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A Problem of Electromagnetic Induction

IN the Editorial of July 1939 we considered among other interesting problems that of the e.m.f. induced in a cylinder made up of alternate segments of brass and iron when rotated in an axial magnetic field. When this was briefly described in a recent lecture it raised some interesting discussion, which indicated that the problem is worth further consideration, especially when the spaces occupied by the brass and iron are not equal. Fig. 1 shows such a cylinder rotating in an axial field between two poles, the field being produced by six magnetizing coils. We have adopted this arrangement to remove any question as to whether the magnetic field may rotate. We assume that the rotating cylinder is much longer than shown, so that end effects may be neglected. Eddy currents in the poles could be made negligible by constructing the poles of laminated strip wound in a spiral.

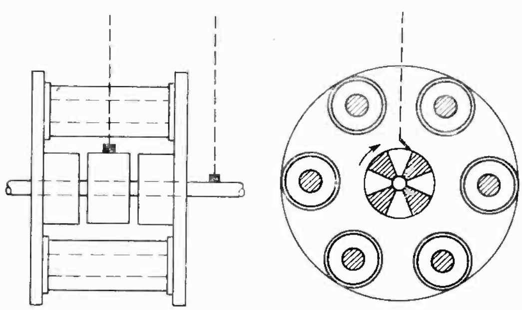

Fig. 1

A clear picture of what happens to the magnetic field can be obtained by the analogy shown in Fig. 2, which represents a race-track made up of alternate patches of land and water. A number of equally spaced runners are supposed to be running from right to left; in the water, on account of their slow progress they are close together, but on emerging from the water their speed is greatly increased and their spacing increases.

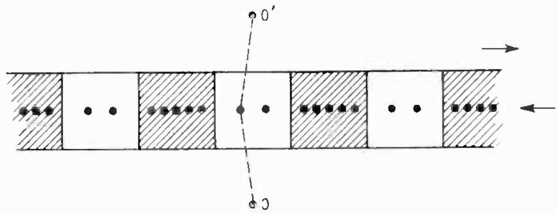

Fig. 2

The number of runners passing any point per minute is the same whether on land or in water, decrease of speed being counteracted by increase in density. If now we picture the whole track moving to the right at the average speed of a runner, to stationary observers such as OO′ each runner will simply move to and fro and could be fastened to the observers by 'stretched elastic threads' as shown by the dotted line. When in the water he will move slowly to the right, and when on dry land, rapidly to the left.

This is exactly what happens to the lines of magnetic induction as the cylinder rotates; in the iron the density is much greater than in the brass and the lines move slowly in the direction of rotation, whereas in the brass they move rapidly backwards, but the number of lines cut per second is exactly the same whether the radius considered is in the iron or in the brass. The smaller flux density in the brass is counteracted by the higher speed with which the lines cut through the material. If the total magnetic flux along the cylinder is Φ and the speed n revolutions per second, then the induced e.m.f. in

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any radius is $n\Phi$. On open circuit, assuming the direction of the magnetic flux in Fig. 1 to be away from us, the surface of the cylinder will be uniformly charged positively, the equivalent negative charge being uniformly distributed throughout the material of the cylinder.

We shall now assume that the cylinder is not made up of equal segments of brass and iron, but that a fraction α is of brass and the remainder $1-\alpha$ of iron of permeability μ. If R is the radius of the cylinder and B the flux density in the brass, then $\Phi = \pi R^2[\alpha B + (1-\alpha)\mu B]$ and for the induced e.m.f. we have $E = \pi R^2 n B[\alpha + (1-\alpha)\mu]$. If $\alpha = \frac{1}{4}$, the quantity in brackets is $(1+3\mu)/4$, if $\alpha = \frac{1}{2}$, it is $(1+\mu)/2$, and if $\alpha = \frac{3}{4}$, it is $(3+\mu)/4$.

If we approach the problem from the more abstract Maxwellian point of view, without the aid of race-track analogies, we find it much more difficult. In the brass the e.m.f. induced in 1 cm of radius *due to its movement* in the magnetic field is Bv, while in the iron it is μBv. To these we have to add the e.m.f. due to the change of flux through any stationary path; otherwise, although the mean value of Bv and μBv taken round the whole cylinder gives the correct value, the e.m.fs induced in the iron and brass are entirely different.

If in Fig. 3 the stationary path a b c d has a radial length of 1 cm, the e.m.f. induced around the path by the changing flux is $(\mu - 1)Bv$, where v is the velocity with which the interface between the brass and iron is sweeping across the stationary path.

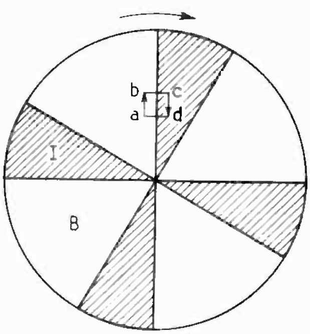

Fig. 3

We now come to the question that led to the writing of this article; how is this e.m.f. to be divided between a b and c d? Since there cannot be a discontinuity in the resultant radial electric force at the interface, the obvious answer is to divide the e.m.f. $(\mu-1)Bv$ into two parts such that the addition to the motional e.m.f. in the brass, and the subtraction from that in the iron give equal resultant e.m.fs. This is where the difficulty arises, for since $(\mu-1)Bv$ is exactly equal to the difference between the two motional e.m.fs, μBv and Bv, equal resultants are obtained however the circuital e.m.f. is divided between the two sides. If it is all added to Bv in the brass the resultant everywhere is μBv, whereas if it is all subtracted from μBv in the iron, the resultant everywhere is Bv. The correct division depends on something quite outside this application of Maxwell's laws; viz., the relative amounts of brass and iron in the cylinder; i.e., on the value of α. We have seen that the resultant in both brass and iron is $Bv[\alpha + (1-\alpha)\mu]$; hence the portion of the circuital e.m.f. in a b must be $Bv[\alpha + (1-\alpha)\mu] - Bv = Bv[(1-\alpha)(\mu-1)]$, while that in c d must be $\mu Bv - Bv[\alpha + (1-\alpha)\mu] = Bv[\alpha(\mu-1)]$. Hence we see that the circuital e.m.f. is divided between the brass and iron inversely as the proportion of these materials in the cylinder.

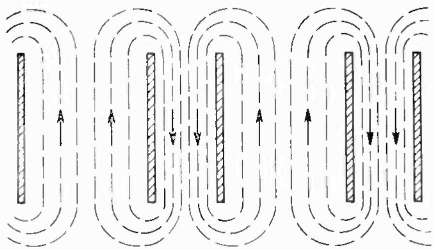
Fig. 4

We feel that the application of the race-track analogy, and the treatment of the lines of magnetic induction as discrete physical entities throws more light on this problem than does the application of Maxwell's laws, which, in their simple form, appear to be incapable of giving a definite solution without the aid of the former. A mathematical correspondent remarked that he was not sure that theory gives a constant e.m.f. as the cylinder rotates, but that it is too complicated for rigorous calculation. This emphasizes the simplicity and the rigorous accuracy of the analogy, which can also be applied to the passage of the magnetic flux across the teeth and slots of a dynamo armature.

It is interesting to note that, if in Fig. 4 the narrow strips represent the cross-sections of iron cores in which the magnetic flux is changing at the same rate but in opposite directions in adjacent strips, the electric field induced in the surrounding medium will be approximately as shown. The values of the induced electric force $\mathcal{E}$ in the spaces between the cores are inversely proportional to the distances between the cores, and the circuital e.m.f. is divided between the two sides in the same way as in Fig. 3.

G. W. O. H.

Readers will note a slight reduction in the number of pages in this issue of *Wireless Engineer*. This reduction has been necessitated by the withdrawal of overtime working by a section of the printing industry and has, in the circumstances, been unavoidable. We hope that it will be only temporary.

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A USEFUL NETWORK PROCEDURE

By Sidney C. Dunn, M.Sc., A.M.I.E.E.

(Paisley Technical College)

THE method proposed establishes a relation between the parallel connection of quadripoles and the star connection of the same quadripoles. While the general network is considered first, the method is most useful in a particular case of frequent occurrence.

Consider the parallel connection of two T-networks as in Fig. 1 (a) working between generator and load each of finite impedance. The transfer response is given by

$$\frac{v}{e} = \frac{Z_8 B_{14}}{D} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where D is the fourth-order determinant of the network (see Appendix) and B_{14} the cofactor formed by striking out row 1 and column 4 in D and multiplying the minor so formed by -1. The network has been redrawn in Fig 1 (b) to disclose the meshes involved.

Now replace the circuit by that of Fig. 2 which is formed by cutting through the network at the dotted line in Fig. 1 (a) and unfolding it so that the load resistor lies in the middle and the circuit is fed at each end by a zero-impedance generator of e.m.f. Ae. We will now find the value A must have in order that the response of this network will be the same as that of the previous one. It will be noted that we have reduced the number of effective meshes in the network from four to one since the circuit may be further rearranged as in Figs. 3 (a) and 3 (b) where the superposition theorem is invoked to find the contribution to the resultant output voltage made by each generator.

If, in order to save space, we use detached suffixes instead of the full impedance symbols the partial responses are found to be

$$\frac{v_a}{Ae} = \frac{38\,(57 + 67 + 56)}{8\,(2 + 3)\,(57 + 67 + 56) + (42 + 23 + 34)\,(85 + 86 + 57 + 56 + 76)}$$

$$\frac{v_b}{Ae} = \frac{68\,(42 + 23 + 34)}{8\,(5 + 6)\,(42 + 23 + 34) + (76 + 65 + 75)\,(82 + 83 + 42 + 23 + 34)}$$

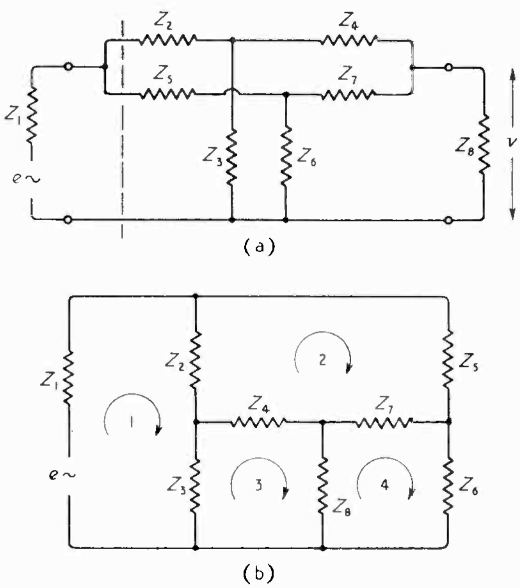

Fig. 1. *Parallel-connected T networks (a) and their meshes (b).*

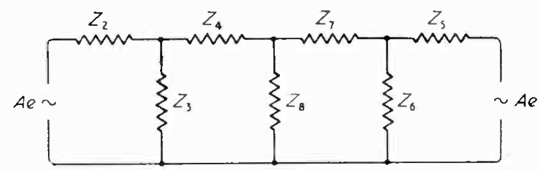

Fig. 2. *Equivalent circuit of Fig. 1.*

If the consequent expression for $(v_a + v_b)/e$ is compared with that for v/e obtained from Equ. (1) it will be found that

$$A = \frac{D'}{D} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where D' is the determinant of the original network when $Z_1 = 0$. The process may be quite generally applied to any number of quadripoles in parallel. The common pair of terminals at the output side is left untouched while at the input side each 'channel' is given its own generator of e.m.f. calculable from Equ. (2). The equivalence is shown in Fig. 4.

The device is most convenient when the generator feeding the quadripoles has zero internal impedance for then $A = 1$ and the operations required to find the transfer response are,

(a) with all the partial generators shorted except one find the partial response of the network to that generator. The denominator of

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the expression when arranged to be present to a degree equal to the number of original meshes is the determinant of the network and does not need to be evaluated again.

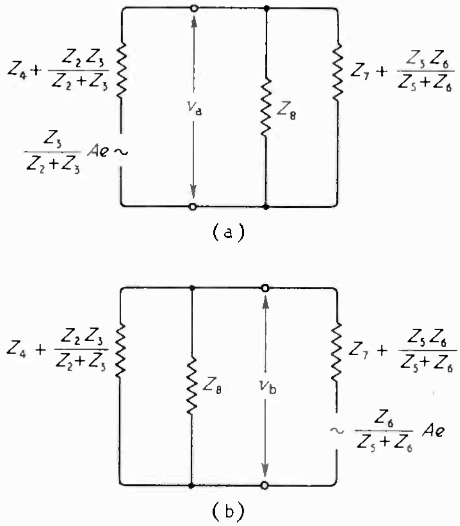
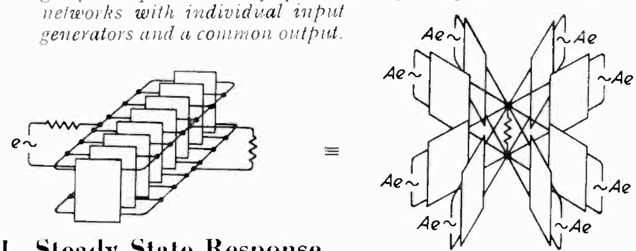


Fig. 3. *Reduction of circuit of Fig. 2 by the super-position theorem.*

(b) Find the partial numerators in the same way for each generator in turn. This can often be done by inspection.

The overall response of the network is the sum of the partial numerators divided by the common determinant. The advantage of the method lies in the fact that rearranging the network as a two-mesh circuit does not alter the value of the determinant and the latter is now calculable as a second-order array instead of the original higher-order expression.

