_a$	$2J_{3a}J_{ob}\cos 3\alpha$	$u\cos\theta\,(\omega_a + 2\omega_b)\,2J_{1a}J_{2b}\cos(\alpha + 2\beta)$
$u\cos\theta$	$3\omega_a$	$2J_{oa}J_{3b}\cos 3\beta$	$u\cos\theta\,(\omega_a - 2\omega_b)\,2J_{1a}J_{2b}\cos(\alpha - 2\beta)$
			$u\cos\theta\,(2\omega_a + \omega_b)\,2J_{2a}J_{1b}\cos(2\alpha + \beta)$
			$u\cos\theta\,(-2\omega_a + \omega_b)\,2J_{2a}J_{1b}\cos(-2\alpha + \beta)$

Harmonics			*Crossmodulation terms*
4th *Order*			
$-u \sin\theta$	$4\omega_a$	$2J_{4a}J_{ob}\sin 4\alpha$	$-u\sin\theta\,(\omega_a+3\omega_b)\,2J_{1a}J_{3b}\sin(3\beta+\alpha)$
$-u \sin\theta$	$4\omega_b$	$2J_{oa}J_{4b}\sin 4\beta$	$+u\sin\theta\,(\omega_a-3\omega_b)\,2J_{1a}J_{3b}\sin(3\beta-\alpha)$
			$-u\sin\theta\,(2\omega_a+2\omega_b)\,2J_{2a}J_{2b}\sin(2\alpha+2\beta)$
			$-u\sin\theta\,(2\omega_a-2\omega_b)\,2J_{2a}J_{2b}\sin(2\alpha-2\beta)$
			$+u\sin\theta\,(3\omega_a-\omega_b)\,2J_{3a}J_{1b}\sin(3\alpha-\beta)$
			$-u\sin\theta\,(3\omega_a+\omega_b)\,2J_{3a}J_{1b}\sin(3\alpha+\beta)$

In general we should have

Harmonic Terms
$$2u\begin{Bmatrix}+\sin\theta\\+\cos\theta\end{Bmatrix} n\,\omega_f\,J_{na}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} n\,\omega_f\,J_{ob} \tag{153}$$

Crossmodulation Terms
$$2u\begin{Bmatrix}-\sin\theta\\+\cos\theta\end{Bmatrix}(n\omega_a\pm m\omega_b)\,J_{na}J_{mb}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(n\,\omega_a\pm m\,\omega_b) \tag{154}$$

If we use the approximation of equation (140) and assume that the tones a and b are equal, then we may write the following generalised equations (neglecting signs).

Harmonics
$$u\begin{Bmatrix}\sin\theta\\ \cos\theta\end{Bmatrix}\omega\frac{y^n}{n!\,2^{n-1}}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\omega(t-\tau) \tag{155}$$

Intermodulation Terms
$$u\begin{Bmatrix}\sin\theta\\ \cos\theta\end{Bmatrix}\omega\frac{y^n}{(n-1)!\,2^{n-1}}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\omega(t-\tau) \tag{156}$$

Note that only terms of frequencies $(n-1)\,\alpha\pm\beta$ and $(n-1)\,\beta\pm\alpha$ are here included and that ω is the frequency of the harmonic, or the intermodulation terms.

We may now extend equation (145) to the case of intermodulation terms, taking into account that each intermodulation term is n times greater than the corresponding harmonic terms of the same frequency, as indicated by equations (155) and (156). Thus

$$P''_n = n^2\,t''_n\,\omega^2\,P_F^{\,2} \tag{157}$$

Equations (147) and (157) are identical with equations (100) and (114), and therefore the total intermodulation power, of n^{th} order, caused by the multichannel signal is

$$T''_n = \frac{2^{n-1}\,(n-1)!\,n^2\,t''_n\,P^n}{3} \tag{158}$$

while the intermodulation power, falling within a small band of width b, is

$$d''_n = \frac{T''_n\,F''_n(\omega)}{N} \tag{159}$$

The spectra $F'(\omega)$ and $F''(\omega)$ are identical, as long as the approximations, made on pages 24 and 26 are valid.

The factor $\begin{Bmatrix}\sin\theta\\ \cos\theta\end{Bmatrix}$ introduces an uncertainty in the determination of the amount of distortion power of all orders. It may be assumed that $\sin\theta=\cos\theta=\frac{\sqrt{2}}{2}$, so that an average condition is obtained.

Intermodulation Power for Signal Input Consisting of n Tones

We may extend equation (149) to the case of an n — tone input signal thus:

$$g_0(t) = a \sin \omega_a t + b \sin \omega_b t + \ldots + n \sin \omega_n t$$

$$- u \frac{d}{dt} \sin \left[\theta + y_a \sin \omega_a (t - \tau) + y_b \sin \omega_b (t - \tau) + \ldots + y_n \sin \omega_n (t - \tau) \right] \qquad (160)$$

the angles $\omega_a(t - \tau)$, $\omega_b(t - \tau)$, ... $\omega_n(t - \tau)$ will be written as α, β ... ν etc. The expansion of $S = \sin \left[\theta + y_a \sin \alpha + y_b \sin \beta + \ldots y_n \sin \nu \right]$ is simplest by De Moivres theorem which gives:

$$S = \frac{1}{j} Im \left[(\cos \theta + j \sin \theta) (\cos \overline{y_a \sin \alpha} + j \sin \overline{y_a \sin \alpha}) \ldots (\cos \overline{y_n \sin \nu} + j \sin \overline{y_n \sin \nu}) \right]$$

$$= \sin \theta \prod \left[\cos \overline{y_a \sin \alpha} \cos \overline{y_b \sin \beta} \ldots \right]$$

this includes the cosines of all angles but θ.

$$+ \cos \theta \sum \prod \left[\sin \overline{y_a \sin \alpha} \,.\, \cos \overline{y_b \sin \beta} \ldots \right]$$

this includes the cosines of all angles but $y_a \sin \alpha$. There are in all $(n - 1)$ cos terms and one sin term in the product.

$$- \sin \theta \sum \prod \left[\sin \overline{y_a \sin \alpha} \,.\, \sin \overline{y_b \sin \beta} \,.\, \cos y_c \sin \gamma \ldots \right]$$

this includes the cosines of all angles but three, one of these is θ. There are $(n - 2)$ cos terms and 2 sin terms in the product.

etc. (161)

We now use the expansions:

$\sin (y_a \sin \alpha) = 2J_1(y_a) \sin \alpha + 2J_3(y_a) \sin 3\alpha + \ldots$ which we write as

$$= 2J_{1a} \sin \alpha + 2J_{3a} \sin 3\alpha + \ldots$$

and $\cos (y_a \sin \alpha) = J_{oa} + 2J_{2a} \cos 2\alpha + 2J_{4a} \cos 4\alpha + \ldots$ (162)

It is seen that each angle α, β etc. can appear only once in any product, so that coherent low order terms, due to high order products, are already accounted for.

First order intermodulation terms come from the second term only of equation (161), where we take the J_1 term of the sine factor, with the J_0 terms of all the cosine factors. Thus we obtain n terms of the form $2J_{1a} \sin \alpha \, J_{ob} J_{oc} \cdots J_{on} \cos \theta$. The contribution to the output signal is obtained by multiplying by $-u$, and differentiating.

Thus

$$\left[g_0(t) \right]_1 = \sum - 2u \cos \theta \, J_{1a} \, \omega_a \cos \alpha \, J_{ob} J_{oc} \cdots J_{on} \qquad (163)$$

and the first order interaction power is

$$p_1 = u^2 \sum J_{1a}^2 J_{ob}^2 J_{oc}^2 \ldots J_{on}^2 \omega_a^2$$

$$= u^2 \left[\omega_a^2 \frac{J_{1a}^2}{J_{oa}^2} + \omega_b^2 \frac{J_{1b}^2}{J_{ob}^2} + \ldots + \omega_n^2 \frac{J_{1n}^2}{J_{on}^2} \right] J_{oa}^2 J_{ob}^2 \ldots J_{on}^2 \qquad (164)$$

p_1 is not the actual first order interaction power, but a quantity proportional to it.

We shall assume here that all the ys are small and the number of terms large. Since the input power is finite, Σy^2 must be finite.

Let then

$$y^2 = \phi(\omega)\, d\omega \qquad (165)$$

and

$$\Sigma y^2 = \int_{\omega_1}^{\omega_2} \phi(\omega)\, d\omega = p \qquad (166)$$

where ω_1 and ω_2 are the limits of the input spectrum

$$\omega_a^2 \frac{J_{1a}^2}{J_{ob}^2} = \omega_a^2 \left(1 - \frac{y_a^2}{4} + \frac{y_a^4}{64}\right)^{-2} \left(\frac{y_a}{2} - \frac{y_a^3}{16}\right)^2$$

$$= \omega_a^2 \left[1 + 0\,(y_a^2)\right] \left[\frac{y_a^2}{4} + 0(y_a^4)\right]$$

$$= \frac{\omega_a^2 y_a^2}{4} \left[1 + 0(y_a^2)\right]$$

$0(y_a^2)$ means terms of the order of y_a^2.

Therefore

$$\sum \frac{\omega_a^2 J_{1a}^2}{J_{oa}^2} = \frac{1}{4} \int_{\omega_1}^{\omega_2} \omega^2 \phi(\omega)\, d\omega + 0(\delta\omega)$$

$$= \frac{1}{4} \int_{\omega_1}^{\omega_2} \omega^2 \phi(\omega)\, d\omega \text{ in the limit.} \qquad (167)$$

The multiplying term may be expanded thus

$$J_{oa}^2 J_{ob}^2 J_{oc}^2 \ldots J_{on}^2 = \left[\left(1 - \frac{y_a^2}{4} + \frac{y_a^4}{64}\right)\left(1 - \frac{y_b^2}{4} + \frac{y_b^4}{64}\right) \ldots \right.$$

$$\left. \left(1 - \frac{y_n^2}{4} + \frac{y_n^4}{64}\right)\right] \qquad (168)$$

since $n = \dfrac{\omega_2 - \omega_1}{\delta\omega}$ it is of the order of $1/\delta\omega$, so that when we expand equation (168) we obtain

$$J_{oa}^2 J_{ob}^2 J_{oc}^2 \ldots J_{on}^2$$

$$= 1 + n \text{ terms like } -\frac{y_a^2}{4}$$

$$+ n \text{ terms like } \frac{y_a^4}{64}$$

$$+ \frac{n(n-1)}{2} \text{ terms like } \frac{y_a^2 y_b^2}{16} \text{ etc.}$$

In the limit $n \to \infty$ and the ys $\to 0$, but the sum of the n y^2s is finite (see (166)). The y^2s then are of the order of $1/n$ and the terms of the expansion give

$$1 + \sum \quad -\frac{y_a^2}{4} \qquad n \times 0\left(\frac{1}{n}\right) \text{ — finite in the limit}$$

$$+\sum \quad \frac{y_a^4}{64} \qquad n \times 0\left(\frac{1}{n^2}\right) \text{ — zero in the limit}$$

$$+\sum\sum \frac{y_a^2\, y_b^2}{16} \qquad 0(n^2) \times 0\left(\frac{1}{n^2}\right) \text{ — finite in the limit}$$

etc.

Thus only the terms formed by multiplying different ys are finite in the limit; all others disappear and we may write

$$J_{oa}^2\, J_{ob}^2 \ldots J_{on}^2 = \left[1 - \frac{1}{4}\Sigma\, y_a^2 + \frac{1}{16}\Sigma\Sigma\, y_a^2 y_b^2 - \frac{1}{64}\Sigma\Sigma\Sigma\, y_a^2 y_b^2 y_c^2 + \ldots\right]^2$$

$$n \text{ terms.} \qquad \frac{n(n-1)}{2} \text{ terms.} \qquad \frac{n(n-1)\,(n-2)}{6} \text{ terms.}$$

$$= \left[1 - \frac{1}{4}p + \frac{1}{16}\frac{p^2}{2!} - \frac{1}{64}\frac{p^3}{3!} + \ldots\right]^2$$

$$= (e^{-p/4})^2$$

$$= e^{-p/2} \tag{169}$$

It follows by (166), (167) and (169) that

$$p_1 = \frac{u^2}{4}\, e^{-p/2} \int_{\omega_1}^{\omega_2} \omega^2\, \phi(\omega)\, d\omega \tag{170}$$

The second order terms come from the first and third terms of equation (161). The first term of equation (161) gives n terms like $2J_{2a} \cos 2\alpha\, J_{ob} \ldots J_{on} \sin\theta$ while the third term gives $\frac{n(n-1)}{n}$ terms like $4J_{1a} \sin\alpha\, J_{1b} \sin\beta\, J_{oc} \ldots J_{on} \sin 0$.

J_{2a} and $J_{1a}\, J_{1b}$ are both of the order of $\delta\omega$ so that the terms containing J_{2a} become insignificant in the limit. This means neglecting harmonic terms compared with intermodulation terms. The terms in $J_{1a}\, J_{1b}$ give

$$-4 \sin\theta\, \Sigma\, J_{1a}\, J_{1b} \cdot J_{oc} \ldots J_{on} \sin\alpha \sin\beta$$

$$= -2 \sin\theta\, \Sigma\, J_{1a}\, J_{1b} \cdot J_{oc} \ldots J_{on} \left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

By differentiating, the contribution to $g_0(t)$ is

$$2\, u \sin\theta\, \Sigma\, J_{1a}\, J_{1b}\, J_{oc} \ldots J_{on} \left[(\omega_a + \omega_b) \sin(\alpha + \beta) - (\omega_a - \omega_b) \sin(\alpha - \beta)\right]$$

and the first order interaction power is

$$p_1 = u^2 \sum J_{1a}^2 J_{ob}^2 J_{oc}^2 \dots J_{on}^2 \omega_a^2$$

$$= u^2 \left[\omega_a^2 \frac{J_{1a}^2}{J_{oa}^2} + \omega_b^2 \frac{J_{1b}^2}{J_{ob}^2} + \dots + \omega_n^2 \frac{J_{1n}^2}{J_{on}^2} \right] J_{oa}^2 J_{ob}^2 \dots J_{on}^2 \tag{164}$$

p_1 is not the actual first order interaction power, but a quantity proportional to it.

We shall assume here that all the ys are small and the number of terms large. Since the input power is finite, Σy^2 must be finite.

Let then

$$y^2 = \phi(\omega)\, d\omega \tag{165}$$

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$$= \omega_a^2 \left[1 + 0\,(y_a^2)\right] \left[\frac{y_a^2}{4} + 0(y_a^4)\right]$$

$$= \frac{\omega_a^2\, y_a^2}{4} \left[1 + 0(y_a^2)\right]$$

$0(y_a^2)$ means terms of the order of y_a^2.

Therefore

$$\sum \frac{\omega_a^2 J_{1a}^2}{J_{oa}^2} = \frac{1}{4} \int_{\omega_1}^{\omega_2} \omega^2 \phi(\omega)\, d\omega + 0(\delta\omega)$$

$$= \frac{1}{4} \int_{\omega_1}^{\omega_2} \omega^2 \phi(\omega)\, d\omega \text{ in the limit.} \tag{167}$$

The multiplying term may be expanded thus

$$J_{oa}^2 J_{ob}^2 J_{oc}^2 \dots J_{on}^2 = \left[\left(1 - \frac{y_a^2}{4} + \frac{y_a^4}{64}\right)\left(1 - \frac{y_b^2}{4} + \frac{y_b^4}{64}\right) \dots \left(1 - \frac{y_n^2}{4} + \frac{y_n^4}{64}\right)\right] \tag{168}$$

since $n = \dfrac{\omega_2 - \omega_1}{\delta\omega}$ it is of the order of $1/\delta\omega$, so that when we expand equation (168) we obtain

$$J_{oa}^2 J_{ob}^2 J_{oc}^2 \dots J_{on}^2$$

$$= 1 + n \text{ terms like } -\frac{y_a^2}{4}$$

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$$+\, \frac{n(n-1)}{2} \text{ terms like } \frac{y_a^2 y_b^2}{16} \text{ etc.}$$

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$$1 + \sum \quad -\frac{y_a^2}{4} \qquad n \times 0\left(\frac{1}{n}\right) \text{ — finite in the limit}$$

$$+ \sum \quad \frac{y_a^4}{64} \qquad n \times 0\left(\frac{1}{n^2}\right) \text{ — zero in the limit}$$

$$+ \sum\sum \frac{y_a^2\, y_b^2}{16} \qquad 0(n^2) \times 0\left(\frac{1}{n^2}\right) \text{ — finite in the limit}$$

etc.

Thus only the terms formed by multiplying different ys are finite in the limit; all others disappear and we may write

$$J_{oa}^2 J_{ob}^2 \ldots J_{on}^2 = \left[1 - \frac{1}{4}\Sigma\, y_a^2 + \frac{1}{16}\Sigma\Sigma\, y_a^2 y_b^2 - \frac{1}{64}\Sigma\Sigma\Sigma\, y_a^2 y_b^2 y_c^2 + \ldots\right]^2$$

$$n \text{ terms.} \qquad \frac{n(n-1)}{2} \text{ terms.} \qquad \frac{n(n-1)(n-2)}{6} \text{ terms.}$$

$$= \left[1 - \frac{1}{4}p + \frac{1}{16}\frac{p^2}{2!} - \frac{1}{64}\frac{p^3}{3!} + \ldots\right]^2$$

$$= (e^{-p/4})^2$$

$$= e^{-p/2} \tag{169}$$

It follows by (166), (167) and (169) that

$$p_1 = \frac{u^2}{4}\, e^{-p/2} \int_{\omega_1}^{\omega_2} \omega^2\, \phi(\omega)\, d\omega \tag{170}$$

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J_{2a} and $J_{1a}\, J_{1b}$ are both of the order of $\delta\omega$ so that the terms containing J_{2a} become insignificant in the limit. This means neglecting harmonic terms compared with intermodulation terms. The terms in $J_{1a}\, J_{1b}$ give

$$-4 \sin\theta\, \Sigma\, J_{1a}\, J_{1b} \cdot J_{oc} \ldots J_{on} \sin\alpha \sin\beta$$

$$= -2 \sin\theta\, \Sigma\, J_{1a}\, J_{1b} \cdot J_{oc} \ldots J_{on} \left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

By differentiating, the contribution to $g_0(t)$ is

$$2\, u \sin\theta\, \Sigma\, J_{1a}\, J_{1b}\, J_{oc} \ldots J_{on} \left[(\omega_a + \omega_b)\sin(\alpha + \beta) - (\omega_a - \omega_b)\sin(\alpha - \beta)\right]$$

and the second order power is

$$2\,u^2\,\Sigma\,J_{1a}^{\ 2}\,J_{1b}^{\ 2}\,J_{oc}^{\ 2}\dots J_{on}^{\ 2}\,(\omega_a^{\ 2}+\omega_b^{\ 2})$$

The term in the second order power multiplying $\omega_a^{\ 2}$ is

$$2\,u^2\,J_{1a}^{\ 2}\left(\frac{J_{1b}^{\ 2}}{J_{ob}^{\ 2}}+\frac{J_{1c}^{\ 2}}{J_{oc}^{\ 2}}+\ \dots\ +\frac{J_{1n}^{\ 2}}{J_{on}^{\ 2}}\right)J_{ob}^{\ 2}\,J_{oc}^{\ 2}\dots J_{on}^{\ 2}$$

$\dfrac{J_{1a}^{\ 2}}{J_{oa}^{\ 2}}$ is of the order of $\delta\omega$ and $J_{oa}^{\ 2}=1+0(\delta\omega)$ so that the above term may be written.

$$2\,u^2\,J_{1a}^{\ 2}\left(\frac{J_{1a}^{\ 2}}{J_{oa}^{\ 2}}+\frac{J_{1b}^{\ 2}}{J_{ob}^{\ 2}}+\dots+\frac{J_{1n}^{\ 2}}{J_{on}^{\ 2}}\right)J_{oa}^{\ 2}\,J_{ob}^{\ 2}\dots J_{on}^{\ 2}$$