Fig. 4. *Equivalence of paralleled quadripoles to networks with individual input generators and a common output.*



1. Steady State Response

As an example of the method we will take the parallel-T RC network of Fig. 5 (a) rearranged according to our scheme in Fig. 5 (b). Considering first the left-hand generator and using Thévenin's theorem we have,

$$\frac{v'}{e \cdot \frac{R}{R+Z}} = \frac{(16R^2 + 8RZ)(R+Z)}{(R+Z)(16R^2+8RZ) + (4R+Z)(Z^2+2RZ)}$$

$$\frac{v'}{e} = \frac{16R^3 + 8R^2Z}{D} \quad \ldots \quad (3)$$

similarly for the right-hand generator,

$$\frac{v''}{e} = \frac{Z^3 + 2RZ^2}{D}$$

whence

$$\frac{v}{e} = \frac{16R^3 + 8R^2Z + 2RZ^2 + Z^3}{16R^3 + 32R^2Z + 14RZ^2 + Z^3}$$

An alternative and possibly preferable method is to consider Fig. 5 (b) as a two-mesh circuit in which the numerator of the partial response to the left-hand generator is $B_{12}Z_0$ where Z_0 is the impedance seen to the right of the dotted line. In this case, by inspection, $B_{12} = R$ and Z_0 is given by $(16R^2 + 8RZ)/(4R+Z)$.

$$(4R + Z)\, B_{12} Z_0 = 16R^3 + 8R^2Z \quad \ldots \quad (4)$$

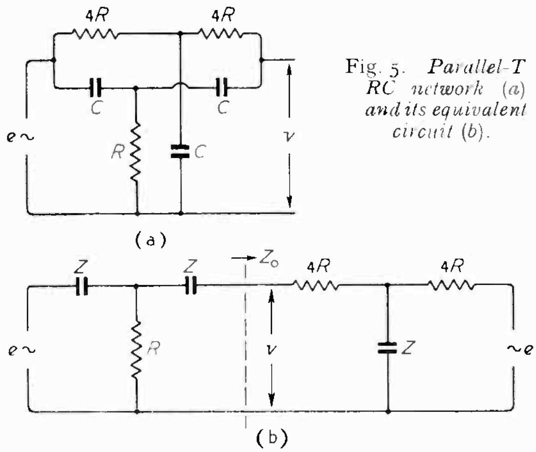


Fig. 5. *Parallel-T RC network (a) and its equivalent circuit (b).*

The reason for the above form is that the determinant of the network is also most neatly expressed when multiplied by $(4R + Z)$. Hence the numerator of the partial response is given by the right-hand side of Equ. (4) and the other partial response is found in a similar manner.

2. Transient Response

The splitting of a network into component channels affords a clear picture of its behaviour under transient conditions. Using the same example as before we can write Equ. (3) as

$$\frac{v'}{e} = \frac{-jx^3 - \sqrt{2}x^2}{-jx^3 - 4\sqrt{2}x^2 + j7x + \sqrt{2}}$$

where $x = \sqrt{8}CR$ is the normalized frequency variable. If we now substitute s/ω_0 for jx,

$$\frac{v'(s)}{e} = \frac{s^3 + \sqrt{2}\omega_0^2}{s^3 + 4\sqrt{2}s^2\omega_0 + 7s\omega_0^2 + \sqrt{2}\omega_0^3}$$

$$= \frac{s^2}{s^2 + 3\sqrt{2}\omega_0 + \omega_0^2}$$

If for e we write $1/s$, the operational form of

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Unit Function, and consult a table of transforms,

$$v'(t) = 1.067 \exp(-4\omega_0 t) - 0.067 \exp(-0.25\,\omega_0 t)$$

The coefficients of $\omega_0 t$ in the above expression are actually $-(3 \pm \sqrt{7})/\sqrt{2}$ or 3.993, 0.2506 but the approximations seem justified on the grounds of elegance. In the same way the partial transient response of the other channel is

$$v''(t) = 1 + 0.067 \exp(-4\omega_0 t) - 1.067 \exp(-0.25\omega_0 t)$$

The resultant transient response of the whole circuit is

$$v(t) = 1 + 1.135 \exp(-4\omega_0 t) - 1.135 \exp(-0.25\,\omega_0 t) \quad .. \quad (5)$$

This curve and the above components are plotted in Fig. 6. Here we have deliberately separated the response into two components to show the physical action within the circuit. In practice Equ. (5) would be written down more directly from the operational form of the complete response.

Equ. (5) is interesting in the present example in that although it is the response of a three-mesh network it involves only two exponential terms. The reason would appear to be connected with the fact that when the network is adjusted to give a complete null at one frequency, as in the present case, then the high-pass and low-pass channels must have equal time constants.

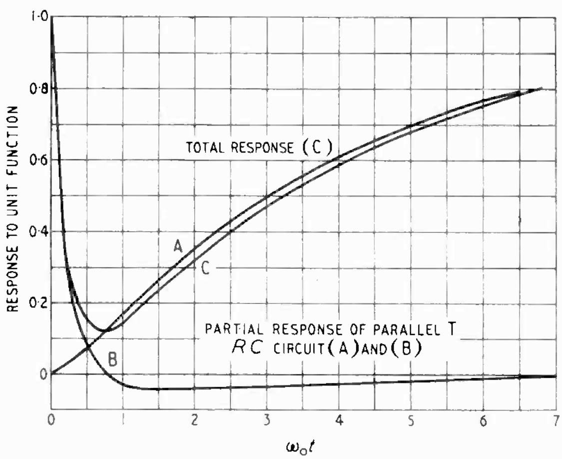

Fig. 6. *Partial and total responses of the network of Fig. 5 to a unit step.*

The mode of operation of parallel-channel circuits is often difficult to explain and the present analysis gives a method of at least describing their structure. The overall response of a parallel-channel network when fed from a zero-impedance generator is the sum of the individual responses of the separated channels when each is terminated on the load side by the short-circuit input impedance of all the other channels in parallel.

3. Impedance Calculations

The method may also be used for finding the input impedance of multi-channel networks. There is, however, one feature which should be pointed out. While any single generator is feeding the network it is putting current through the temporarily passive meshes belonging to other generators and thus reducing their effectiveness. For the previous example, in the equivalent circuit of Fig. 5 (b), for the left-hand generator, we have for unit applied voltage,

$$D = \begin{vmatrix} R + Z & -R \\ -R & 5R + Z + \dfrac{4RZ}{4R + Z} \end{vmatrix}$$

$$(4R + Z)\,D = 16R^3 + 32R^2Z + 14RZ^2 + Z^3$$

$$(4R + Z)\,B_{11} = 20R^2 + 13RZ + Z^2$$

$$(4R + Z)\,B_{12} = R\,(4R + Z)$$

$$\text{input current} = \frac{20R^2 + 13RZ + Z^2}{D} = I_1$$

$$\therefore \text{current in second mesh} = \frac{R\,(4R + Z)}{D}$$

$\therefore$ current through shorted generator loop is

$$\frac{R\,(4R + Z)}{D} \cdot \frac{Z}{(4R + Z)} = \frac{RZ}{D} = I_2$$

In the same way for the other generator,

$$\text{input current} = \frac{4R^2 + 7RZ + 2Z^2}{D} = I_3$$

$\therefore$ current through shorted-generator loop

$$= \frac{RZ}{D} = I_4$$

Total input current to original network

$$= I_1 + I_2 + I_3 + I_4 = I_0$$

input impedance

$$= \frac{1}{I_0} = \frac{16R^3 + 32R^2Z + 14RZ^2 + Z^3}{24R^2 + 18RZ + 32Z^2}$$

It is probable that the procedure is most useful in finding the impedance of complicated structures. Consider the network of Fig. 7 (a) which may be thought of as a variant on the "cube of one ohm wires" theme. In this case we have a prism and the symmetry which aids the solution of the other problem is lacking. This circuit has the advantage however that it can be squashed flat as at Fig. 7 (b), rearranged as at (c) and finally given our form in (d). In this particular instance, with equal resistances the problem is also fairly easy if the star-mesh transformation is used but in the general case with unequal complex impedances the transformation is very laborious while our method remains fairly simple. In Fig. 7 (c) the method

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should be noted of untwisting lattice arms and feeding their ends from a generator of reversed polarity.

Conclusion

The analysis of a particular class of networks, namely those consisting of a number of channels in parallel, fed from a common zero-impedance generator, is simplified by the device of replacing the one generator by a number of identical generators. Since the total current flowing through the original generator can produce no modification of the voltage applied to the network we can split off these separate generators and allot them to each channel. Applying the superposition theorem then reduces the problem to adding the responses of a number of single-mesh circuits. The procedure has also been found helpful in finding the input impedance of circuits which prove to be somewhat intractable by conventional methods.

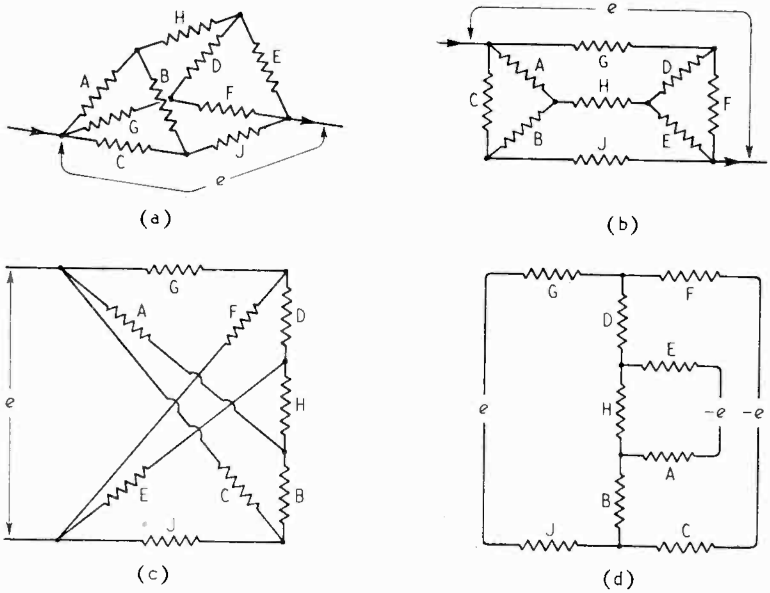

Fig. 7. *Variant of "cube of one-ohm wires" (a); squashed flat (b); re-arranged (c) and brought to its final form with independent generators (d).*

APPENDIX

The Use of Determinants in Network Analysis.

Network problems involve the solution of simultaneous equations and unless this is systematized the work becomes very tedious. Consider the equations

$$a_{11}x + a_{12}y = k_1 \quad \ldots \quad (6)$$

$$a_{21}x + a_{22}y = k_2 \quad \ldots \quad (7)$$

To find x multiply (6) by a_{22}, (7) by a_{12} and subtract (2) from (1), then

$$(a_{11}a_{22} - a_{12}a_{21})\, x = a_{22}k_1 - a_{12}k_2$$

i.e., $$x = \frac{a_{22}k_1 - a_{12}k_2}{a_{11}a_{22} - a_{12}a_{21}}$$

and y is given by a similar expression. The denominator of the right-hand side may be found by writing the coefficients in an array

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

and cross-multiplying so that products are taken diagonally to the right, positive downwards and negative upwards. To find the numerator we use the array

$$\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}$$

in which the right-hand side of the equation has been substituted for the column containing the variable to be found. If we now solve a third-degree equation by the usual methods and set up an analogous array of coefficients we find that the rules for expanding this array are slightly more complicated but will now apply to an array of any order. Such an array is called a determinant and for a third-degree equation would be

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

If the variables here are x, y and z then by the ordinary way,

$$y = \frac{a_{11}k_2a_{33} - a_{11}k_3a_{23} - k_1a_{21}a_{33} + k_1a_{31}a_{23} + a_{13}k_3a_{21} - a_{13}a_{31}k_2}{a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{32}a_{21} - a_{13}a_{31}a_{22}}$$

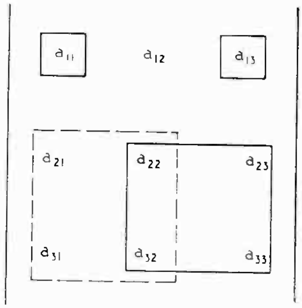

Fig. 8

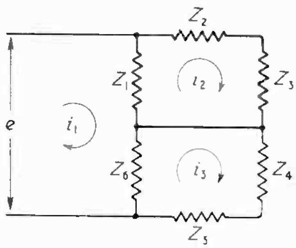

Fig. 9

We can reconcile the first and last two products in both the numerator and denominator with our previous rules by thinking of them as a common factor times a second-order determinant. This is shown in Fig. 8. The middle pair in each case however need to have their signs reversed before they can be interpreted in this way. The 'sub-arrays' concerned here are called minors and a rule is required to determine the sign which must prefix them when expanded to form the denominator or numerator of the root of an equation. This rule is as follows. If a minor is formed by striking out the i-th row and j-th column of a determinant then in

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an expansion it must be multiplied by $(-1)^{i+j}$. A signed minor is called a cofactor. To find the denominator of a root these operations are performed on the determinant of the original equation and for the numerator on the same array but with the right side of the equation substituted as in the second-order case.

In solving a network problem the work is further reduced by the fact that the various elements in the determinant have a simple physical significance. This enables us to write down the determinant of a network without the necessity of setting up the equations. In the case of Fig. 9 if we go through all the usual steps with Maxwell's cyclic currents we arrive at an array,

$$\begin{vmatrix} Z_1 + Z_6 & -Z_1 & -Z_6 \\ -Z_1 & Z_1 + Z_2 + Z_3 & 0 \\ -Z_6 & 0 & Z_4 + Z_5 + Z_6 \end{vmatrix}$$

The elements in the 'positive' diagonal are respectively the mesh impedances of mesh 1, mesh 2, mesh 3. The other elements represent common or mutual impedances between the various meshes. For example Z_1, the element common to row 1 and column 2 or to row 2 and column 1 is the impedance common to mesh 1 and mesh 2. The negative sign is due to the fact that the current in these two meshes flows in opposite directions. In the majority of cases, there is only one source of e.m.f. in this circuit, hence $k_1 = e$ and the other k terms are zero. This simplifies the numerator of the expression for current: e.g.,

$$I_1 = \frac{eB_{11}}{D}$$

where B_{11} is the cofactor formed by striking out row 1 and column 1 and giving the minor so formed a positive sign. This gives a simple formula for input impedance,

$$Z_{in} = D/B_{11}$$

NEGATIVE-RESISTANCE CHARACTERISTICS

Graphical Analysis

By A. W. Keen, A.M.I.E.E., M.I.R.E.

1. Introduction

IN view of the dissipative character of resistive components of passive electrical networks it is appropriate that the property of negative-resistance should be attributed to active elements. The negative-resistance concept has been employed frequently in the literature in connection with oscillatory circuits, and from time to time descriptions have been given of the occurrence of portions of negative slope in the volt-ampere characteristics of otherwise passive devices.

The first comprehensive account of the subject was published in 1935 by E. W. Herold[1]. It had been demonstrated by G. Crisson[2] that positive feedback over a telephone repeater gave rise to negative-resistance effects and that these differed in kind according as the feedback voltage was directly related to the output voltage or the output current of the circuit. It is of interest to observe that a similar distinction was discovered later in the behaviour of amplifiers subject to negative feedback. Herold used this distinction as a basis for the classification of known negative-resistance devices and for the analysis of their properties.

In the present paper a qualitative study of negative-resistance characteristics will be undertaken with the object of finding a more suitable element for analysis than the inadequate idealized resistor of conventional linear-network theory. The element so derived allows synthesis of actual characteristics by the superposition of graphically simple slope functions and leads to equivalent networks which facilitate the understanding of the physical behaviour of negative-resistance devices and their performance in actual networks.