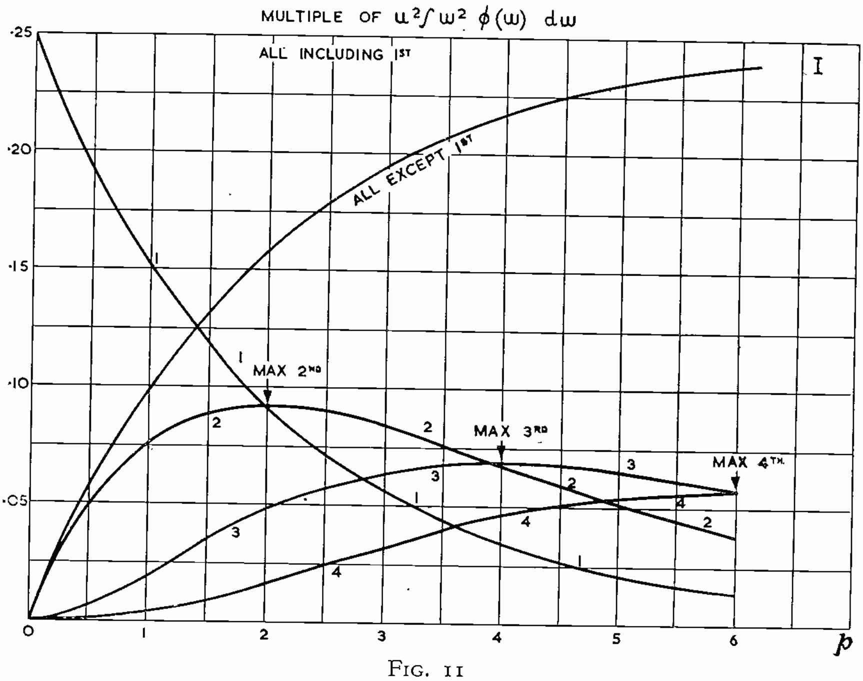

FIG. 11

Total intermodulation power of all orders due to weak echoes.

The total second order power is therefore

$$\begin{aligned} p_2 &= 2\,u^2\,\Sigma\,J_{1a}^{\ 2}\,\omega_a^{\ 2}\left(\frac{1}{4}\sum y_a^{\ 2}+0(\delta\omega)\right)e^{-p/2} \\ &= \frac{1}{2}\,u^2\,p\,e^{-p/2}\,\Sigma\,J_{1a}^{\ 2}\,\omega_a^{\ 2} \\ &= \frac{1}{8}\,u^2\,p\,e^{-p/2}\int_{\omega_1}^{\omega_2}\omega^2\,\phi(\omega)\,d\omega \end{aligned} \qquad (171)$$

In exactly the same way the total third order power is

$$p_3=\frac{u^2}{32}\,p^2\,e^{-p/2}\int_{\omega_1}^{\omega_2}\omega^2\,\phi(\omega)\,d\omega \qquad (172)$$

The total intermodulation power of all orders is therefore

$$p'_{\mathrm{i}} = \frac{u^2}{4} \int_{\omega_1}^{\omega_2} \omega^2 \, \phi(\omega) \, d\omega \; e^{-p/2} \left[1 + \frac{p}{2} + \frac{1}{2!}\left(\frac{p}{2}\right)^2 + \frac{1}{3!}\left(\frac{p}{2}\right)^3 + \ldots \right]$$

$$= \frac{u^2}{4} \int_{\omega_1}^{\omega_2} \omega^2 \, \phi(\omega) \, d\omega \qquad (173)$$

The total intermodulation power excluding first order power, which does not count as distortion, since it cannot be distinguished from the signal itself, is

$$p_{\mathrm{i}} = \frac{u^2}{4} \left(1 - e^{-p/2} \right) \int_{\omega_1}^{\omega_2} \omega^2 \, \phi(\omega) \, d\omega \qquad (174)$$

Fig. 11 shows how the powers of successively higher orders become dominant as p increases. If we assume here that the input spectrum extends from 0 to 1, we have

$$\int_0^1 \phi(\omega) \, d\omega = p, \qquad \int_0^1 \omega^2 \, \phi(\omega) \, d\omega = p/3 \qquad (175)$$

and the total n^{th} order intermodulation power is

$$p_{\mathrm{n}} = \frac{u^2}{4} \frac{p}{3} e^{-p/2} \frac{1}{(n-1)!} \left(\frac{p}{2} \right)^{n-1} \qquad (176)$$

The n^{th} order spectrum distribution of intermodulation powers, for amplitude non-linearity, may be obtained, at one step, by treating the signal spectrum as extending from $\omega = -1$ to $\omega = +1$, with a uniform amplitude of 0·5. This spectrum has a mean of zero and variance of 1/3. The complete n^{th} order spectrum is the distribution of the sum of n independent elements, picked out at random, from the signal distribution. It has thus a mean of zero and variance $n/3$, and its form quickly approaches the Gaussian as n increases. The maximum of the Gaussian curve is $1/\sqrt{2\pi n/3}$, so that when the negative area is turned over on to the positive area, the maximum is $\sqrt{6/n\pi}$.

The value of the spectrum at $\omega = 1$ is the ordinate of the Gaussian curve at abcissa $\sigma\sqrt{3/n}$, where σ is the standard deviation. Thus:

$$F_{\mathrm{n}}(1) = \sqrt{6/n\pi} \; e^{-3/2n} \qquad (177)$$

From (177) we find $F_2(1) = 0{\cdot}462$, $F_3(1) = 0{\cdot}484$, $F_4(1) = 0{\cdot}475$, which are quite close to the values 0·50, 0·50 and 0·479, respectively, obtained from Fig. 7.

The intermodulation spectra, for echo distortion, must be weighted according to ω^2. When this is done we obtain

$$F'_{\mathrm{n}}(1) = \frac{3}{n} \sqrt{6/n\pi} \; e^{-3/2n} \qquad (178)$$

The total echo distortion power of all orders but the first, at the top of the signal band, and within a channel bandwidth is, from (176) and (178)

$$p_{\mathrm{e}} = \frac{\omega^2}{N} \frac{u^2}{4} \frac{p}{3} e^{-p/2} \left[\frac{p}{2} \frac{3}{2} \sqrt{\frac{3}{\pi}} \, e^{-3/4} + \frac{1}{2} \frac{p^2}{4} \sqrt{\frac{2}{\pi}} \, e^{-1/2} + \ldots \right] \qquad (179)$$

$$p_e = \frac{\omega^2 u^2}{4N} \Phi(\tau\Delta\omega) \tag{180}$$

where $\Phi(\tau\Delta\omega)$ is the quantity containing p in equation (179). The quantity $\sqrt{p_e}$ is in the nature of an angular deviation while p is a numerical quantity, proportional to the r.m.s. deviation $\Delta\omega$ of the input signal.

From equation (175)

$$p = \int_0^1 \phi(\omega)\, d\omega = \int_0^1 y^2$$

However $y^2 = 4\delta'\omega^2\tau^2 \left(\frac{\sin\,\omega\tau}{\omega\tau}\right)^2$ where $\delta'\omega$ is the peak deviation due to an individual tone, ω its frequency and 2τ the echo delay. If we neglect the fraction $\frac{\sin\,\omega\tau}{\omega\tau}$, we have

$$p = 4\tau^2\Delta'\omega^2 = 8\tau^2\Delta\omega^2 \tag{181}$$

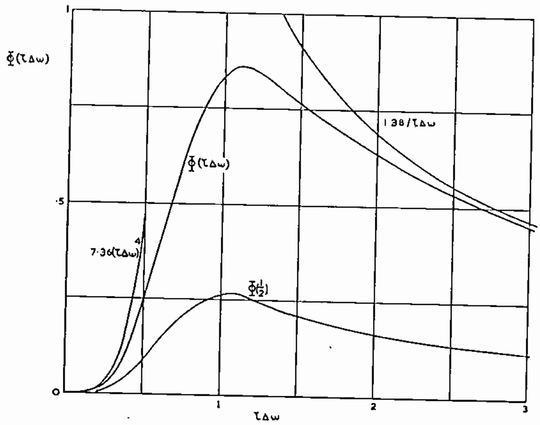

FIG. 12

Intermodulation power in the top channel and in the middle channel due to weak echoes.

The ratio of intermodulation power in a channel of frequency ω, to the total signal power is

$$\frac{p_e}{\Delta\omega^2} = \frac{u^2}{4N}\,\frac{\omega^2}{\Delta\omega^2}\,\Phi(\tau\Delta\omega) \tag{182}$$

The ratio may be expressed, according to the previous notation as d''/P where P is the total fundamental output power, in mW. Therefore

$$d'' = \frac{u^2}{4}\left(\frac{f}{\Delta f}\right)^2 \frac{P}{N}\Phi(\tau\Delta\omega) \tag{183}$$

The quantity $\Phi\,(\tau\Delta\omega)$ is, from equation (179) for the top channel

$$\Phi(\tau\Delta\omega) = \frac{p}{3} e^{-p/2} \left[\frac{3p\sqrt{3}}{4\sqrt{\pi}} e^{-3/4} + \frac{p^2\sqrt{2}}{8\sqrt{\pi}} e^{-1/2} + \dots\right] \tag{184}$$

where p is given by equation (181). $\Phi(\tau\Delta\omega)$ is plotted in Fig. 12 and it agrees with the curve for $\phi(x)$ given by Lewin.

DISTORTION FREE ZONES

General

The two-tone testing method, explained on page 152 of Part I, using two separate modulators, affords a useful means of obtaining the distortion figure of a demodulator, which is independent to a very large degree on the distortion generated by the modulators.

The use of discrete tones is, however, not completely representative or trouble free and it would be preferable to use noise signals for this test, if it can be shown that a distortion figure for the demodulator may be obtained which is independent, in a large measure, of the distortion produced by the modulators.

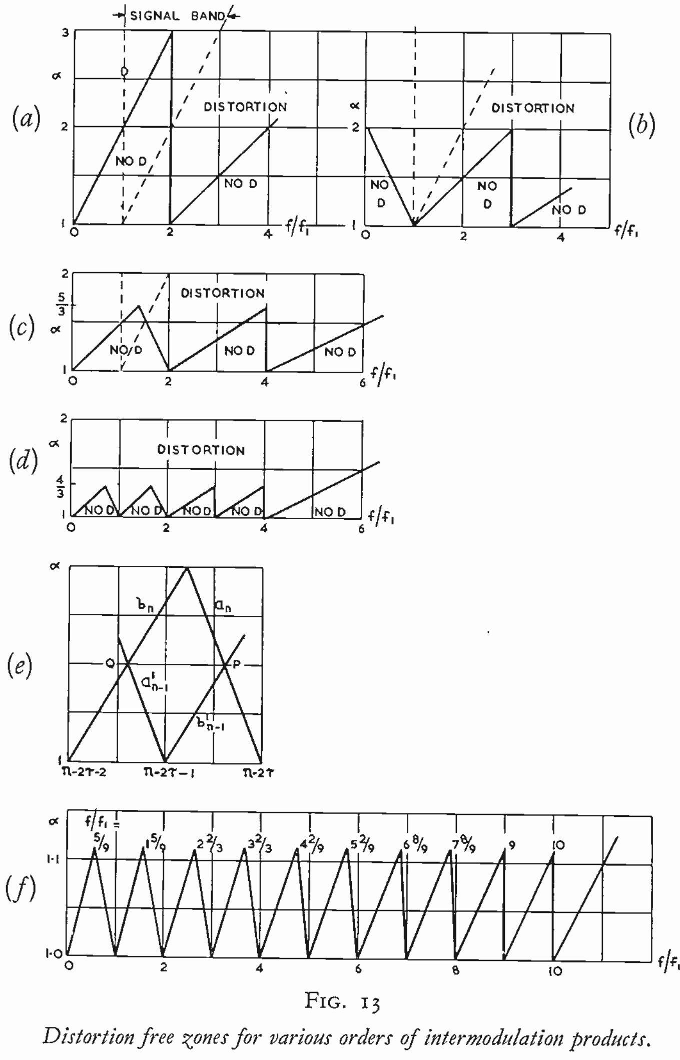

FIG. 13

Distortion free zones for various orders of intermodulation products.

Intermodulation Spectra

If the signal input extends continuously between the frequencies f_1 and f_2 ($f_2 > f_1$), the frequencies at which intermodulation products of a given order are found, are independent of the shape of the input spectrum, although the amount of intermodulation power, at any selected frequency, depends very strongly on it.

Second order intermodulation spectrum

The ++ contribution extends from $2f_1$ to $2f_2$ while the +− extends from $f_1 - f_2$ to $f_2 - f_1$, or folding back the negative frequencies on the positive axis, from zero to $f_2 - f_1$.

These bands overlap if $f_2/f_1 = \alpha > 3$. Therefore if $\alpha < 3$, the intermodulation spectrum extends from zero to $f_2 - f_1$ and $2f_1$ to $2f_2$, while if $\alpha > 3$, it extends from zero to $2f_2$.

This is shown in Fig. 13a where the distortion free zones are plotted as a function of α.

Third order intermodulation products

The third order intermodulation spectrum is distributed as follows:

+++ terms	$3f_1$ to $3f_2$
++− terms	$2f_1 - f_2$ to $2f_2 - f_1$

thus, for $\alpha < 2$ the spectrum extends from $2f_1 - f_2$ to $2f_2 - f_1$ and from $3f_1$ to $3f_2$ and for $\alpha > 2$ the spectrum extends from 0 to $3f_2$ (Fig. 13b).

Fourth order intermodulation products

It is easily seen that, for

$\alpha < 5/3$ the spectrum covers 0 to $2f_2 - 2f_1$, $3f_1 - f_2$ to $3f_2 - f_1$ and $4f_1$ to $4f_2$. and for $\alpha > 5/3$ the spectrum covers 0 to $4f_2$. (Fig. 13c).

All orders up to the fourth

If we combine Figs. 13a, 13b and 13c, we obtain Fig. 13d, in which are plotted the zones free of all intermodulation products, up to the fourth order, in terms of α.

The maximum value of α, for which distortion free zones exist, is 4/3, excluding the frequency band above $4f_2$.

General Case—n^{th} Order Distortion

The n^{th}-order distortion will be found in the zones nf_1 to nf_2, $(n-1)f_1 - f_2$ to $(n-1)f_2 - f_1$, etc. down to the band $\frac{n+1}{2}f_1 - \frac{n-1}{2}f_2$ to $\frac{n+1}{2}f_2 - \frac{n-1}{2}f_1$ for n odd or $\frac{n}{2}f_1 - \frac{n}{2}f_2$ to $\frac{n}{2}f_2 - \frac{n}{2}f_1$ for n even, i.e. 0 to $\frac{n}{2}\left(f_2 - f_1\right)$. This last zone is symmetrical about zero frequency.

Neighbouring zones will overlap, and the lowest zone, with n odd, will extend to zero if

$$\alpha > \frac{n+1}{n-1}$$

Thus the n^{th} order intermodulation spectrum has the following distribution:

n even.	$\alpha < \frac{n+1}{n-1}$	Distortion in $n/2$ zones of width $n(f_2 - f_1)$ and one zone from 0 to $\frac{n}{2}(f_2 - f_1)$. No distortion in $n/2$ zones of width $(n+1)f_1 - (n-1)f_2$.
	$\alpha > \frac{n+1}{n-1}$	Distortion from zero to nf_2.

n odd. $\alpha < \dfrac{n+1}{n-1}$ Distortion in $\dfrac{n+1}{2}$ zones of width $n(f_2 - f_1)$.

No distortion from 0 to $\dfrac{n+1}{2} f_2 - \dfrac{n-1}{2} f_2$ and in $\dfrac{n-1}{2}$ zones of width $(n+1)f_1 - (n-1)f_2$.

$\alpha > \dfrac{n+1}{n-1}$ Distortion from zero to nf_2.

Intermodulation Spectrum for all Orders up to the n^{th}

Consider the n^{th} and $(n-1)^{th}$ orders only. Let $f/f_1 = x$ (Fig. 13e). In the frequency range $n - 2t - 2 < x < n - 2t$, a n^{th} order distortion zone starts at $f = (n-t) f_1 - tf_2$, i.e. $x = n - t - t\alpha$ (line a_n) and ends at $f = (n-t-1)f_2 - (t+1)f_1$, i.e. $x = (n-t-1)\alpha - (t+1)$ (line b_n) provided $\alpha < \dfrac{n+1}{n-1}$.

Maximum values of α, for the occurrence of distortion free zones, up to the n^{th} order, will occur at points such as P where a_n and b'_{n-1} meet, or Q where b_n and a'_{n-1} meet.

We may write

b'_{n-1} is $x = (n-t-1)\alpha - t$ and the co-ordinates of P are

$$x = n - 2t - \frac{t}{n-1}, \quad \alpha = \frac{n}{n-1}$$

a'_{n-1} is $x = (n-t-1) - t\alpha$ and the co-ordinates of Q are

$$x = n - 2t - 1 - \frac{t}{n-1}, \quad \alpha = \frac{n}{n-1}$$

Thus $\alpha = \dfrac{n}{n-1}$ is the maximum value of α giving distortion free zones, up to the n^{th} (4/3 for orders up to the fourth).

The frequencies which are just clear of distortion are given below for $n = 4$ and $\alpha = 4/3$

For $t = 0$, $x = 4$ for P

$x = 3$ for Q

For $t = 1$, $x = 2 - \dfrac{1}{3} = 5/3$ for P

$x = 1 - \dfrac{1}{3} = 2/3$ for Q.

Points of maximum α are thus separated by frequency intervals of f_1 and $\dfrac{n}{n-1} f_1$ alternately.

Fig. 13f gives the distortion free zones, for all orders up to the 10^{th}.

Overlapping Distortion Free Zones for Two Signal Bands

In order to measure the distortion produced by a receiver demodulator and to obtain a figure which is independent of the modulator distortion up to the fourth order, say, we choose the signal bands A and B of each modulator in the following way.

A and B have distortion free zones which overlap when these signals are applied separately to two modulators.

A and B together, considered as a single signal, have second, third and fourth order intermodulation products at frequencies falling within the distortion free bands of the signals A and B, applied separately. This then enables us to examine the output distortion without interference from the input distortion of A and B (Fig. 14).

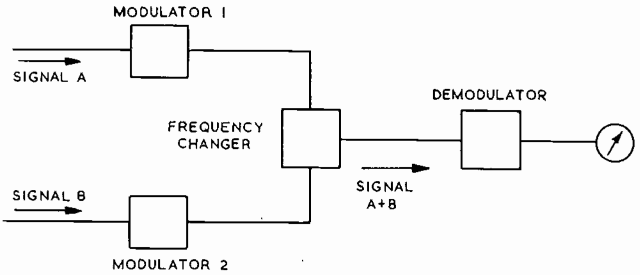

FIG. 14

Two band noise test on demodulator.

If we take the bandwidths of signals A and B to be equal, a graphical treatment of this problem is possible. Let A and B extend from a to $a + \lambda$, and b to $b + \lambda$, respectively, where $b > a + \lambda$. As was shown on page 38 λ should be smaller than $a/3$, if free zones are to exist for signal A. If they do, they also exist for signal B.

Fig. 15a shows the distortion free zones for A and B separately, for $a = 10$, $\lambda = 2$ and b from 12 to 26. The distortion zones of A are:

3rd order: $a - \lambda$ to $a + 2\lambda$, $3a$ to $3a + 3\lambda$

4th order: 0 to 2λ, $2a - \lambda$ to $2a + 3\lambda$ and $4a$ to $4a + 4\lambda$.

These appear in Fig. 15a as white horizontal bands. Second order distortion zones are covered by the fourth order and therefore need not be considered separately.

The distortion zones of B are given by similar expressions in b and appear as oblique bands. Distortion free zones common to both signals appear in the shaded areas.

When A and B are taken together, intermodulation terms may be generated in a variety of ways, but their enumeration is simplified when A and B have the same width. When this is the case each mode of interaction generates a sub-spectrum of the same width and the same shape, for uniform inputs, as can be seen at once by a transformation, such as that used for the determination of intermodulation spectra for a " white noise " input.