An approximation to the ideal unilateral characteristic chosen is provided by the diode valve so that, if a diode having a negative slope in its volt-ampere characteristic existed, actual negative resistors could be represented by networks containing both kinds of diode. Negative resistance effects are experienced in valves having three or more electrodes; these are examined and found to be accurately representable by combinations of unilateral positive and negative elements or their diode equivalents.

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2. Actual Negative-Resistance Characteristics

The difference between the two kinds of negative resistance is clearly illustrated in a comparison of their volt-ampere characteristics; the examples shown in Fig. 1 are repeated from Herold's paper. The significant feature of the curves is their form, from which Herold deduced that the type shown at (a) is due to voltage control, while that at (b) indicates current as the controlling factor. Moreover he noted that the relationship between them is analogous to that between resistance and conductance; i.e., a reciprocal one.

It will be clear from Fig. 1 that the behaviour

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of the two kinds of resistor differs not over the range of negative slope but at the ends of this part of the characteristic. Accordingly, if the voltage range over which the negative slope occurs could be extended indefinitely the two types would become indistinguishable except in the magnitude of their slope. Conversely, if this range could be reduced to zero the negative resistance property would vanish. These two transformations leave the resistance purely negative and entirely positive respectively and suggests that the actual resistance contains both positive and negative elements which predominate alternately over the voltage axis in the voltage-controlled type and over the current axis in the current-controlled case ; also that the two types correspond to fundamentally different combinations of pure positive and negative portions.

A feature of both characteristics, whose importance has not been adequately recognized in the literature, is the inequality of the absolute (E/I) and differential (dE/dI) values of the resistor. This discrepancy implies non-linearity of the resistor (i.e., functional dependence upon voltage or current) and is an essential feature of resistors whose characteristics contain both positive- and negative-going portions. The characteristics possessed by certain screen-grid valves, as illustrated by Fig. 2, shows that both positive and negative absolute resistance may be accompanied by either positive or negative differential resistance (cf. points 1-4) at any given operating point.

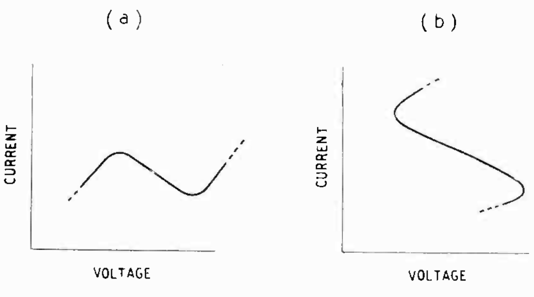

Fig. 1. *Negative-resistance characteristic due to voltage control (a) and current control (b).*

Over the current range occupied by the negative slope of Fig. 1(a) there are, within the defined range of the characteristic, three values of applied voltage corresponding to each value of current. Accordingly, stability requires that the negative resistor be energized by a constant-voltage generator (i.e., one of zero internal impedance) for, if sufficient resistance is present in the generator, the current drawn from it may cause the p.d. developed between the resistor terminals to shift over more than one of the permissible values. The required generator characteristic may be approached by application of negative voltage feedback ; a voltage proportional to the p.d. between the resistor terminals is applied to the voltage source in such a manner that fluctuations of its output voltage tend to be self-cancelling

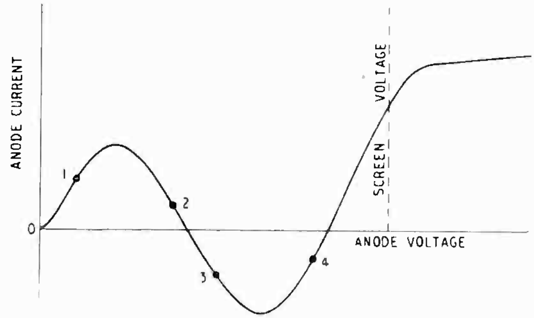

Fig. 2. *Characteristic of a screen-grid valve at constant control-grid voltage.*

and the output impedance is correspondingly reduced. Analogously, the multi-valued nature of the current in the characteristic of Fig. 1(b) over the voltage range occupied by the negative slope requires (ideally) a constant-current source (i.e., one of infinite internal impedance) such as may be approximated in practical generators by the application of heavy negative current feedback.

The foregoing brings out the need in an idealized negative resistor of the following characteristics :—

(i) Absence of portions in the volt-ampere characteristic having positive slope.

(ii) Possibility of inequality of absolute and differential values of resistance.

(iii) Capability of association with idealized positive resistors in such a manner that accurate equivalents of actual resistors may be synthesized. On the other hand the idealized resistor should have, at least to a fair approximation, a frequently occurring practical equivalent. Ideally the two types of pure resistance should be corresponding special cases of a more general idealized resistor.

3. Modified Ideal Positive Resistor

The volt-ampere characteristic of the idealized resistor of conventional network theory is essentially linear, the absolute and differential resistance values being identical for all values of impressed voltage or current. Analytically this restriction confines the characteristic to the first and third quadrants of the volt-ampere plane, for if it had a non-zero intercept with either axis the equality of actual and differential resistances would be destroyed and the resistor would be (electrically) non-linear. The behaviour of the linear resistor is bilateral (i.e., independent

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of the polarity of the impressed quantity), as shown in Fig. 3. The graphical criterion of bilaterality involves successive rotation of the volt-ampere characteristic about the axes $i = 0$, $e = 0$ (in either order), which should leave the characteristic unchanged, and is useful in the analysis of complex cases.

The simple characteristic just described is inadequate for the analysis of resistors of the type under consideration since any finite series-parallel combination of linear elements cannot produce a non-linear resultant. Moreover nothing is gained in this respect by assigning the idealized pure negative resistor a characteristic of the same type as Fig. 3 but with negative slope, as discussed, for example by D. M. Tombs[3].

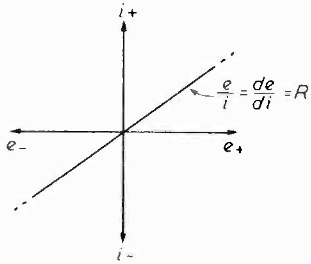


Fig. 3. *Characteristic of ideal positive resistor.*

Analytically the characteristic of Fig. 3 (or its negative-slope equivalent) needs only displacement from the origin to introduce electrical non-linearity but an element having such a characteristic is not available in useful two-terminal form since for physical reasons e and i must pass through zero together. As a result the intercept must be introduced effectively.

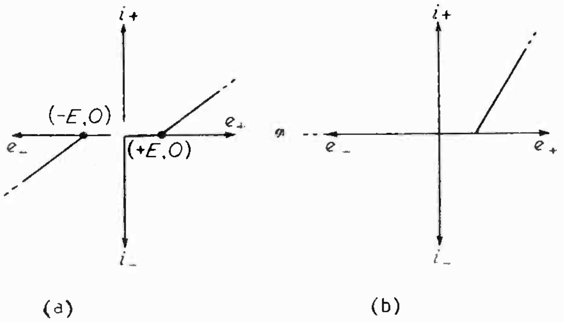
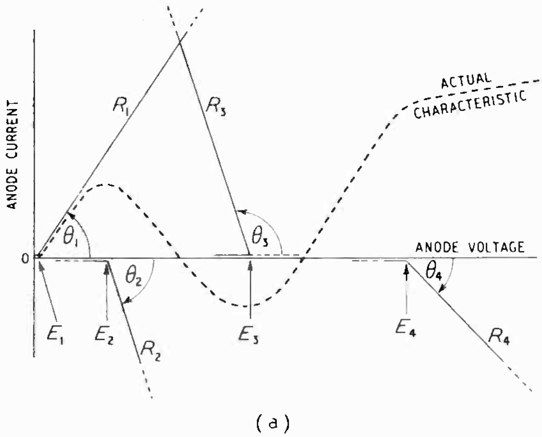


Fig. 4. *Symmetrical modified characteristic of a positive resistor (a), unilateral modified characteristic (b).*

The obvious method of allowing i to be a function containing higher powers of e than the first (or vice versa when i is the independent variable) will be discarded because of the difficulty, both graphically and analytically, of handling series-parallel combinations of such elements. Instead the characteristic of Fig. 3(a) will be modified by expanding the point at the origin over the segment of the abcissa bounded by the points $(-e, 0)$, $(+e, 0)$. It will be obvious that if this expansion is not performed symmetrically the bilateral property will be lost [See Fig. 4(a)]. On the other hand practical resistors generally have characteristics that are not bilaterally symmetrical and may even be entirely unilateral. Accordingly the 'broken' characteristic of Fig. 4(a) will be modified to the unilateral form shown at (b) to obtain a more versatile element which needs two parameters for its specification: (i) the potential at which the discontinuity occurs and (ii) the value (magnitude and sign) of its slope.



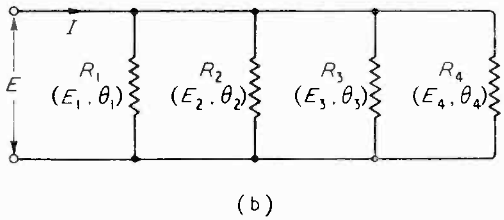


Fig. 5. *First-order synthesis of negative-resistance characteristic (a), and equivalent circuit (b). Where $|\Theta| > \pi/2$ [cf. R_3] it must be regarded as the resultant of two or more parallel components for each of which $|\Theta| < \pi/2$; this case should be avoided by taking the turn in two steps.*

4. Analysis of Complex Resistors

It will be clear that characteristics of the form of Fig. 1(a) are readily synthesized by the superposition of suitably chosen slope functions of the type shown in Fig. 4(b), while the current-controlled case illustrated by Fig. 1(b) may be dealt with similarly after inversion. In both cases the degree of approximation may be improved without limit by increasing the number of slope functions employed; fortunately it is worst at the turning points, which are avoided in practical applications for stability reasons.

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A first approximation to the characteristic of Fig. 2 requires four slope functions, two of each sign, as shown in Fig. 5(a). It will be noted that the method of synthesis just described involves only direct superposition of the basic functions; accordingly, actual negative resistors may be represented by simple combinations of the idealized unilateral resistor. In the voltage-controlled case the elements add in parallel to give the required resultant current, while in the current-controlled case the elements add in series to give the resultant voltage. The equivalent network so derived is shown for the first case in Fig. 5(b).

5. Analysis of Valve Characteristics

The unilateral element derived in Section 3 is physically appropriate to electronic circuits, since an approximate equivalent exists in the hard diode valve. This is shown in Fig. 6(a) together with the corresponding representation [Fig. 6(b)] of the bilateral characteristic of Fig. 4(a). The value of this equivalence is limited from the point of view of actual synthesis from separate elements by the fact that the negative counterpart of the ordinary (positive) diode is not available as a distinct element. On the other hand, it is of importance in bringing out the existence of such elements in association with positive diodes in more complex valves. Thus equivalent networks, containing, in general, both positive and negative diode constituents, may be derived for the more complex valves and, in reducing the latter to combinations of the same basic elements, bring out clearly their physical behaviour and relative characteristics and point to the possibility of replacing the more complex types by two or more simpler types.*

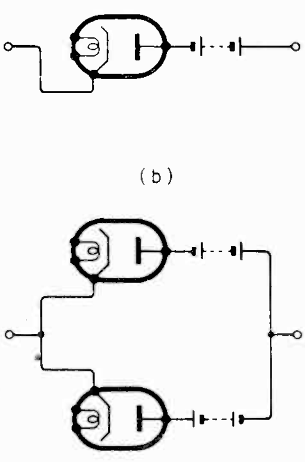

Fig. 6. *Illustrating the use of diodes to obtain the characteristic of Fig. 4.*

* It may be mentioned that the characteristic of Fig. 4(b) has frequently been used in the analysis of networks containing diodes, and the analytical difficulty of dealing with the discontinuity has been overcome by the use of suitable transforms. Thus, the unilateral characteristic may be represented by the product of a step voltage of the form of Heaviside's Unit Function and the characteristic of Fig. 3. Alternatively, where the unilateral resistor need be represented over a limited voltage range the characteristic of Fig. 3 may be multiplied by a square wave, thereby allowing Fourier Series rather than Fourier Integral analysis[4, 5]. Again, the discontinuity may be taken out by Laplace transformation, which (apart from a pole at zero) converts the function into a relatively simple algebraic function of the complex variable[6].

When a positive (with respect to cathode) potential is applied to an electrode of a valve, current is drawn from an interior space charge together with corresponding components initiated by other electrodes, the whole forming the total 'space' current. On approaching the electrode concerned this component may be absorbed (i.e., collected), reflected (or effectively so, as when secondary emission occurs) or transmitted (i.e., allowed to pass on to a succeeding electrode), as illustrated by Fig. 7. In an equivalent network both reflection and transmission require negative admittance to be attributed to the paths taken by the reflected and transmitted components.

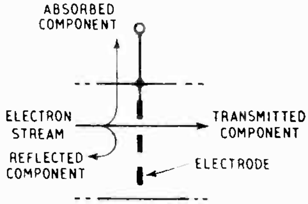

Fig. 7. *The components of electron current in a valve are shown.*

It is interesting to compare the network of Fig. 5(c) with the actual valve whose anode characteristic it represents. The tetrode concerned may, from the point of view of the anode-cathode terminals, be regarded as a complex 'diode'; i.e., one whose characteristics are modified in a (generally) complex manner by the introduction of the intermediate electrodes. Accordingly, the equivalent network of idealized positive and negative diodes and their associated generators may be regarded as a transformation of the actual 'diode.' In this respect the slope analysis provides an aid to the understanding of the physical behaviour of the tetrode valve.

The possibility of extending this method of treatment to three-terminal devices may be demonstrated by using the triode valve as an example. Normally the grid is biased negatively to prevent grid current flow and the entire space current is collected by the anode. To a first approximation the currents are:

$$i_a = (\mu/r_a)V_{cg} + (1/r_a)V_{ca} \quad [i_a \geqslant 0]$$

$$i_g = 0.$$

The anode current may, therefore, be resolved into two unilateral slope functions

$$i_a(g) = (\mu/r_a)\,V_{cg} \quad [i_a(g) \geqslant 0]$$

$$i_a(a) = (1/r_a)\,V_{ca} \quad [i_a(a) \geqslant 0]$$

and, correspondingly, the valve contains two positive diodes, one between cathode and grid and the other between cathode and anode, the former having a slope μ times that of the latter. It has been shown[7] that, ignoring space-charge considerations,

$$\mu = c_{cg}/c_{ca}$$

where c_{cg} = cathode-grid capacitance

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and c_{ca} = cathode-anode capacitance.