Second order terms A + A give frequencies $2a$ to $2a + 2\lambda$

Second order terms A − A give frequencies $-\lambda$ (or 0) to $+\lambda$

Second order terms A + B give frequencies $a + b$ to $a + b + 2\lambda$

Second order terms A − B give frequencies $a - b - \lambda$ to $a - b + \lambda$

Second order terms B + B give frequencies $2b$ to $2b + 2\lambda$

Second order terms B − B give frequencies $-\lambda$ (or 0) to $+\lambda$

In finding the zones with 2nd, 3rd and 4th order terms in Fig. 15b, only the zones involving A and B together are needed. Those with A or B alone are useless for our purpose, since they also give interaction with A or B singly.

The third order interaction terms involving A and B together are as follows:

A + A + B from $(2a + b)$ to $(2a + b + 3\lambda)$

A + A − B from $(2a - b - \lambda)$ to $(2a - b + 2\lambda)$

A + B + B from $(a + 2b)$ to $(a + 2b + 3\lambda)$

B + B − A from $(2b - a - \lambda)$ to $(2b - a + 2\lambda)$

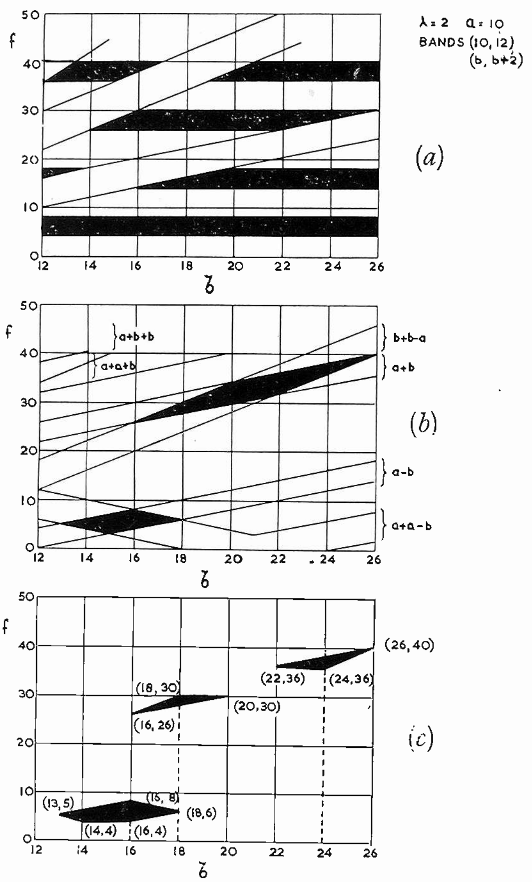

Fig. 15

Free zones for two signals A and B of equal bandwidth.

There are two more "mixed" interaction zones, A + B − A extending from $(b - \lambda)$ to $(b + 2\lambda)$ and A + B − B extending from $(a - \lambda)$ to $(a + 2\lambda)$ but they cannot be used for testing, since they cover the same zones as B + B − B and A + A − A respectively.

Zones with fourth order interactions must cover those with second order interactions, so that only second and third order terms need be considered.

The frequencies suitable for testing are contained in the shaded areas of Figs. 15a and 15b. These zones are shown separately in Fig. 15c.

As λ increases, the zones of Fig. 15a become narrower, while those of Fig. 15b become wider. The values $a = 10$, $\lambda = 2$ give wider testing zones than $a = 10$, $\lambda = 1$ and $a = 10$, $\lambda = 3$ (Figs. 16 and 17).

From Fig. 15 the zones available for test are: halving frequencies for simplicity: 2 to 4 for A 5 to 6 and B 8 to 9; 14 to 15 for A 5 to 6 and B 9 to 10; 18 to 19 for A 5 to 6 and B 12 to 13.

Intermodulation Spectra for Two Noise Bands

The intermodulation spectrum for two white noise bands of equal width and power may be found simply; it consists of a number of sub-spectra of identical width and shape and different weights.

Take the pair A(5,6) and B(8,9). The second order spectrum is obtained from the interactions A + A, A − A, A + B, A − B, B + B. Each of these gives a sub-spectrum of total width 2, of the form shown in Fig. 2. To find the weight to be

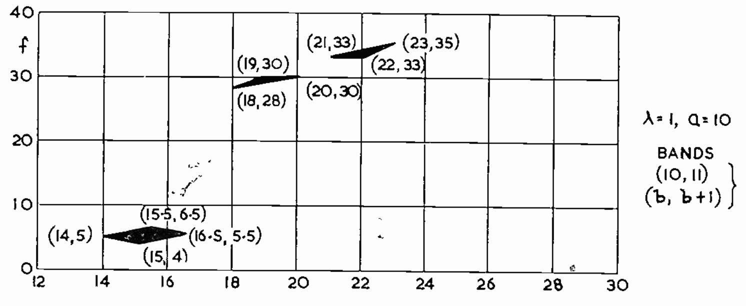

FIG. 16

Testing zones for a = 10, λ = 1.

attached to these, we take into account the fact that a ++ interaction and the corresponding +− interaction have equal weights, and that an (A,B) interaction has twice the weight of a (A,A) or (B,B) interaction.

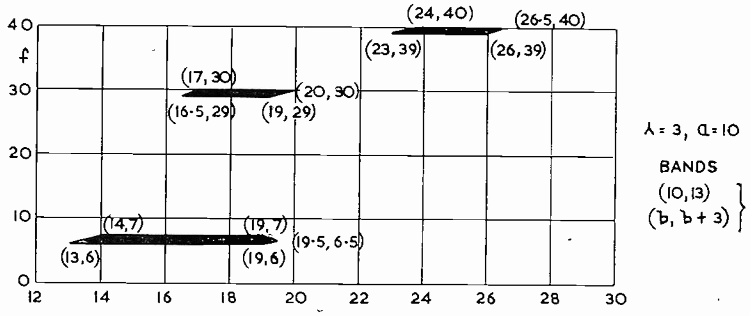

FIG. 17

Testing zones for a = 10, λ = 3.

This gives us weights in the ratio 1, 1, 2, 2, 1, 1, respectively, all divided by 8. We thus obtain

Interaction	Weight	Range
$A+A$	1/8	10 to 12
$A-A$	1/8	−1(0) to 1
$A+B$	1/4	13 to 15
$A-B$	1/4	2 to 4
$B+B$	1/8	16 to 18
$B-B$	1/8	−1(0) to 1

The second order spectrum is shown in Figs 18a to 18c. Of the total intermodulation power, 1/4 lies in the frequency range 2 to 4, the maximum occurring at 3.

The possible third order interactions are $+++$ and $++-$, with relative weights of 1/4 and 3/4. In each type we may have interactions as (AAA), (AAB), (ABB) and (BBB) with relative weights of 1, 3, 3, 1, respectively. Thus we obtain the following interaction table.

Interaction	Weight	Range
$A+A+A$	1/32	15 to 18
$A+A-A$	3/32	4 to 7
$A+A+B$	3/32	18 to 21
$A+A-B$	3/32	1 to 4
$A-B+B$	6/32	4 to 7
$A-A+B$	6/32	7 to 10
$A+B+B$	3/32	21 to 24
$-A+B+B$	3/32	10 to 13
$B+B-B$	3/32	7 to 10
$B+B+B$	1/32	24 to 27

These spectra are shown in Figs. 18d to 18f. The range 2 to 4 holds part of a sub-spectrum stretching from 1 to 4. According to Fig. 2 the power in the band 2 to 4 is 5/6 of the power in the band 1 to 4, which represents 3/32 of the total third order intermodulation power. Therefore the band 2 to 4 contains 7·8% of the total third order power and it has a maximum at frequency 2·5.

For the fourth order the possible interactions are $++++$, $+++-$ and $++--$, in the ratio 1/8, 1/2 and 3/8 respectively and in each of these the following combinations, AAAA, AAAB, AABB, ABBB and BBBB are found in the ratio 1 : 4 : 6 : 4 : 1 respectively, so that the weights of the interactions are as shown below.

Interaction	Weight	Range
$B+B+B+B$, $A+B-A-B$, $A+A-A-A$	18/128	-2 to $+2$
$A+B-A-A$, $B+B-A-B$	24/128	1 to 5
$B+B-A-A$	6/128	4 to 8
$A+A+A-B$	4/128	6 to 10
$A+A+A-A$, $A+A+B-B$	16/128	9 to 13
$A+A+B-A$, $A+B+B-B$	24/128	12 to 16
$A+B+B-A$, $B+B+B-B$	16/128	15 to 19
$B+B+B-A$	4/128	18 to 22
$A+A+A+A$	1/128	20 to 24
$A+A+A+B$	4/128	23 to 27
$A+A+B+B$	6/128	26 to 30
$A+B+B+B$	4/128	29 to 33
$B+B+B+B$	1/128	32 to 36

The fourth order intermodulation spectra are given in Figs. 18g to 18i.

Take the pair A(5,6) and B(8,9). The second order spectrum is obtained from the interactions A + A, A − A, A + B, A − B, B + B. Each of these gives a sub-spectrum of total width 2, of the form shown in Fig. 2. To find the weight to be

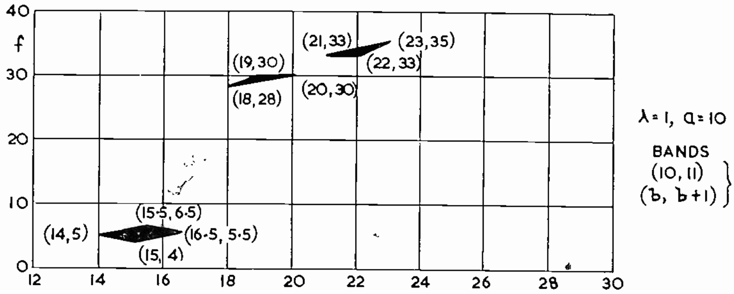

FIG. 16

Testing zones for $a = 10$, $\lambda = 1$.

attached to these, we take into account the fact that a ++ interaction and the corresponding +− interaction have equal weights, and that an (A,B) interaction has twice the weight of a (A,A) or (B,B) interaction.

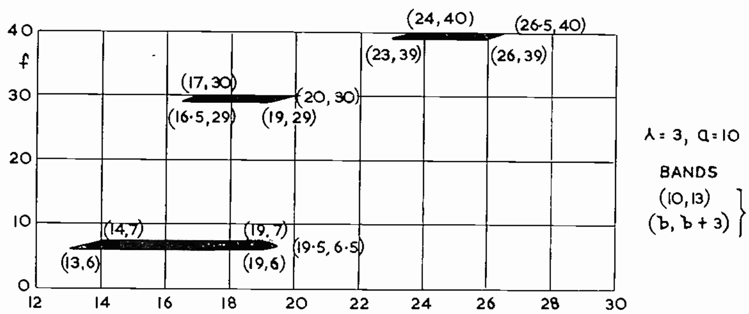

FIG. 17

Testing zones for $a = 10$, $\lambda = 3$.

This gives us weights in the ratio 1, 1, 2, 2, 1, 1, respectively, all divided by 8. We thus obtain

Interaction	Weight	Range
$A+A$	1/8	10 to 12
$A-A$	1/8	−1(0) to 1
$A+B$	1/4	13 to 15
$A-B$	1/4	2 to 4
$B+B$	1/8	16 to 18
$B-B$	1/8	−1(0) to 1

The second order spectrum is shown in Figs 18a to 18c. Of the total intermodulation power, 1/4 lies in the frequency range 2 to 4, the maximum occurring at 3.

The possible third order interactions are $+++$ and $++-$, with relative weights of 1/4 and 3/4. In each type we may have interactions as (AAA), (AAB), (ABB) and (BBB) with relative weights of 1, 3, 3, 1, respectively. Thus we obtain the following interaction table.

Interaction	Weight	Range
$A+A+A$	1/32	15 to 18
$A+A-A$	3/32	4 to 7
$A+A+B$	3/32	18 to 21
$A+A-B$	3/32	1 to 4
$A-B+B$	6/32	4 to 7
$A-A+B$	6/32	7 to 10
$A+B+B$	3/32	21 to 24
$-A+B+B$	3/32	10 to 13
$B+B-B$	3/32	7 to 10
$B+B+B$	1/32	24 to 27

These spectra are shown in Figs. 18d to 18f. The range 2 to 4 holds part of a sub-spectrum stretching from 1 to 4. According to Fig. 2 the power in the band 2 to 4 is 5/6 of the power in the band 1 to 4, which represents 3/32 of the total third order intermodulation power. Therefore the band 2 to 4 contains 7·8% of the total third order power and it has a maximum at frequency 2·5.

For the fourth order the possible interactions are $++++$, $+++-$ and $++--$, in the ratio 1/8, 1/2 and 3/8 respectively and in each of these the following combinations, AAAA, AAAB, AABB, ABBB and BBBB are found in the ratio 1 : 4 : 6 : 4 : 1 respectively, so that the weights of the interactions are as shown below.

Interaction	Weight	Range
$B+B+B+B$, $A+B-A-B$, $A+A-A-A$	18/128	-2 to $+2$
$A+B-A-A$, $B+B-A-B$	24/128	1 to 5
$B+B-A-A$	6/128	4 to 8
$A+A+A-B$	4/128	6 to 10
$A+A+A-A$, $A+A+B-B$	16/128	9 to 13
$A+A+B-A$, $A+B+B-B$	24/128	12 to 16
$A+B+B-A$, $B+B+B-B$	16/128	15 to 19
$B+B+B-A$	4/128	18 to 22
$A+A+A+A$	1/128	20 to 24
$A+A+A+B$	4/128	23 to 27
$A+A+B+B$	6/128	26 to 30
$A+B+B+B$	4/128	29 to 33
$B+B+B+B$	1/128	32 to 36

The fourth order intermodulation spectra are given in Figs. 18g to 18i.

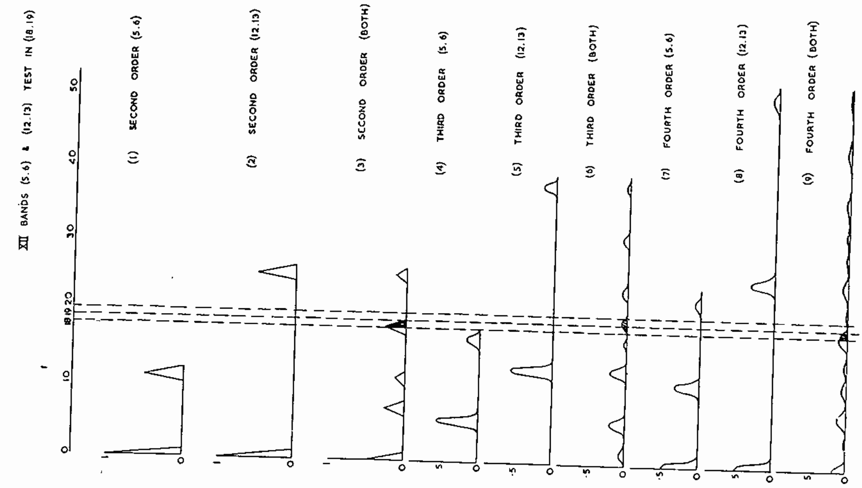

FIG. 20
Intermodulation spectra for two equal signal bands 5, 6 and 18, 19.

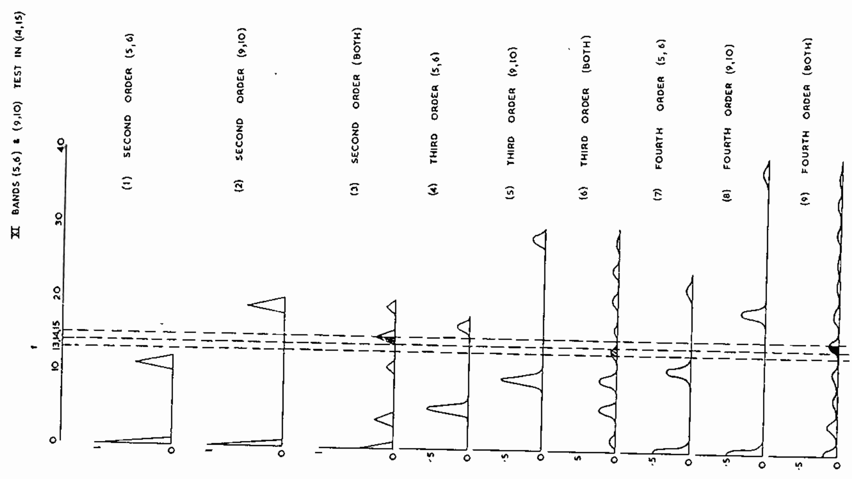

FIG. 19
Intermodulation spectra for two equal signal bands 5, 6 and 9, 10.

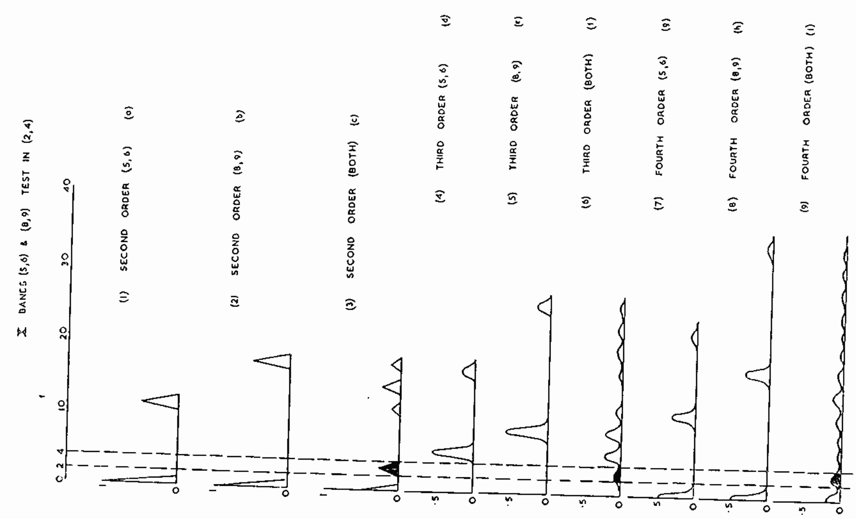

FIG. 18
Intermodulation spectra for two equal signal bands 5, 6 and 8, 9.

The range 2 to 4 contains the middle part of the distribution 1 to 5 and no others. From Fig. 2 the 4th order intermodulation power in the band 2 to 4 is 11/12 of 24/128, i.e. 17·2% of the total 4th order intermodulation power.

The amounts of intermodulation power, falling in the three selected test bands are as follows:

Input bands		Test band	Order of inter-action	% of total intermod power of order n	Maximum of n^{th} order spectrum	Frequency of maximum	Interaction
A	B	C	n	K	P_{max}	f_{max}	
5,6	8,9	2,4	2	25	0·250	3·0	A—B
			3	3·8	0·070	2·5	A+A—B
			4	17·2	0·125	3·0	A+B—A—A A+A—A—B
5,6	9,10	14,15	2	12·50	0·250	15	A+B
			3	1·56	0·047	14	B+B—A
			4	8·60	0·125	15	A+A+B—A A+B+B—B
5,6	12,13	18,19	2	12·50	0·250	18	A+B
			3	1·56	0·047	19	B+B—A
			4	8·60	0·125	18	A+A+B—A A+B+B—B

The intermodulation spectra for the two additional sets of signal bands given in the above table are shown in Figs. 19 and 20.

The test bands (13,14) in the second set and (19,20) in the third set are free of second order intermodulation power for A and B combined. The maximum of third order intermodulation power is, however, 50% higher than in the table above.

The powers in the enlarged bands are given below

A	B	C	n	K	P_{max}	f_{max}
5,6	9,10	13,15	2	12·5	0·250	15·0
			3	7·8	0·070	13·5
			4	9·4	0·125	15·0
5,6	12,13	18,20	2	12·5	0·250	18·0
			3	7·8	0·070	19·5
			4	9·4	0·125	18·0

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BOOK REVIEW

An Introduction to Colour Television, by G. G. Gouriet. Published for the Television Society by Norman Price, Ltd. Price 8/6d.