In addition, the failure of the grid to collect current requires the connection of a third element, a negative 'image' of the grid-cathode diode, between grid and anode. The complete equivalent network is shown in resistance form as Fig. 8. It differs from conventional equivalents in being of transmission form.

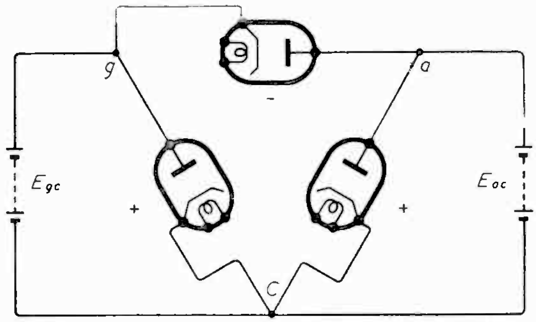

Fig. 8. *Diode equivalent circuit of a triode valve. The hypothetical "negative" diode assumes such an impedance value that the required operating condition (usually zero current) obtains in the grid-cathode circuit.*

When the triode is employed as an amplifier the steady potentials (V_{cg}, V_{ca}) applied to its electrodes are chosen to prevent grid current flow and keep anode current from falling to the discontinuity in the anode characteristic on negative excursions of the input signal. Moreover the anode characteristic not being linear [it is more accurately represented by

$$i_a = \{(\mu/r_a)\,V_{cg} + (1/r_a)\,V_{ca}\}^{3/2}$$

than by the previous equation] and the transition from anode current cut-off to the 'linear' regime being, relatively gradual, there is, consequently, an optimum mean anode current for minimum non-linearity and the signal input must be restricted in amplitude to keep distortion to small proportions. Under these conditions the r_a used in the above relationships will differ in its absolute and differential values but, provided the latter is preferred and the valve operated substantially linearly the network of Fig. 8 may also be employed to represent the alternating-current behaviour of the valve. Since the anode circuit is loaded it is necessary to add in the grid-anode branch an impedance of such a value that the zero grid-current condition is preserved. For the conventional amplifier (i.e., grid input–anode output) this value will be the parallel resultant of r_a and the load impedance. This additional element, being characteristic of the external network rather than of the valve itself may be termed the 'reflected' component of the grid-anode impedance. Its presence may be shown to be necessary in maintaining zero total impedance around the mesh formed by the valve elements and the external impedances.

6. Conclusion

The methods and ideas introduced in the foregoing are capable of extension to more complex valve types than the triode and lead to developments which, it is hoped, will be dealt with in subsequent papers.

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X-CUT QUARTZ CRYSTAL

Equivalent Circuit as an Ultrasonic Generator

By F. M. Leslie, M.Sc., Ph.D.

(*Electrical Engineering Dept., Leeds University*)

SUMMARY.—A new equivalent circuit for a crystal driving a mechanical load is devised in terms of an electrical transmission line instead of the usual lumped components of inductance and capacitance.

Introduction

THE general theoretical treatment of crystal ultrasonic generators has received a fair amount of attention and the presentation of the equivalent electrical circuit in terms of lumped components of inductance and capacitance is now well known. For understanding the behaviour of the generator the usual equivalent circuit is perhaps not quite so simple as it might be, and with this in mind a new equivalent circuit has been devised which lends itself very readily to the conditions under which the crystal operates when employed for ultrasonic generation.

List of Symbols

C_0	=	capacitance of the crystal when clamped.
E	=	voltage applied to the crystal.
f, f_0	=	frequency.
I	=	current to the crystal.
j	=	$\sqrt{-1}$.
m	=	a constant.
S	=	stress in the plane distance x from one face.
S_1, S_2	=	stress at the crystal face.
t	=	crystal thickness.
v	=	velocity of propagation.
x	=	distance measured from one face of the crystal.
Z_m, Z_{me}	=	electrical impedance.
z_0	=	characteristic impedance of the crystal.
z_1, z_2	=	mechanical load impedance.
β	=	phase shift constant.
ξ'	=	velocity of the plane distance x from one face.
ξ'_1, ξ'_2	=	velocity of the crystal face.

Equivalent Electrical Circuit

The general theoretical treatment of crystals is adequately dealt with in the literature[1, 2, 3] and the fundamental equations which will be employed in determining the equivalent circuit for a crystal having unit area are as follows, the losses in the crystal being ignored :—

$$\xi' = \xi'_1 \cos \beta x - j\left(\frac{S_1 + mE}{z_0}\right) \sin \beta x \quad .. \quad (1)$$

$$(S + mE) = (S_1 + mE) \cos \beta x - j\xi'_1 z_0 \sin \beta x \quad .. \quad .. \quad .. \quad .. \quad (2)$$

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$$I = j2\pi f C_0 E + m(\xi_1' - \xi'_2) \quad .. \quad .. \quad (3)$$

Fig. 1 depicts a typical crystal, the electrodes being on the major surfaces and the distance x measured to the right of the left-hand face. As is usual in the development of the equivalent lumped circuit, the presence of subsidiary vibrational modes will be ignored. If z_1 and z_2 are the mechanical loads on the two faces then $-\xi'_1 S_1 = z_1$ at $x = 0$ and $\xi'_2 S_2 = z_2$ at $x = t$, thus equation (2) may be written

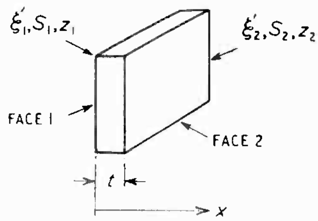

Fig. 1. *Form of typical crystal.*

$$\xi'_2 = \frac{z_1}{z_2} \xi'_1 \cos \beta t + \frac{mE}{z_2} (\cos \beta t - 1) - j\xi'_1 \frac{z_0}{z_2} \sin \beta t \quad .. \quad .. \quad .. \quad (4)$$

Putting $x = t$ in equation (1) and then combining with equation (4),

$$\xi'_1 = \frac{\dfrac{mE}{z_2}(\cos \beta t - 1) + \dfrac{jmE}{z_0} \sin \beta t}{(1 + z_1/z_2) \cos \beta t + j\,(z_1/z_0 + z_0/z_2) \sin \beta t} \quad .. \quad .. \quad .. \quad (5)$$

From equation (1)

$$(\xi'_1 - \xi'_2) = \xi'_1\,[(1 - \cos \beta t) - jz_1/z_0 \sin \beta t] + j\frac{mE}{z_0} \sin \beta t \quad .. \quad .. \quad .. \quad (6)$$

Substituting equation (5) in equation (6),

$$(\xi'_1 - \xi'_2) = mE\left[\frac{2\,(\cos \beta t - 1) + j\,(z_2/z_0 + z_1/z_0) \sin \beta t}{(z_1 + z_2) \cos \beta t + j\left(z_0 + \dfrac{z_1 z_2}{z_0}\right) \sin \beta t}\right] \quad (7)$$

It is seen from equation (3) that the input impedance to the crystal can be considered as made up of two components in parallel. One component due to the capacitance C_0 and the other due to the motion of the crystal, this latter component will be termed the motional impe-

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dance. From equation (7) it follows that the motional impedance for unit area of the crystal will be given by,

$$Z_m = \frac{E}{m(\xi'_1 - \xi'_2)} =$$

$$\frac{1}{m^2}\left[\frac{(z_1 + z_2)\cos\beta t + j(z_0 + z_1 z_2/z_0)\sin\beta t}{2(\cos\beta t - 1) + j(z_2/z_0 + z_1/z_0)\sin\beta t}\right] \quad (8)$$

or

$$Z_m = \frac{z_0}{m^2}\left[\frac{(z_1 + z_2) + j\left(z_0 + \frac{z_1 z_2}{z_0}\right)\tan\beta t}{2z_0\left(\frac{\cos\beta t - 1}{\cos\beta t}\right) + j(z_1 + z_2)\tan\beta t}\right] \quad (9)$$

Consider now the input impedance to an electrical transmission line of length t having a phase shift constant β, characteristic impedance $z_0/4m^2$ and terminated by a load $\frac{z_1 + z_2}{4m^2}$. The input impedance to the line would be,

$$Z_{me} = \frac{z_0}{m^2}\left[\frac{(z_1 + z_2) + jz_0\tan\beta t}{4z_0 + 4j(z_1 + z_2)\tan\beta t}\right] \quad .. (10)$$

Suppose now that $\beta t \approx \pi$ and further that,

$$z_0 \gg \frac{z_1 z_2}{z_0}$$

$$z_0 \gg (z_1 + z_2)\tan\beta t$$

then equations (9) and (10) become identical, equation (9) reducing to

$$Z_m = \frac{1}{4m^2}[(z_1 + z_2) - jz_0\Delta\theta] \quad .. \quad .. (11)$$

where

$$\Delta\theta = \frac{2\pi\Delta f t}{v}$$

$$\Delta f = f_0 - f$$

and the resonant frequency f_0 is given by

$$f_0 = v/2t$$

With regard to the requirement that $z_0 \gg z_1 z_2/z_0$ this is readily fulfilled in the case of a quartz crystal, and for example, water loads [$z_1 = z_2 = 1.5 \times 10^5 g/(cm^2\ sec)$, $z_0 = 14.4 \times 10^5\ g/(cm^2\ sec)$], further, as $\beta t \approx \pi$ the second condition will also be satisfied. For an X-cut quartz crystal the constant m has a value $0.528 \times 10^5/t$ where t is in cm, expressing z_0, z_1 and z_2 in $g/(cm^2\ sec)$, then Z_m is obtained in statohms.

Thus, from the equivalence of equations (9) and (10) it follows that the behaviour of the loaded crystal in the vicinity of the resonant frequency follows very closely that of an electrical transmission line, provided the load impedances are not too great. The equivalent circuit in terms of a line is shown in Fig. 2.

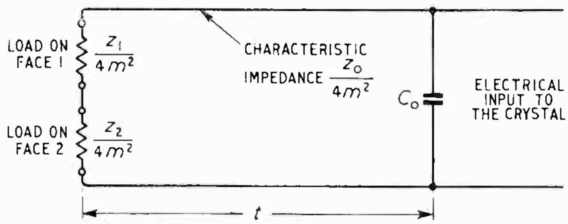

Fig. 2. *Equivalent transmission-line circuit for the crystal near resonance.*

Conclusion

An approximate equivalent circuit in terms of an electrical transmission line has been devised for the X-cut quartz crystal as used for ultrasonic generation. The equivalent circuit behaviour is almost identical with that for the crystal in the vicinity of resonance, provided the crystal-face loads are small, as they would be in the case of liquid loads. The equivalent circuit is considerably simpler than the usual lumped arrangement, and the effects of non-resonant operation or the coating of the crystal major surfaces with a thin metallic layer are readily interpreted from it.

Acknowledgment

The writer desires to express his thanks to Professor Carter for suggestions in the preparation of the paper, and to Imperial Chemical Industries Ltd. for their Fellowship.

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TRANSIENT RESPONSE CALCULATION

Use of Poisson Probability Tables

By J. E. Flood, B.Sc.(Eng).

THE usual criterion adopted for judging the transient response of an amplifier circuit is the response at the output of the amplifier when the input voltage is a unit step (i.e., a voltage which is zero when $t<0$ and is unity when $t>0$). The response to an input waveform of any other shape can, in general, be determined by means of Duhamel's integral[1]. The response to a unit step of some simple amplifier circuits having a number of identical stages can be expressed in terms of the Poisson exponential probability summation. Tables and charts of this function have been published, thus simplifying the calculation of the response of the amplifier circuits.

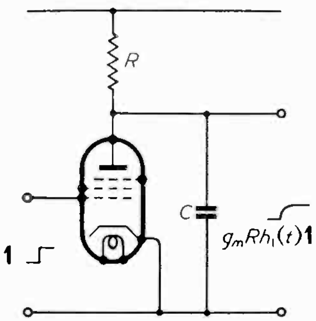

Fig. 1. *Simple resistance-coupled stage.*

(*Fig. 2 is on page 183.*)

1. Resistance-coupled Amplifier

The single-stage resistance-coupled pentode amplifier shown in Fig. 1 has an anode load resistance R, and C is the total stray capacitance between anode and earth.

The Heaviside operational expression for the output voltage when the input voltage is a unit step is

$$g_mR\,h_1(p)\,\mathbf{1} = g_m\frac{R}{1+pCR}\mathbf{1} = g_mR\frac{\alpha}{p+\alpha}\mathbf{1} \quad \ldots \quad (1)$$

where $\alpha = 1/RC$

This gives the well known result for the response as a function of time

$$g_mR\,h_1(t)\,\mathbf{1} = g_mR\,(1-e^{-\alpha t}) \text{ for } t>0 \ldots \quad (2)$$

The response to a unit step of an amplifier having n similar stages is

$$(g_mR)^nh_n(p)\,\mathbf{1} = [g_mR\,h_1(p)]^n\,\mathbf{1}$$
$$= (g_mR)^n\left(\frac{\alpha}{p+\alpha}\right)^n\mathbf{1} \quad \ldots \quad (3)$$

It is shown in Appendix 1 that the response as a function of time is

$(g_mR)^n\,h_n(t)\,\mathbf{1}$ where

$$h_n(t) = 1 - e^{-\alpha t}\left(1 + \frac{\alpha t}{1!} + \frac{\alpha^2t^2}{2!} + \ldots + \frac{\alpha^{n-1}t^{n-1}}{(n-1)!}\right)$$
$$= 1 - e^{-\alpha t}\sum_{r=0}^{n-1}\frac{\alpha^rt^r}{r!} \quad \ldots \quad (4)$$

Equation (4) is the expansion of the incomplete Gamma Function

$$\frac{1}{(n-1)!}\int_0^{\alpha t} e^{-\alpha t}(\alpha t)^{n-1}\,d(\alpha t)$$

which has been tabulated by Pearson[2] for values of n up to 50. Equation (4) is also of the same form as the Poisson exponential probability summation.

It is shown in Appendix 2 that the probability $\{p_r(a)\}$ that an event will occur exactly r times in a large number of trials, for which a is the average number of occurrences, is

$$p_r(a) = \frac{a^r}{r!}e^{-a} \quad \ldots \quad (5)$$

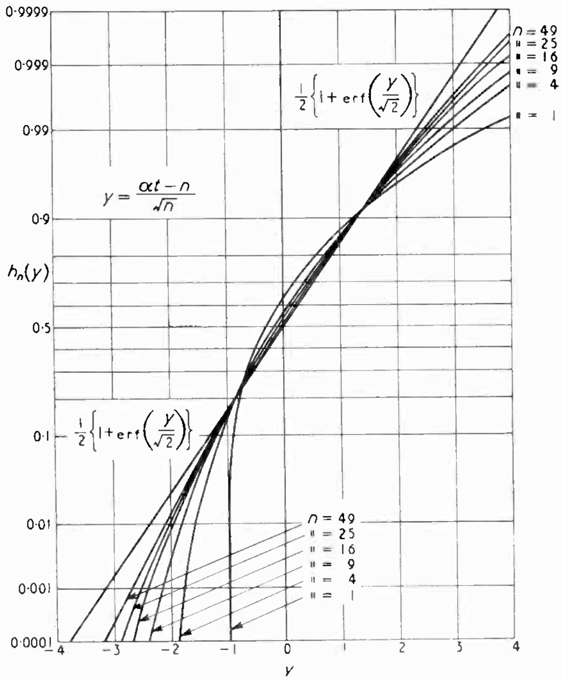

Fig. 3. *Response to a unit step of a resistance-coupled n-stage amplifier.*

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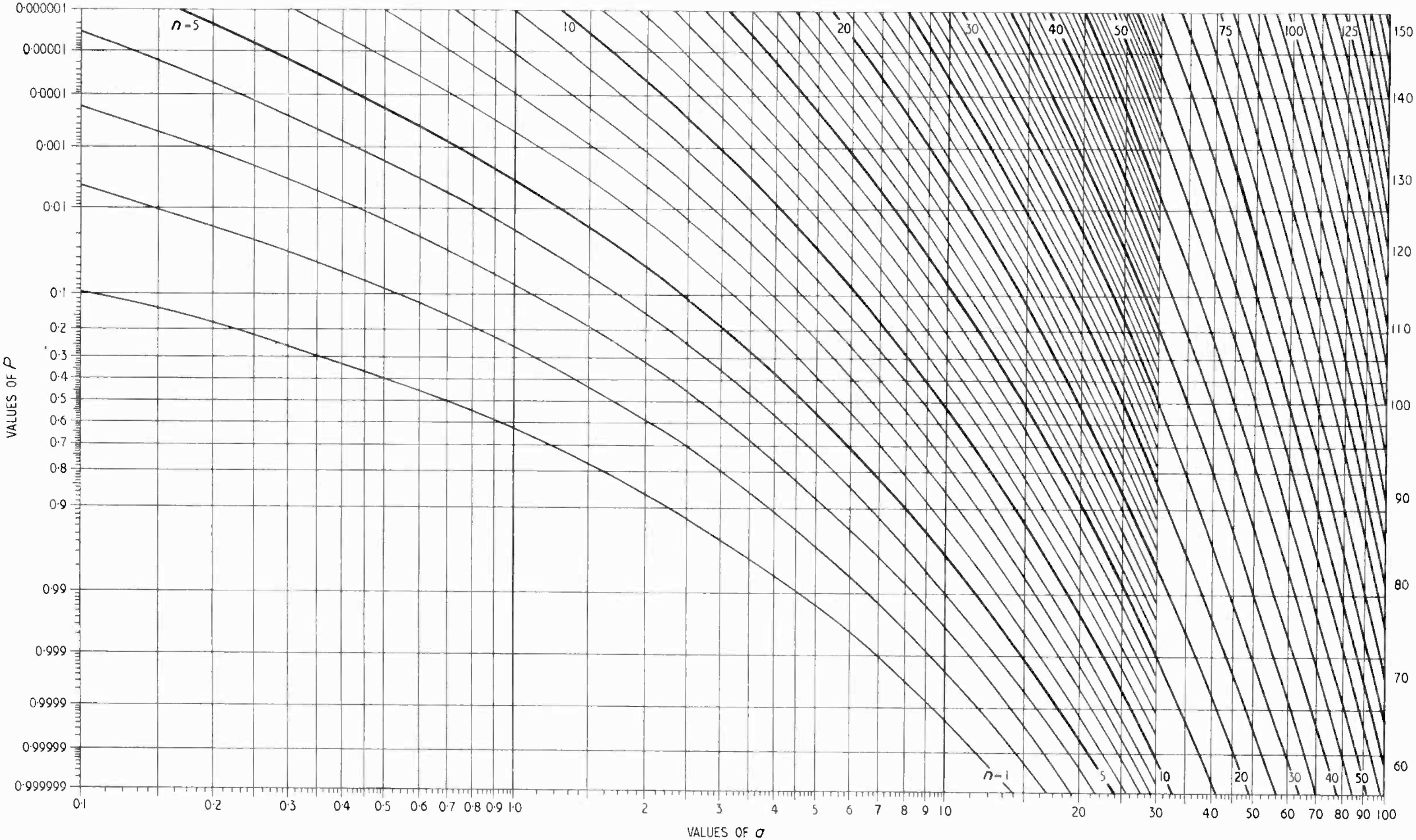


Fig. 2. *Probability curves showing Poisson's exponential summation* $P = 1 - \left[1 + \frac{a}{1!} + \frac{a^2}{2!} + \ldots + \frac{a^{n-1}}{(n-1)!}\right] e^{-a}$ *for the probability* **P** *that an event occurs at least n times in* **a** *large group of trials for which the average number of occurrences is* **a**. *A scale proportional to the normal probability integral is used for* **P**, *a logarithmic scale for* **a**.

and the probability that the event will occur at least n times is given by the summation

$$P_n(a) = \sum_{r=0}^{n-1} p_r(a)$$

$$= 1 - e^{-a}\left(1 + \frac{a}{1!} + \frac{a^2}{2!} + \ldots + \frac{a^{n-1}}{(n-1)!}\right) \quad (6)$$

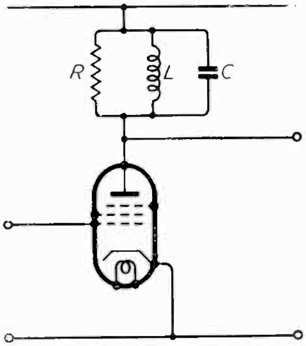
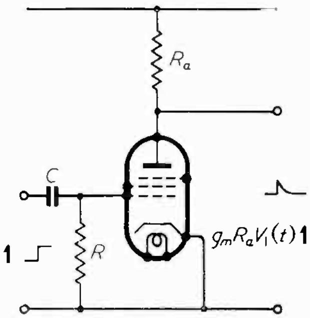


Fig. 4 (above). *Single tuned-circuit coupling.*

Fig. 5 (below). *Resistance-capacitance coupling.*

Fig. 6 (right). *Response to a unit step of an n-stage resistance-capacitance coupled amplifier.*



The expression for the response to a unit step of the n-stage resistance-coupled amplifier is seen to be identical in form with the Poisson exponential summation and so can be evaluated from the tables by Molina[3] or charts by Campbell[4]. Molina's tables give the values of both p_n and P_n. The chart reproduced as Fig. 2 (on page 183). enables the response of amplifiers to be simply determined for any number of stages up to 150 and values of αt up to 100.

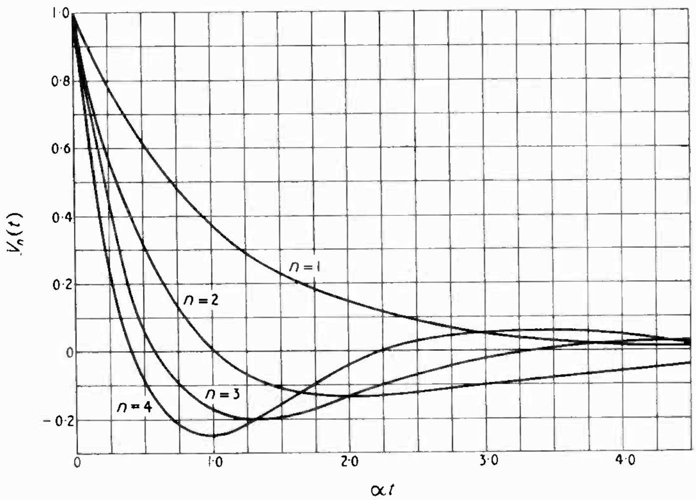
It is shown in Appendix 5 that when the number of stages is large, the response to a unit step is approximately given by

$$h_n(t) \approx \tfrac{1}{2}\left\{1 + \operatorname{erf}\left(\frac{\alpha t - n}{\sqrt{n}}\right)\right\}$$

Tables of the error function can be found in text-books on statistics[12, 13] and enable the response to be calculated approximately when the number of stages is large. Fig. 3 shows the above equation and the response calculated from equation (4) for different numbers of stages: the error function expression is seen to represent the response with increasing accuracy as the number of stages is increased.

2. Amplifier With Single-tuned Circuits

The circuit of the amplifier shown in Fig. 4 can be obtained from the circuit of Fig. 1 by means of the low-pass to band-pass transformation.[5] The envelope of the response of the band-pass circuit to an input of its mid-band frequency, amplitude-modulated by a unit step, is approximately the response to a unit step of the equivalent low-pass circuit. It has been shown[6] that the envelope of the response of



an amplifier with n identical tuned circuit stages to an input of their natural frequency, modulated by a unit step, is also given by equation (4), where $\alpha = 1/2RC = \omega_0/2Q$. The response can therefore be rapidly determined by means of the tables or charts.

3. Stages of Resistance-Capacitance Coupling

It is well known that the response to a unit step of the coupling circuit shown in Fig. 5 is given by

$$\left.\begin{aligned} v_1(t)\,\mathbf{1} &= \left[1 - h_1(t)\right]\mathbf{1} = e^{-\alpha t}\,\mathbf{1} \\ v_1(p)\,\mathbf{1} &= \mathbf{1} - h_1(p)\,\mathbf{1} = \frac{p}{p+\alpha}\,\mathbf{1} \end{aligned}\right\} \quad \ldots \quad (7)$$

where $\alpha = 1/RC$

The response of an n-stage resistance-capacitance coupled amplifier to an input consisting of a unit step is $(g_m R_a)^n v_n(t)\,\mathbf{1}$

where $v_n(p)\,\mathbf{1} = \left(\dfrac{p}{p+\alpha}\right)^n \mathbf{1}$

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It is shown in Appendix 3 that

$$v_n(t) = e^{-\alpha t} \sum_{r=0}^{n-1} {}_{n-1}C_r\,(-1)^r \frac{\alpha^r t^r}{r!} \quad \ldots \quad (8)$$

$$= \sum_{r=0}^{n-1} {}_{n-1}C_r\,(-1)^r\, p_r(\alpha t) \quad \ldots \quad (9)$$

also
$$v_n(t) = 1 + \sum_{r=1}^{n} {}_nC_r\,(-1)^r\, h_r(\alpha t) \quad \ldots \quad (10)$$

$$= 1 + \sum_{r=1}^{n} {}_nC_r\,(-1)^r\, P_r(\alpha t) \quad \ldots \quad (11)$$

where ${}_nC_r$ is the binomial coefficient

$$\frac{n\,.\,(n-1)\,.\,.\,.\,(n-r+1)}{r!}$$

Equation (9) is convenient for computation from the tables of the individual terms $p_r(a)$ and equation (11) is convenient for computation from the curves shown in Fig. 2. Fry[13] gives tables of the binomial coefficient ${}_nC_r$ for values of n up to 100. Fig. 6 shows $v_n(t)$ for values of n up to 4.

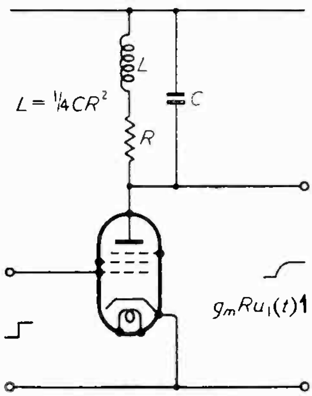

Fig. 7. *Critically-damped inductance - compensated stage.*

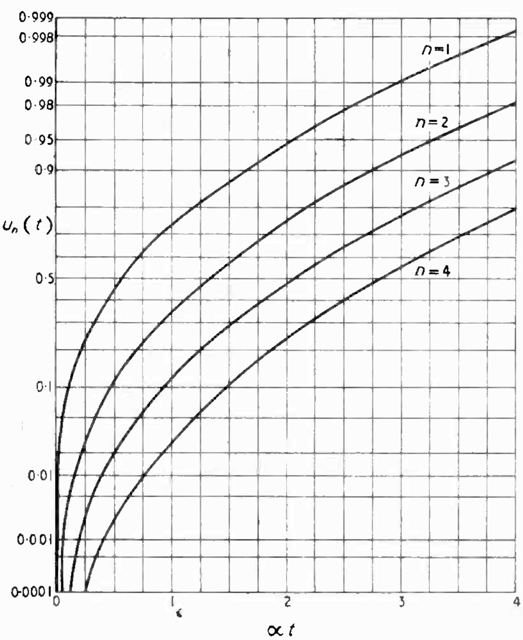

Fig. 8. *Response to a unit step of a critically-damped inductance-compensated n-stage amplifier.*

Equation (7) can often be used to represent the error in the output of a servo-mechanism, such as an automatic gain control, when a sudden change is made in the input.[11] Equations (8) and (10) therefore represent the error in the output of the *n*th servo of a chain of servo-mechanisms connected in tandem. For instance, if a sudden change is made in the input level of the pilot signal to a carrier telephone line containing *n* repeaters, each fitted with a pilot-operated automatic gain control, the change in the level of the output signal is given by $v_n(t)$. Although each servo operates without any overshoot when alone, the output from the *n*th servo-mechanism overshoots $n-1$ times before reaching the correct value.

4. Critically-damped Inductance-compensated Amplifier

The circuit is shown in Fig. 7. It has been shown that the response to a unit step has the quickest build-up which is possible without overshoot when $L = \frac{1}{4}CR^2$, which is the condition

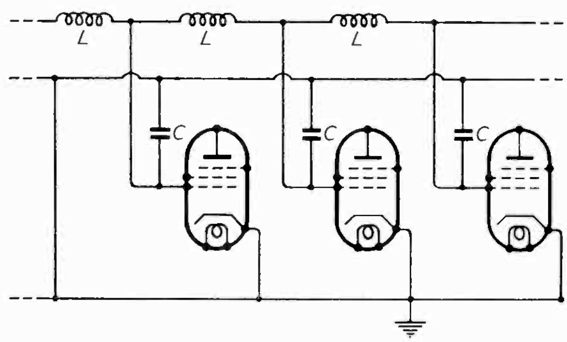

Fig. 9. *Filter-coupled amplifier.*

for critical damping. The response to a unit step is then[7]

$$g_m R\, u_1(t)\,\mathbf{1} = g_m R\,[1 - e^{-2\alpha t}(1 + \alpha t)]\,\mathbf{1} \qquad (12)$$

where $\alpha = 1/CR = R/4L$

it is shown in Appendix 4 that the response of an amplifier having n identical stages is

$$u_n(t)\,\mathbf{1} = 1 - e^{-2\alpha t} \sum_{r=0}^{2n-1} {}_nb_r \frac{\alpha^r t^r}{r!},\; t > 0 \quad \ldots \quad (13)$$

where ${}_nb_0 = 1$, ${}_nb_{2n-1} = 2^n$,

and ${}_nb_r = 2_{n-1}\,b_{r-2} + {}_{n-1}b_{r-1}$

$$\therefore u_n(t) = 1 - e^{-\alpha t} \sum_{r=0}^{2n-1} {}_nb_r\, p_r(\alpha t) \quad \ldots \quad \ldots \quad (14)$$

The values of $p_r(t)$ are readily obtainable from the tables and ${}_nb_r$ is easily calculated. Table 1 gives the values of ${}_nb_r$ for values of n up to 8 and Fig. 8 shows $u_n(t)$ for values of n up to 4.