This seventy-two page monograph is a reproduction of two lectures given by the author to the Television Society. The time available in two lectures is hardly sufficient for the purpose in hand, but the casual reader will find that the book provides an inexpensive and handy introduction to the N.T.S.C. system if not to the subject in general.

Part One deals with the physical aspects of colour and draws much of its material from W. D. Wright's book, *Normal and Defective Colour Vision*. Here Mr. Gouriet displays a lucid style, and the newcomer is given an easy journey through the elementary stages of colorimetry. For the more serious reader a specific reference should have been made to W. T. Wintringham's article, "Colour Television and Colorimetry," Proc. I.R.E., October, 1951, which contains an excellent bibliography.

Part Two, entitled "Colour Systems," is actually devoted mainly to a description of the N.T.S.C. system. This is well written, and the author makes an interesting comment on a minor failing of the system, but the emphasis is more on compatibility aspects than on the performance as a colour system. Without discussing the effects of noise, or mentioning the fact that the colour receiver has to discard information in the upper part of the luminance band, the author somehow concludes that the N.T.S.C. system is "a first-class example of how a communication channel may be used efficiently." More should have been said about the methods and problems of instrumentation, a subject which is proving to be the major drawback in colour television.

Before concluding, a word of praise is due to the Television Society and to the publishers for their part in producing this book. The typography is good and the addition, or rather subtraction, of colour in a few of the illustrations lends attraction to the general appearance.

LONG DISTANCE PROPAGATION OF 16 KILOCYCLE WAVES

By N. M. Rust.

Reference is made to two recent publications dealing with long distance propagation of signals radiated from Rugby station on 16 kilocycles.

The phenomena involved are simply explained on a qualitative basis by means of the well-known ray hop theory.

The results obtained from this, admittedly very incomplete, explanation indicate certain important points that require both careful practical and experimental investigation and theoretical analysis.

Introduction

TWO papers have recently been published giving experimental results obtained on Rugby 16 kilocycle transmissions at great distances.

The first of these by Budden published in *The Philosophical Magazine*, Vol. XLIV, page 508 *et seq*, is entitled " Propagation of Very Low Frequency Radio Waves to Great Distances," and applies a waveguide propagation theory, previously developed by Budden, of the propagation of atmospherics arriving from great distances, to the analysis of a Hollingsworth interference pattern measured on Rugby 16 kc signals on a flight to Cairo.

The second paper, by Pierce of Crufts College, Harvard, is entitled " The Diurnal Carrier-Phase Variation of a 16 kc Transatlantic Signal," and was published in " The Proceedings of the Institute of Radio Engineers " for May 1955. This paper records, without any attempt to interpret the results in relation to the phenomena involved, the diurnal variations of Rugby Transmission times, as measured at Harvard by comparison with a very stable crystal reference oscillator.

An attempt was made by the Propagation Section of the Research Division of the M.W.T. Co. to apply Budden's theoretical results to estimate probably diurnal transmission time variations at different distances and check these with Pierce's results. This has thrown doubt on the validity of the details of Budden's theory. It occurred to the author of the present article, therefore, to investigate to what extent a very simple empirical analysis employing ray theory, using ground wave and first, second, third and fourth hops, would account for the main features of the phenomena involved. As the results obtained are of some interest, and point to the necessity of caution in any attempt to apply Pierce's results for distance comparisons involving signals measured over long and short paths respectively, it was thought that it might be of interest to record them. They do not go farther than to suggest that simpler explanations would fit in with the experimental facts as well as, or better than, the much more complicated analyses that have been developed from time to time to account for the phenomena.

It should be quite obvious that there is insufficient experimental data to formulate or check any theory. The outstanding point that emerges is the necessity of obtaining more experimental data. In this connection, it would seem to be important to record both amplitude and time delay changes, from hour to hour and from day to day, at different distances. The effect of polarization should also receive careful attention, as there is reason to think that the time delay of horizontal and vertical polarization components reflected from the ionosphere may prove substantially different.

Simple Ray Hop Theory

Fig. 1 shows to scale the earth's surface with two extreme positions of the Ionised Reflecting layer, namely at 70 km, shown by a full line, and at 85 km, shown by a dotted line. It will be obvious that from the point of view of any ray theory

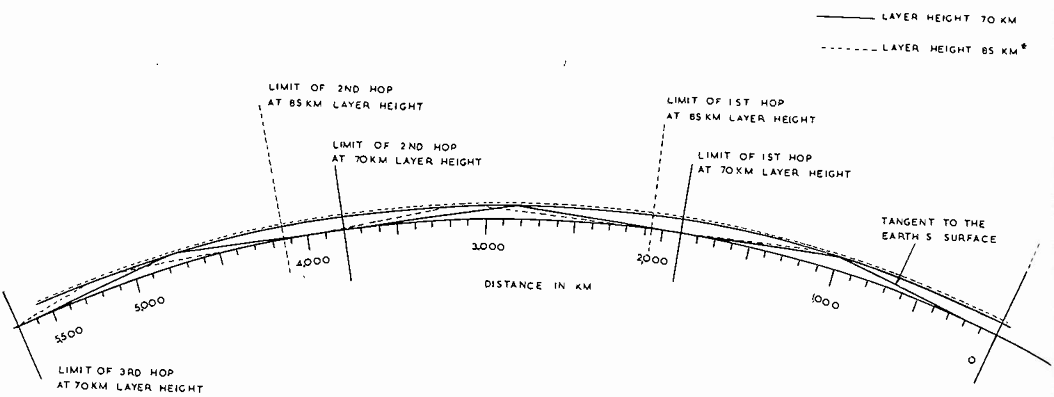


FIG. 1

the limiting distance of a single hop will be defined by the tangent to the earth's surface from the transmitting point for any greater distance would involve transmission through the earth. On this basis the limiting distance of first, second and third hops have been shown for the extreme layer heights. These distances were derived by calculation, as graphical methods would be subject to large errors. Thus, on these assumptions, a receiving station at 4,000 km distance would receive a very weak ground wave component, and second, third and possibly fourth and fifth hop signals with an equivalent layer height of 85 km. With an equivalent layer height of 70 km, the second hop transmission would be cut off.

No attempt has been made to estimate the relative strength of the various components, but it would seem reasonable to suppose that, up to 1,000 kms, the ground wave and first hop wave would predominate. At night the hop waves would be strengthened owing to the improved reflecting properties of the layer and of course the hop regions would extend owing to the layer height rising. As the distance increased the hop waves would tend to predominate over the ground wave and the higher order hops would replace the lower, until, say at 5,000 km, it would prove very difficult to resolve or to measure the ground wave, and the transmission effects would be in the main determined by third and fourth hop transmission. Let us therefore see how this rather crude picture will fit in with the phenomena recorded in the papers mentioned above.

Fig. 2 has been prepared using results obtained by the Research Division of the Marconi Company in work on long range position fixing systems. The curves have been calculated from a useful approximation formula developed by Millington, and are plotted in terms of transmission phase delay in cycles with reference to the ground wave for the first five hops (I, II, III, IV and V respectively) up to 5,500 kms.

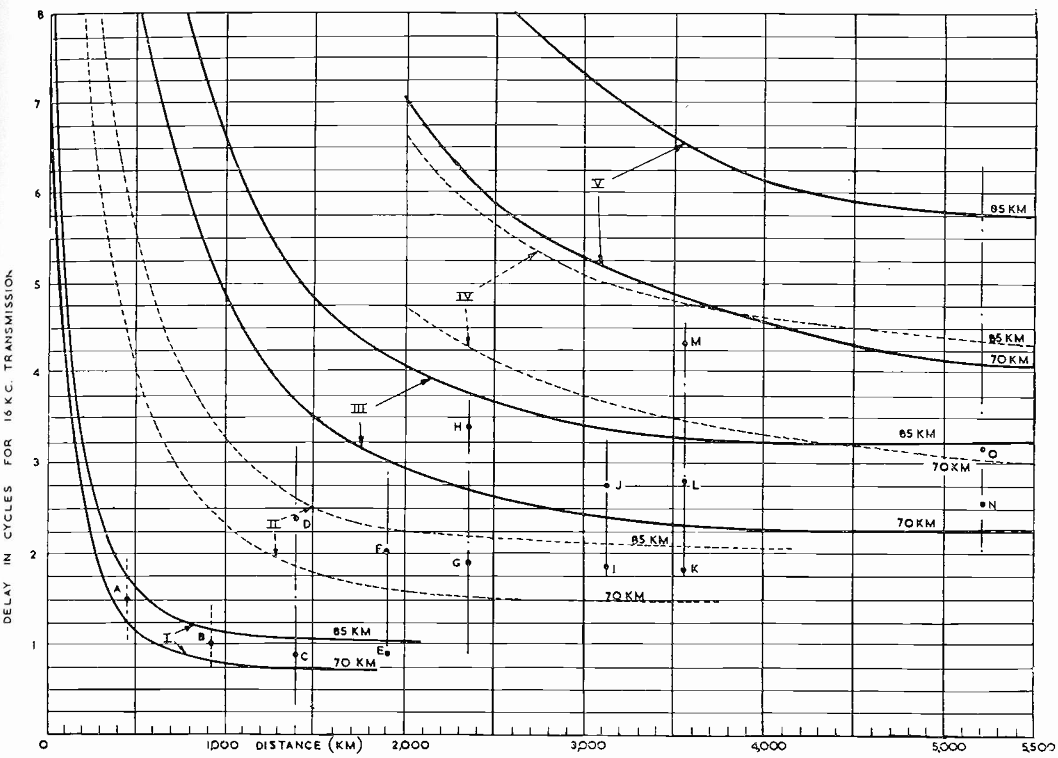


Fig. 2

Application to Field Strength Curve—Flight to Cairo

Fig. 3 reproduces the results quoted in Budden's paper mentioned above. This is a Hollingsworth type of curve recording measurements taken on a flight to Cairo during daylight. The dotted line represents an average following a signal amplitude $X\sqrt{\text{distance}}$ law. The first interference effect is a sharp dip at about 450 kms distance. Referring to Fig. 2 it will be seen that this could be accounted for by interference between the ground wave and a first hop arriving $1\frac{1}{2}$ cycles later due to reflection from a layer approximately midway in height between 70 and 85 kms, as shown at A.