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5. A Constant-resistance Network

When a number of valves are to be connected together, a method of avoiding connecting all the valve capacitances in parallel is to connect a low-pass filter section between each pair of valves, the valve capacitances forming the shunt arms of the filter, as shown in Fig. 9. In order to make the delay distortion fairly small within the pass band, mutual inductance between adjacent coils is sometimes used to provide a mid-series derived type of filter[8]. The response to a unit step of any practical dissipationless filter circuit is, however, oscillatory[9], and so is the response of an 'ideal' filter[10].

In some applications it is necessary for the response of the coupling network to be free from overshoot, and a simple way of achieving this is to connect damping resistors across the coils of the filter. If a resistance equal to the design resistance of the network is connected in parallel with each inductance, the circuit of Fig. 9 becomes equivalent to a number of the constant-resistance sections shown in Fig. 10 connected in tandem. The response to a unit step of the constant-resistance network of Fig. 10 is

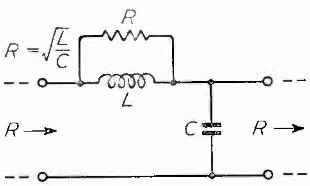

Fig. 10. *Constant-resistance network.*

$$\phi_1(p)\,\mathbf{1} = \frac{1/pC}{R + 1/pC}\mathbf{1} = \frac{1}{1 + pCR}\mathbf{1}$$

$$= \frac{\alpha}{p + \alpha}\mathbf{1} \text{ where } \alpha = 1/RC$$

$$= h_1(p)\,\mathbf{1} \text{ (equation 1)}$$

The response to a unit step of the constant-resistance network is seen to be the same as the response of a single-stage resistance-coupled amplifier. The response at the end of the nth section of the coupling network is therefore the same as the response of an n-stage resistance-coupled amplifier and is readily calculated from equation (4) by means of the tables or charts.

TABLE I

Coefficients ${}_nb_r$

r \ n	1	2	3	4	5	6	7	8
1	1	2	2	2	2	2	2	2
2		3	4	4	4	4	4	4
3		2	7	8	8	8	8	8
4			8	15	16	16	16	16
5			4	22	31	32	32	32
6				20	52	63	64	64
7				8	64	114	127	128
8					48	168	240	255
9					16	176	396	494
10						112	512	876
11						32	464	1304
12							256	1488
13							64	1184
14								576
15								128

Acknowledgment

Acknowledgment is made to the Engineer-in-Chief of the G.P.O. for permission to publish this paper. The writer's thanks are due to Mr. W. E. Thomson for his help and advice.

APPENDIX 1

Response to a Unit Step of a Resistance-coupled Amplifier

Let the response to a unit step of an amplifier having n similar stages be $(g_mR)^n\,h_n(t)\,\mathbf{1}$

Assume that $h_n(t) = 1 - e^{-\alpha t}\left[1 + \frac{\alpha t}{1!} + \frac{\alpha^2 t^2}{2!} + \ldots\ldots + \frac{\alpha^{n-1} t^{n-1}}{(n-1)!}\right]$

$$= 1 - e^{-\alpha t}\sum_{r=0}^{n-1}\frac{\alpha^r t^r}{r!} \qquad \ldots \qquad \ldots \qquad (4)$$

$$\therefore h_{n+1}(t)\,\mathbf{1} = \frac{\alpha}{p+\alpha}\cdot h_n(t)\,\mathbf{1}$$

$$= \frac{\alpha}{p+\alpha}\left\{1 - e^{-\alpha t}\left[1 + \frac{\alpha t}{1!} + \frac{\alpha^2 t^2}{2!} + \ldots + \frac{\alpha^{n-1}t^{n-1}}{(n-1)!}\right]\right\}\mathbf{1}$$

But $f(p)\,e^{-\alpha t}g(t)\,\mathbf{1} = e^{-\alpha t}f(p-\alpha)\;g(t)\,\mathbf{1}$ by the shifting theorem[1]

$$\therefore h_{n+1}(t) = \frac{\alpha}{p+\alpha}\mathbf{1} - e^{-\alpha t}\frac{\alpha}{p}\left[1 + \frac{\alpha t}{1!} + \frac{\alpha^2 t^2}{2!} + \ldots + \frac{\alpha^{n-1}t^{n-1}}{(n-1)!}\right]\mathbf{1}$$

But $\frac{\alpha^r t^r}{r!}\mathbf{1} = \frac{\alpha^r}{p^r}\mathbf{1}$

$$\therefore h_{n+1}(t)\,\mathbf{1} = \frac{\alpha}{p+\alpha}\mathbf{1} - e^{-\alpha t}\frac{\alpha}{p}\left[1 + \frac{\alpha}{p} + \frac{\alpha^2}{p^2} + \frac{\alpha^3}{p^3} + \ldots + \frac{\alpha^{n-1}}{p^{n-1}}\right]\mathbf{1}$$

$$= \frac{\alpha}{p+\alpha}\mathbf{1} - e^{-\alpha t}\left[\frac{\alpha}{p} + \frac{\alpha^2}{p^2} + \frac{\alpha^3}{p^3} + \ldots\ldots + \frac{\alpha^n}{p^n}\right]\mathbf{1}$$

$$= (1 - e^{-\alpha t})\,\mathbf{1} - e^{-\alpha t}\sum_{r=1}^{n}\frac{\alpha^r t^r}{r!}\mathbf{1}$$

$$\therefore h_{n+1}(t) = 1 - e^{-\alpha t}\sum_{r=0}^{n}\frac{\alpha^r t^r}{r!}$$

$\therefore$ if equation (4) is valid for n stages, it is valid for $n+1$ stages. But it is well known for $n = 1$ and is simply proved for $n = 2$ and 3; therefore, it is valid for any number of stages.

APPENDIX 2

The Poisson Exponential Probability Summation

If the probability that an event will occur in any one trial is p_1 and a large number (m) of trials is made, then the average number of events (a) occurring in m trials is mp_1.

The probability that the event will occur in each of r trials is $p_1{}^r$ and the probability that the event will fail to occur in the remaining m-r trials is $(1-p)^{(m-r)}$. But the number of ways in which the event may happen exactly r times in m trials is ${}_mC_r = \frac{m(m-1)(m-2)\ldots(m-r+1)}{r!}$

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$\therefore$ The probability (p_r) that the event will happen exactly r times in m trials is

$$p_r = {}_mC_r\, p_1{}^r\, (1 - p_1)^{m-r}$$

But $p_1 = a/m$

$$\therefore p_r = {}_mC_r \left(\frac{a}{m}\right)^r \left(1 - \frac{a}{m}\right)^{m-r}$$

$$= \frac{m(m-1)(m-2)\ldots(m-r+1)}{r!} \cdot \frac{a^r}{m^r}\left(1 - \frac{a}{m}\right)^{m-r}$$

$$= \frac{a^r}{r!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\ldots\left(1 - \frac{r-1}{m}\right)\left(1 - \frac{a}{m}\right)^{-r}\left(1 - \frac{a}{m}\right)^{m}$$

Since m is very large, the terms inside each bracket except the last tend to unity

$$\therefore p_r = \frac{a^r}{r!}\left(1 - \frac{a}{m}\right)^m$$

But when m is very large, $\left(1 - \frac{a}{m}\right)^m \rightarrow e^{-a}$

$$\therefore p_r = \frac{a^r}{r!} e^{-a} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

which gives the probability that the event will occur exactly r times out of a large number of trials for which the average number of occurrences is a.

Let the probability that the event will occur at least n times in a large number of trials be P_n.

$$\therefore P_n = p_n + p_{n+1} + p_{n+2} + \ldots\ldots$$

$$= \sum_{r=n}^{\infty} p_r$$

But $\sum_{q=0}^{\infty} p_r$, the sum of all the possible chances, is equal to unity.

$$\therefore P_n = 1 - \sum_{r=0}^{n-1} p_r = 1 - e^{-a} \sum_{r=0}^{n-1} \frac{a^r}{r!} \qquad \ldots \quad \ldots \quad (6)$$

APPENDIX 3

Resistance-capacitance Coupled Stages

The response to a unit step of a single coupling network is

$$\left.\begin{aligned} v_1(t)\, \mathbf{1} &= [1 - h_1(t)]\, \mathbf{1} = e^{-\alpha t}\, \mathbf{1} \\ v_1(p)\, \mathbf{1} &= \mathbf{1} - h_1(p)\, \mathbf{1} = \frac{p}{p+\alpha}\, \mathbf{1} \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (7)$$

$\therefore$ The response of an n-stage resistance-capacitance coupled amplifier is

$$(g_m R_a)^n\, v_n(t)\, \mathbf{1}$$

$$\text{where } v_n(p)\, \mathbf{1} = \left(\frac{p}{p+\alpha}\right)^n \mathbf{1}$$

But by the shifting theorem, $f(p)\, \mathbf{1} = e^{-\alpha t} \frac{p}{p-\alpha} f(p-\alpha)\, \mathbf{1}$

$$\therefore v_n(p)\, \mathbf{1} = e^{-\alpha t} \frac{p}{p-\alpha} \frac{(p-\alpha)^n}{p^n}\, \mathbf{1}$$

$$= e^{-\alpha t}\left(1 - \frac{\alpha}{p}\right)^{n-1} \mathbf{1}$$

$$= e^{-\alpha t}\left[\mathbf{1} - {}_{n-1}C_1 \frac{\alpha}{p} + {}_{n-1}C_2 \frac{\alpha^2}{p_2} + \ldots\right.$$

$$\left. + (-1)^r\, {}_{n-1}C_r \frac{\alpha^r}{p^r} + \ldots (-1)^{n-1} \frac{\alpha^{n-1}}{p^{n-1}}\right] \mathbf{1}$$

$$\therefore v_n(t) = e^{-\alpha t} \sum_{r=0}^{n-1} {}_{n-1}C_r\, (-1)^r \frac{\alpha^r t^r}{r!} \qquad \ldots \quad \ldots \quad (8)$$

Alternatively

$$v_n(p)\, \mathbf{1} = [1 - h_1(p)]^n\, \mathbf{1}$$

$$= [1 - {}_nC_1 h_1(p) + {}_nC_2 h_2(p) + \ldots$$

$$+ (-1)^r\, {}_nC_r h_r(p) + \ldots + (-1)^n h_n(p)]\, \mathbf{1}$$

$$\text{since } [h_1(p)]^r\, \mathbf{1} = \frac{\alpha^r}{(p+\alpha)^r}\, \mathbf{1} = h_r(p)\, \mathbf{1}$$

$$\therefore v_n(t) = 1 + \sum_{r=0}^{n} {}_nC_r\, (-1)^r\, h_r(t) \qquad \ldots \quad \ldots \quad (10)$$

APPENDIX 4

Response to a Unit Step of a Critically-damped Inductance-compensated Amplifier

The response to a unit step of the single-stage amplifier shown in Fig. 7 is $g_m R u_1(t)\, 1$ where

$$\left.\begin{aligned} u_1(p)\, \mathbf{1} &= \frac{4\alpha^2 + \alpha p}{(2\alpha + p)^2}\, \mathbf{1} \\ u_1(t)\, \mathbf{1} &= [1 - e^{-2\alpha t}(1 + \alpha t)]\, \mathbf{1} \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (12)$$

where $L = \frac{1}{4} CR^2$, $\alpha = 1/RC = R/4L$.

$$\therefore u_n(p)\, \mathbf{1} = \left[\frac{4\alpha^2 + \alpha p}{(2\alpha + p)^2}\right]^n \mathbf{1}$$

Assume that $u_n(t)\, \mathbf{1} = \left[1 - e^{-2\alpha t}\left(1 + {}_nb_1 \frac{\alpha t}{1!} + \right.\right.$

$$\left.\left. {}_nb_2 \frac{\alpha^2 t^2}{2!} + \ldots\ldots + {}_nb_{2n-1} \frac{\alpha^{2n-1}}{(2n-1)!}\right)\right] \mathbf{1} \qquad (13)$$

If this is so,

$$u_{n+1}(t)\, \mathbf{1} = \frac{4\alpha^2 + \alpha p}{(2\alpha + p)^2}\, \mathbf{1} -$$

$$\frac{4\alpha^2 + \alpha p}{(2\alpha + p)^2} \cdot e^{-2\alpha t} \sum_{r=0}^{2n-1} {}_nb_r \frac{\alpha^r t^r}{r!}\, \mathbf{1}$$

But $f(p)\, e^{-\alpha t} g(t)\, \mathbf{1} = e^{-\alpha t} f(p - \alpha)\, g(t)\, \mathbf{1}$

$$\therefore u_{n+1}(t)\, \mathbf{1} = \frac{4\alpha^2 + \alpha p}{(2\alpha + p)^2}\, \mathbf{1} -$$

$$e^{-2\alpha t} \cdot \frac{2\alpha^2 + \alpha p}{p^2} \sum_{r=0}^{2n-1} {}_nb_r \frac{\alpha^r}{p^r}\, \mathbf{1}$$

$$= [1 - e^{-2\alpha t}(1 + \alpha t)]\, \mathbf{1} -$$

$$e^{-2\alpha t} \sum_{r=0}^{2n-1} {}_nb_r \left[2 \frac{\alpha^{r+2}}{p^{r+2}} + \frac{\alpha^{r+1}}{p^{r+1}}\right] \mathbf{1}$$

$$= \mathbf{1} - e^{-2\alpha t} \sum_{r=0}^{2n+1} [2\, {}_nb_{r-2} + {}_nb_{r-1}] \frac{\alpha^r}{p^r}\, \mathbf{1}$$

$$= \left[1 - e^{-2\alpha t} \sum_{r=0}^{2n+1} {}_{n+1}b_r \frac{\alpha^r t^r}{r!}\right] \mathbf{1}$$

where ${}_{n+1}b_0 = 1$ and ${}_{n+1}b_r = 2\, {}_nb_{r-2} + {}_nb_{r-1}$.

Therefore if equation (13) is valid for n stages, it is valid for $n + 1$ stages, the coefficients having the above relationship. But the equation is valid for $n = 1$ (Equ. 12), so it is valid for any value of n. The coefficient ${}_nb_1 = 1$, therefore, ${}_2b_1 = 2$, ${}_2b_2 = 3$, ${}_2b_3 = 2$ and the coefficients for larger values of n can be similarly calculated.

APPENDIX 5

Response to a Unit Step of Resistance-coupled Amplifier with Many Stages

The response of the n-stage amplifier to a unit impulse (i.e., the limiting case of a pulse whose duration tends to zero, the product of its amplitude and duration being kept constant at unity[1]) is obtained by differentiating $h_n(t)$ and is given by

$$q_n(t) = \frac{(\alpha t)^{n-1}}{(n-1)!} e^{-\alpha t},\ t > 0 \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

It can be shown that the mean value of αt for the curve of $q_n(\alpha t)$ is n and the standard deviation is $\sqrt{n}$.

$$\therefore \text{Putting } y = \frac{\alpha t - n}{\sqrt{n}}$$

and transforming equation (15) in terms of the new variable,

$$q_n(y) = \frac{n^{n-\frac{1}{2}} \left(1 + \frac{y}{\sqrt{n}}\right)^{n-1} e^{-\sqrt{n}y - n}}{(n-1)!}$$

$$\text{since } \frac{dy}{d(\alpha t)} = \frac{1}{\sqrt{n}}$$

If n is large, the factorial can be replaced by the Stirling approximation

$$n! = \sqrt{2\pi n}\, n^n e^{-n}$$

and $\left(1 + \frac{y}{\sqrt{n}}\right)^n = e^A$

where $A = \log_e \left[1 + \frac{y}{\sqrt{n}}\right]^n$

$$= n\left[\frac{y}{\sqrt{n}} - \tfrac{1}{2}\frac{y^2}{n} + \ldots\ldots\ldots\right]$$

Hence if $y^2 \ll n$,

$$q_n(y) \approx \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

which is the expression for the normal or Gaussian law of distribution. Substituting for y,

$$q_n(t) \approx \frac{1}{\sqrt{2\pi n}} \exp\left[\frac{(\alpha t - n)^2}{2n}\right] \quad \ldots \quad \ldots \quad (16)$$

We have shown that the shape of the response of a resistance-coupled amplifier to a very short pulse tends towards a Gaussian distribution curve as the number of stages is made very large. Equation (16) is only approximate for a finite number of stages and is inaccurate for small values of αt; moreover, it gives a value of $q_n(t)$ which is not zero for $t \leqslant 0$, which is physically impossible, but equation (16) represents $q_n(t)$ with increasing accuracy as the number of stages is increased.

The response of the amplifier to a unit step is

$$h_n(t) = \int_0^{\alpha t} q_n(\alpha t)\, d(\alpha t)$$

$$\therefore\ h_n(y) = \int_{-\infty}^{y} q_n(y)dy = \int_{-\infty}^{0} q_n(y)dy + \int_0^y q_n(y)dy$$

But $\displaystyle\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy = \tfrac{1}{2}$

and $\displaystyle\frac{1}{\sqrt{2\pi}}\int_0^y e^{-y^2/2}\, dy = \tfrac{1}{2}\,\mathrm{erf}\left(\frac{y}{\sqrt{2}}\right)$

$$\therefore\ h_n(y) = \tfrac{1}{2}\left\{1 + \mathrm{erf}\left(\frac{y}{\sqrt{2}}\right)\right\}$$

Substituting for y,

$$\therefore\ h_n(t) = \tfrac{1}{2}\left\{1 + \mathrm{erf}\left(\frac{\alpha t - n}{\sqrt{2n}}\right)\right\} \quad \ldots \quad \ldots \quad \ldots \quad (17$$

This result has been given by Grant.[11] Tables of the error function can be found in text books on statistics[12, 13] and enable the response of resistance-coupled amplifiers to be calculated approximately when the number of stages is large.

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C.C.I.R. TELEVISION DELEGATION

Representatives of fifteen countries were in the delegation of the International Radio Consultative Committee which has been studying British television. Their visit was the last of a series, for they had previously been to France, Holland and the U.S.A. to see the systems used in those countries. Various demonstrations and visits to manufacturers were arranged and some of the outstanding items which they saw are briefly described below.

There was a demonstration of projection television on 29th April at the Odeon Cinema, Penge, by Cinema-Television Ltd., when the B.B.C. broadcast of the Cup Final was shown on a screen 20 ft by 15 ft. The projection tube was operated at 50 kV with an average beam current of 1-2 mA and a Schmidt optical system was used. The highlight brightness was stated to be 7 ft-lamberts.

A direct comparison between 405-line and 625-line television could be made in a closed-circuit demonstration at the E.M.I. works, the bandwidths used being respectively 3 Mc/s and 5.5 Mc/s. Two cameras were used side by side and the pictures appeared simultaneously on adjacent 15-in c.r. tubes. It was particularly interesting to observe that the improvement in picture definition with the greater number of lines was quite small. The importance of correct 'gamma' and its effect on picture quality were also demonstrated.

In a visit to the Marconi Company at Great Baddow, cameras in the grounds and marquee produced pictures which were shown on sets in all parts of the building and the high sensitivity of the camera tubes was evident by the good pictures secured in the dusk. A large number of demonstrations of an instructive nature were also given and included one of 'spot wobble' by Cinema Television.

At the B.B.C. research station 405-line and 625-line pictures were shown with a 3-Mc/s bandwidth and also with superimposed delay and echo distortions of known magnitude. It was shown that a 625-line picture was degraded by the limited bandwidth to a lower level than the 405-line and that it suffered much more severely from the distortions. A delay distortion of only 0.02 μsec, which it is hard to prevent in cable links, produced intolerable distortion of the 625-line picture, but permitted a usable 405-line picture to be obtained.

Demonstrations of the effect of thermal noise, and also of single-frequency interference, on a picture were given at the Post Office Research Station, Dollis Hill. A signal/thermal-noise ratio of about 30 db was shown to be necessary completely to avoid traces of thermal noise, but up to 60 db was needed in some cases for single-frequency interference. A television microwave-relay link was shown in operation and a method of measuring phase-delay on cables with a remote termination was shown and explained.

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AMPLIFIER WITH NEGATIVE-RESISTANCE LOAD

Measurement of Stage Gain

By D. M. Tombs, M.Sc., A.M.I.E.E., and M. F. McKenna, B.Sc., A.C.G.I.

(Department of Electrical Engineering, Imperial College of Science and Technology, London)

SUMMARY.—The gain of a single-stage amplifier with a resistive load is always less than μ, the amplification factor of the valve. When a negative-resistance element is used as load, the gain can exceed μ and, theoretically, be made to increase indefinitely as the value of the negative-resistance approaches the differential anode resistance of the valve.

This paper describes experimental measurements of the stage gain up to 2.5 μ for a triode amplifier loaded with a negative-resistance element. An account is also given of the restrictions imposed on the operation of this type of amplifier by conditions of stability and the limitation of the gain due to the self-capacitance of the load.

1. Introduction

FIG. 1(a) represents the load-line diagram of a triode valve amplifier having a load resistance R_L, with a quiescent or operating point at Q and a grid excursion between voltages V_{g1} and V_{g2}. E_{HT} is the battery supply voltage. The intercepts of the load line with the static characteristics give projections on the anode voltage (V_a) axis, at V_{a1} and V_{a2}. The voltage gain is given as $(V_{a2}-V_{a1})/(V_{g2}-V_{g1})$ and by inspection the output is in antiphase with the input. The maximum stage gain attainable is μ (i.e., when the load line is horizontal) and occurs when R_L becomes infinite. For triode amplifiers of this kind this is never achieved in practice because of the infinite steady voltage drop across the load resistance.

MS accepted by the Editor, December 1949.

Voltage gains greater than μ can be obtained, however, by using a negative-resistor as load, the load line for this case is shown in Fig. 1 (b). For the same grid excursion, the projections of the intercepts on the V_a axis become V'_{a1} and V'_{a2} and consequently the voltage gain is the ratio $(V'_{a2}-V'_{a1})/(V_{g2}-V_{g1})$, a value which approaches infinity as the load line becomes tangential to the valve characteristic at the point Q; i.e., when the magnitude of the negative-resistance approaches the differential anode resistance of the valve. By inspection the output is again in antiphase with the input, although the anode current has changed phase by 180°.

An alternative method of achieving the same

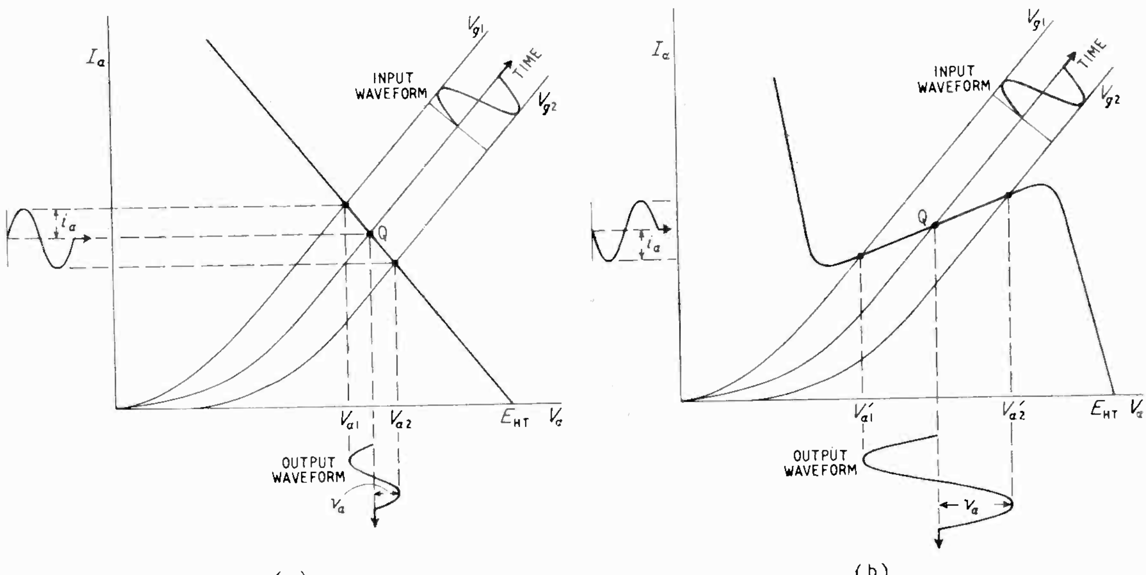

Fig. 1. *Load-line diagrams (a) with positive- and (b) with negative-resistance loads. The anode currents are in opposite phase.*

result, as suggested by Dowling[1], is to operate a tetrode valve on the negative slope of its I_a/V_a characteristic in conjunction with a resistive load. If the magnitude of the load resistance is adjusted to a value less than, but approaching, the differential anode resistance of the valve at its operating point, gains greater than μ are attainable.

The object of this paper is to confirm, by experiment, that stage gains greater than μ can be obtained with a triode valve amplifier when a negative resistance is used as load, and that the circuit is stable.

2. Theory

2.1. *Gain of Single-stage Amplifier.*

The expression for the voltage amplification of a single-stage amplifier is

$$A = -\frac{\mu Z}{r_a + Z} = -\frac{\mu}{1 + r_a/Z} \quad \ldots \quad (1)$$

where Z is the impedance of the anode circuit, and r_a is the differential anode resistance of the valve.

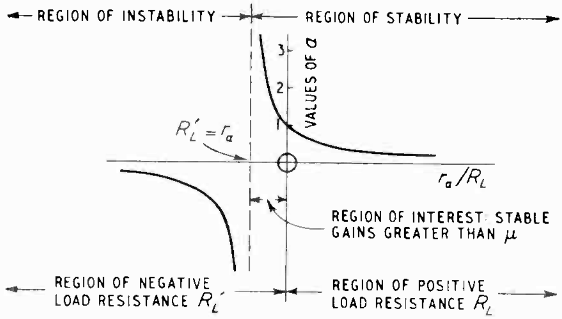

Fig. 2. *Relation of normalized amplification a to* r_a/R_L.

Introducing a new symbol a, the normalized amplification and defining it as $a = A/(-\mu)$ equation (1) may be expressed in the form

$$a = \frac{1}{1 + r_a/Z} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

For increasing values of anode load impedance, a tends to unity, and when the anode load becomes resistive (i.e., for $Z = R_L$)

$$a = \frac{1}{1 + r_a/R_L} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

The curve of a as a function of r_a/R_L is plotted to the right of the origin in Fig. 2.

2.2. *Single Stage with Negative-resistance Load*

If, however, R_L is made negative, say $-R_L'$, then

$$a = \frac{1}{1 + r_a/(-R_L')} = \frac{1}{1 - r_a/R_L'} \quad \ldots \quad (4)$$

and evidently a can have values greater than unity tending to infinity as R_L' tends to r_a. The effect of varying the value of R_L' is shown to the left of the origin in Fig. 2.

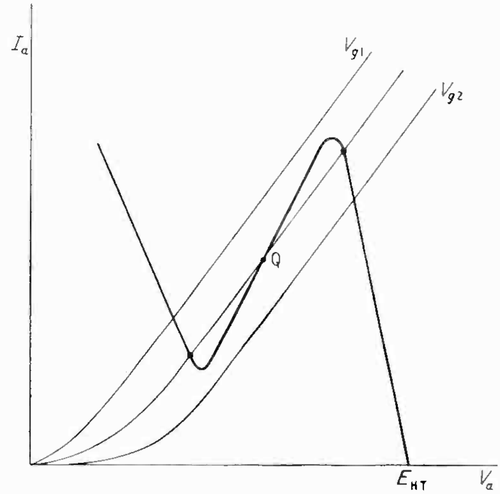

Fig. 3. *Triple intersection.*

2.3. *Stability.*

Now R_L' can approach r_a in two ways, from a larger value than r_a or from a smaller. Which of the two, if either, is stable can be readily determined from consideration of Fig. 1(a) and (b). Referring to Fig. 1(a), for any given grid potential there is a single point of intersection between the valve characteristic and the load line. This is the single-valued solution of the Kirchhoff equation for the anode mesh

$$E_{HT} + V_a + V_L = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where V_a is a non-linear function of I_a given by the curve, and V_L is a linear function of I_a given by the load line; viz., $V_L = I_a R_L$.