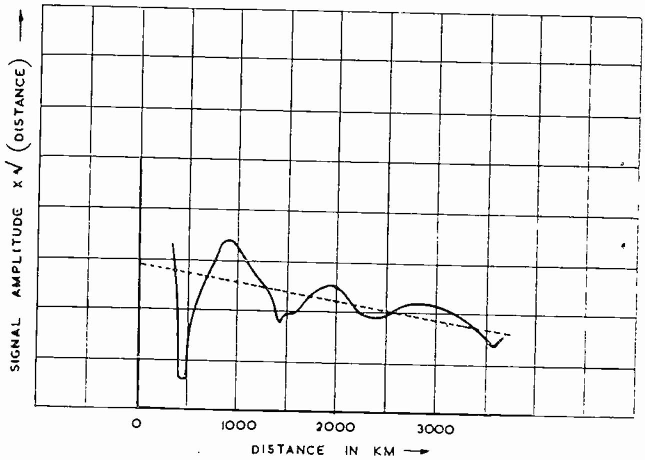
We next have an interference maximum at about 920 kms. Point B shows that this could be caused by the first hop from the same height of layer coming into phase with the ground wave.

The dip at 1,400 km can be explained on the basis of interference between ground wave, first hop (Point C) and second hop (Point D). It will be noted that D is delayed $1\frac{1}{2}$ cycles relative to C. It might be assumed that, at this distance, the ground wave has faded out to the extent that the interference between first and second hop predominates.

At 1,900 km the first hop is weakening (E) but is practically in phase with the second hop arriving one cycle later (F). This would account for the maximum. At 2,350 the first hop has for all practical purposes faded out, and we get a dip due to the fact that the third hop, which is now having an effect, arrives $1\frac{1}{2}$ cycles later than the second hop (H and G respectively).

FIG. 3

The product "signal strength × square root of distance" is plotted against "distance" for the signal from the sender GBR 16 (kc/s) during the day time in June and July 1950.



The blunt maximum at above 3,000 km would be accounted for by the third hop (J) now arriving 1 cycle behind the second hop (I) and thus reinforcing it.

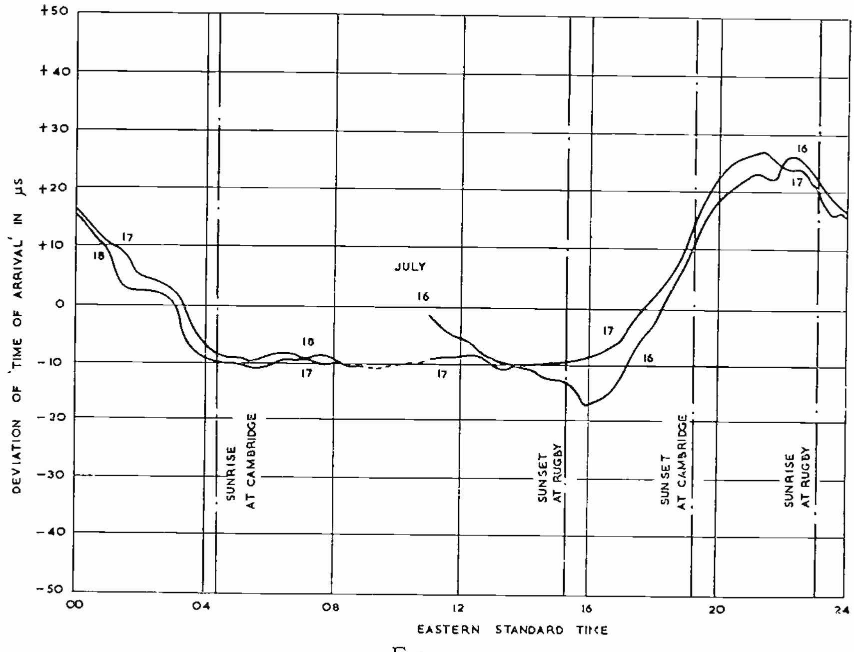


FIG. 4

The diurnal variation of the GBR transmission time, reduced from Fig. 4. The difference in behaviour on July 16 and 17, in the interval between 11h and 18h EST, is probably caused by an oscillator anomaly.

Finally, at 3,500 km the second hop (K) is weakening, and the fourth hop coming up. The latter (M) arrives $1\frac{1}{2}$ cycles behind the third hop (L), thus creating a minimum.

It should be pointed out that the explanation given above involves considerably higher reflecting layers than the approximately 70 km layer which Budden calculates with his analysis.

Application to Transmission Time Comparisons at Harvard

Figs. 4 and 5 have been abstracted from Pierce's paper mentioned above. The distance from Rugby to Harvard is given as 5,180 kms.

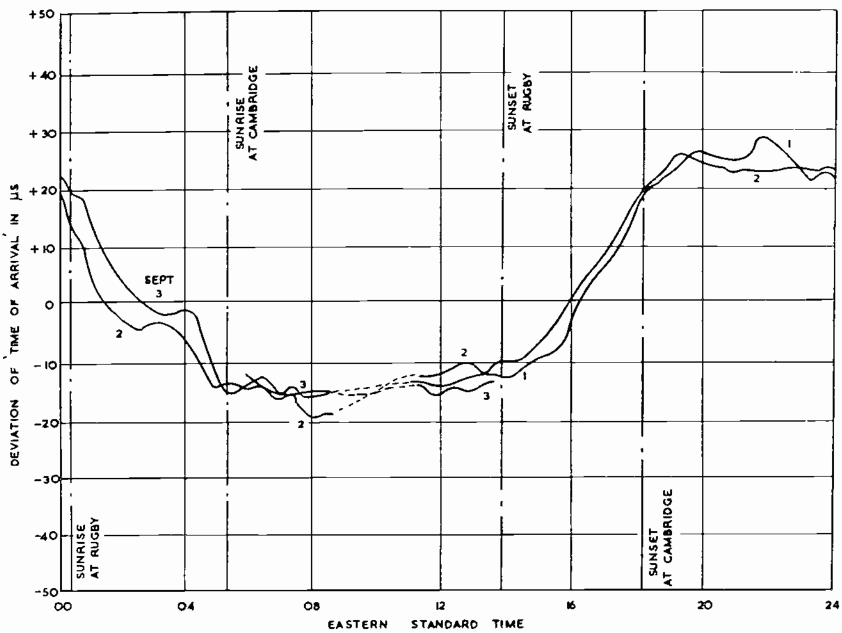


FIG. 5

The diurnal variations of GBR's transmission time for September 1 *to* 3, 1954. *Note that the anomaly of Fig.* 4 *is not present here.*

Two features stand out in these curves.

The first is an increase in delay time of approximately 40 microseconds between full daylight and full dark conditions. This, expressed in cycles of 16 kc., would correspond to a delay increase of 0·64 cycles (as Fig. 2 is plotted in cycles delay).

Referring to Fig. 2 it will be seen that the two hops which will predominate are the third and fourth. At this distance they would tend to act together, and the change in delay time would probably be accounted for merely by the alteration of equivalent reflecting layer height as we go from full daylight to dark, or vice versa.

Thus, assuming that the third hop predominates, a change in layer height from that corresponding to N to that corresponding to O, would account for the 0·64 cycle change required.

The other feature is the flat parts of the curves half-way between sunrise Rugby and sunrise Cambridge (Harvard). Pierce points out that this change is accompanied by a very marked drop in carrier field strength, and suggests it is related to change in third hop conditions. Reference to Fig. 1 shows that at this time the shadow band is crossing the ionosphere at the region in mid-Atlantic at which the third hop signals are being reflected. The effect could therefore be explained by a weakening of third hop signals owing to the shadow band crossing this reflecting surface. Fourth hop signals, which are perceptibly over 1 cycle extra delay, then take charge of the phase. The combined effect of alteration of layer height, and strengthening of fourth hop signals relative to third hop, tends to hold the resulting time delay nearly constant for some time. Then, when the shadow band has passed over, the third hop signals strengthen relative to the fourth and pull the time delay of the resultant signal relatively sharply back to that corresponding to third hop transmission.

It will be noticed that (see Fig. 5), although there is, in September, a slight flattening of the curve between sunset Rugby and sunset Harvard, in July this is hardly noticeable. The probable reason for the smaller effect at sunset is in connection with the angle the shadow band makes with the Rugby-Harvard path. Especially at midsummer, the inclination of the shadow band to the path will vary very much according to whether it is sunrise or sunset. It is suggested, therefore, that the respective inclinations are such that, at sunrise the reflecting surface is more disturbed than at sunset.

Conclusion

The interference phenomena recorded in Budden's paper, and Pierce's Transmission Time curves can be very simply explained by the interference of Ground, and first, second, third and fourth hop components, which progressively " tail off " as distance is increased.

An important point, therefore, arises, which should not be lost sight of. This is, that in considering transmission comparisons which involve long distances to one station, and relatively short to another, it is not safe to assume that the maximum delay differences will be defined by the changes of delay time to one or both stations. For example, to take an extreme case, a receiving station 1,000 kms from Rugby, and 5,000 kms from an American counterpart, might experience, as between daylight and dark conditions, changes of delay time of the order of only 50 microseconds. But the nearer station might be received by predominant ground wave and the distant station by predominant third hop. The result might therefore be to add at least 150 microseconds delay time to the more distant transmission, although the change of delay due to day-night effects would only be 50 microseconds.

The added time delay (due to the fact that the equivalent velocity is slowing up as distance increases) which would result if the simple assumptions taken above can be regarded as basically correct, must therefore not be neglected in Radio Navigational Aids employing V.L.F. when used over long distances.

It is of course realized that the explanation given above is very much over simplified, and cannot be used to obtain quantitative data of any kind. The most important point to check, in developing a theory that will account for the phenomena, is whether the equivalent velocity decreases with distance in the way that the simple explanation would suggest that it should do. It would seem clear, that to obtain the clearest possible understanding of these phenomena, it should be possible to interpret them in terms of both phase and group velocity, just as waveguide phenomena may be interpreted.

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