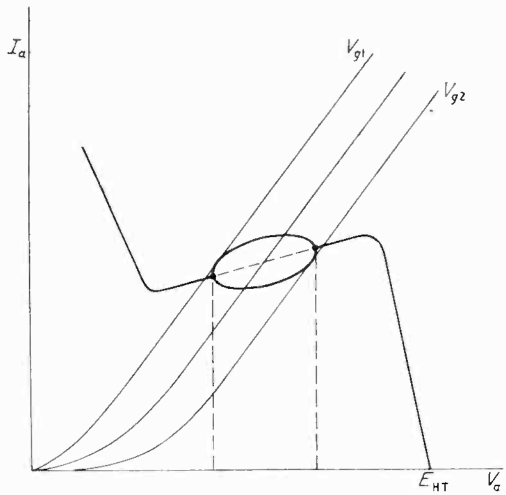

Fig. 4. *Elliptical excursion.*

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In Fig. 1(b) the intersection of the load line and the characteristic for a given grid potential is seen to be single valued and the circuit is stable. This condition arises when $R_L' > r_a$. When, however, the load line is seen to have multiple intercepts with the characteristic as in Fig. 3, the amplifier is unstable, a condition for which $R_L' < r_a$.[2, 3] These conditions apply only to voltage-controlled negative resistors ; the reverse is true for the current-controlled types.

2.4. *Effect of Shunt Capacitance.*

The inevitable presence of the shunt capacitance of the load means that equation (2) may be wrttten

$$a = \frac{1}{1 + r_a/R_L + j\omega C r_a} \quad \ldots \quad \ldots \quad (6)$$

where C is the load self-capacitance, $\omega = 2\pi f$ and f the frequency of the applied signal, and equation (4) as

$$a = \frac{1}{1 - r_a/R_L' + j\omega C r_a} \quad \ldots \quad \ldots \quad (7)$$

For stability under static control conditions R_L' must be greater than r_a, and within this limitation, the magnitude of a cannot be greater than $1/\omega C r_a$. For this case the excursion on the load-line diagram becomes elliptic and is shown qualitatively in Fig. 4. The expression for a in equation (6) may be plotted on a vector diagram giving a locus for variations of r_a/R_L and r_a/R'_L. This is shown in Fig. 5(a) and (b) on the assumption that f, C, and r_a are constant. It will be seen from Fig. 5(b) that the locus of a is a semi-circle of diameter $1/\omega C r_a$, which is the maximum value a can have under the assumed conditions.

3. Experimental

3.1. *Negative-Resistance Elements.*

Two forms of voltage-controlled negative resistors were considered :

(a) A tetrode valve operating on the negative slope of its I_a/V_a characteristic.

(b) A cathode-coupled negative-resistance circuit.

The first of these was used initially, but was later replaced by (b). The circuit diagram of this device, together with the current-voltage characteristic at its input terminals AB, are shown in Fig. 6(a) and (b) respectively.*

In order to obtain the requisite variation of resistance, a calibrated resistance R was connected across AB, the combination providing positive or negative values according to the relative magnitudes of R and the negative input-resistance of the device :

viz., $R_L' = -\dfrac{R R_n}{R - R_n}$ ohms, for R greater than R_n

or $R_L = \dfrac{R R_n}{R - R_n}$ ohms, for R_n greater than R

where R_n is the magnitude of the negative input-resistance.

The apparatus used for the experiment on the stage gain is shown in Fig. 7. The negative-resistance element was isolated from the d.c. circuit of the amplifier by a 1.0-μF capacitor. As R was changed to afford the desired variation of R_L and R_L', the h.t. supply voltage was also changed so as to ensure operation about

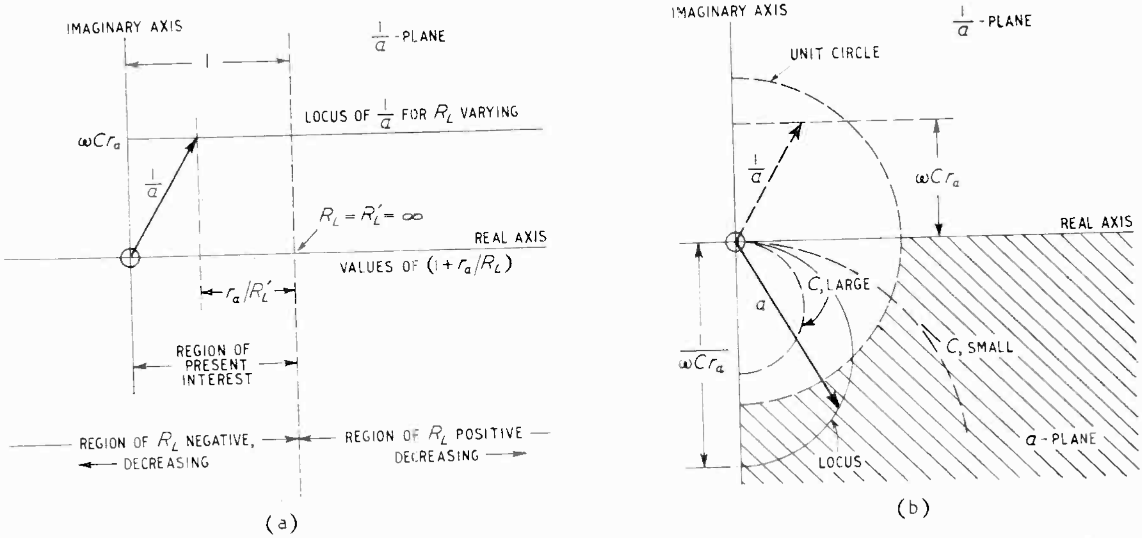

Fig. 5. *(a) and (b) show respectively the vector loci of 1/a and a, for R_L varying and an assumed shunt capacitance C, (stage gain is proportional to a). The shaded region in (b) shows the area in which gains greater than μ occur.*

* For a full discussion of the behaviour of the circuit, see reference 4.

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the same point on the valve characteristic.

The grid circuit comprised a d.c. potential divider for biasing the amplifying valve to the desired operating point and a calibrated attenuator permitted the adjustment of the 1,000-c/s input signal.

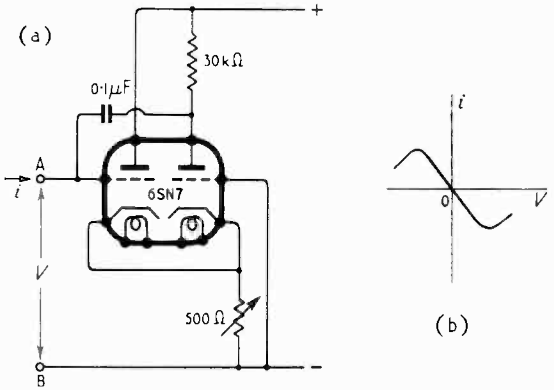

Fig. 6. (*a*) *Cathode-coupled negative-resistance circuit and* (*b*) *static i/V characteristic at input terminals A and B.*

3.2. *Measurements.*

The constants μ and r_a of the amplifying valve at the quiescent point were obtained from measurements on a Miller Bridge[5] and were found to be, respectively, 14.99 and 8776 ohms.

The technique employed for the measurement of stage gain consisted of adjusting, to a constant value on an oscilloscope, the input voltage to the valve so as to make the amplitude of the output voltage equal to that across the supply points (a) and (b) of the attenuator. The gain of the stage expressed in decibels was then read directly from the scale of the attenuator.

The load resistance was measured on an audio-frequency bridge operated on the same frequency as that of the signal applied to the amplifier; viz., 1,000 c/s. To facilitate measurements of positive and negative values of load resistance on the same bridge, a known resistance of 1,851 ohms was placed in shunt with the load to give a combination whose resistance value remained positive. The load had a shunt self-capacitance of the order of 0.002 μF, the susceptance of which, at the operating frequency, was comparable with the conductance of the load over the range of R_L' and was included in the calculations of the performance of the amplifier.

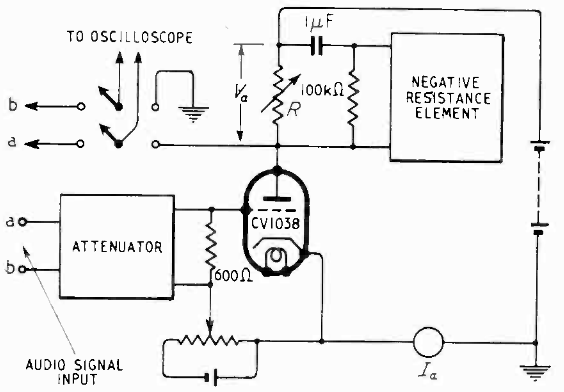

Fig. 7. *Diagram of measuring equipment.*

3.3. *Experimental Results.*

Detailed experimental results are given in tabular form in the Appendix and the experimental curve relating a and r_a/R_L is shown, alongside the theoretical one, in Fig. 8.

For values of gain above 2.5μ the output waveform became noticeably distorted and prevented reliable results from being obtained. Smaller input signals were used in an attempt to overcome this, but the pick-up voltages in the grid circuit were sufficient to make the results inaccurate.

The effect of the shunt self-capacitance of the load could be neglected for values of r_a/R_L greater than 2, but for smaller values, and for values of

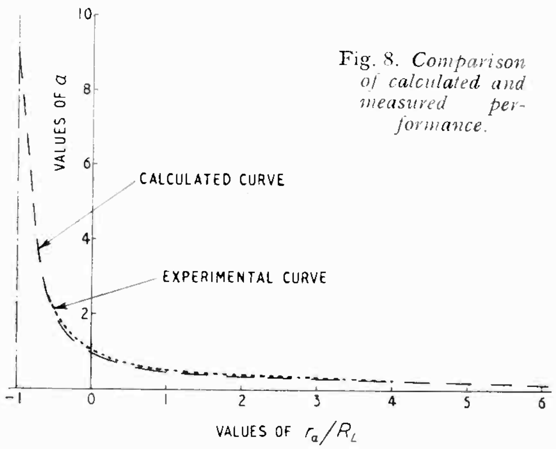

Fig. 8. *Comparison of calculated and measured performance.*

r_a/R_L' less than unity it had to be taken into account. The limit of a for $r_a/R_L' = 1$ was determined by the self-capacitance of the load, as previously indicated, and was approximately 9.

4. Conclusions

The object of this paper was to confirm experimentally that stage gains in excess of μ could be obtained with a triode amplifier using a negative-resistance load. The experimental and theoretical results agree within 6% for gain of 2.5μ with a closer agreement for smaller values. The occurrence of distortion prevented the determination of higher values of gain, but even in the absence of distortion the self-capacitance of the load would have limited the gain to approximately 9μ.

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The restrictions imposed on the amplifier for stable operation, necessitated the magnitude of R_L' being greater than r_a and if higher gains are required, the input must diminish if distortionless output is to be maintained.

Acknowledgments

The authors wish to express their thanks to Prof. Willis Jackson and Dr. J. Lamb for suggestions made when reading the manuscript of the paper and also to A. C. Nicolls for the help given with the experimental work.

REFERENCES

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[2] J. G. Brainerd: "Ultra High Frequency Techniques," Chapman & Hall, p. 169.
[3] A. A. Andrenov & C. E. Chaikin: "Theory of Oscillations," Princeton.
[4] P. G. Sulzer: "A Cathode-coupled Negative-Resistance Circuit," *Proc. Inst. Radio Engrs*, Aug. 1948.
[5] J. M. Miller: *Proc. Inst. Radio Engrs*, 1918, Vol. 6, No. 3., p. 141.

APPENDIX

Values of Stage gain (db)	Values R_n (Ω)	Values of R_L (Ω)	Values of r_a/R_L	Calculated values of a	Exptl. values of a
6.0	2142	1,440	6.1	0.143	0.133
9.0	2142	1,878	4.67	0.176	0.187
13.5	2150	4,010	2.185	0.310	0.315
16.5	2155	6,210	1,412	0.415	0.445
19.0	2155	10,920	8.803	0.552	0.594
22.0	2146	29,420	0.298	0.769	0.839
23.5	2145	100,200	0.088	0.917	0.906
24.0	2142	− 576,000	− .015	1.01	1.055
25.0	2140	− 78,500	− .112	1.12	1.186
26.0	2145	− 46.000	− .191	1.22	1.330
27.0	2120	− 27,100	− .325	1.46	1.491
28.0	2120	− 21,670	− .405	1.65	1.672
29.5	2120	− 18,200	− .482	1.87	1.990
30.0	2120	− 17,100	− .514	2.00	2.105
31.0	2120	− 15,710	− .560	2.20	2.360

NEW BOOKS

Leitfaden der Elektrotechnik. Vol. 1. Grundlagen der Elektrotechnik.

By Moeller and Wolff. Pp. 358 + x. With 257 illustrations. B. G. Teubner Verlagsgesellschaft, Leipzig.

This is the fourth edition of this textbook on the fundamentals of electrical engineering. The authors are professors at Brunswick and Nuremberg respectively. The foundations of the subject are laid with great care and thoroughness; the mathematical treatment is very clear, and diagrams are used wherever they can be of any help. In the earlier editions coloured diagrams were used, but this is not possible under present conditions. An interesting innovation is the numbering of the illustrations by the page on which they appear, thus a diagram on p. 129 in Fig 129.1; this makes it very easy to find any given figure, but it is difficult to see why the .1 need be added when there is only one figure on the page, as is nearly always the case. Many of the vector diagrams would look much neater if the Gothic type were replaced by Roman type. As an example of the detailed information given, there is a table on p. 19 giving not only the resistivities and conductivities of the various metals and alloys, but also both α and β in the formula $R_2 = R_1[1 + \alpha(\theta_2 - \theta_1) + \beta(\theta_2 - \theta_1)^2]$ in which the initial temperature θ_1 is taken as 20°C. In another column is also given the value of τ (235 for copper) which, if added to the Centigrade temperature, gives an 'absolute' temperature to which, over the ordinary working range, the resistance is proportional.

The subjects dealt with cover electric circuits, energy and power, thermo-electricity, electrolytes and cells, magnetic and electric fields, alternating currents including locus diagrams and resonance, choking coils and air-core transformers, waveforms, and the measurement of three-phase power. A new chapter has been inserted on the calculation of single-phase and three-phase transmission lines, but unfortunately the index was not corrected and the references to subsequent items are all eight pages wrong; thus 'Schaltzeichen' is not on p. 342, as stated, but on p. 350.

The magnetic field is developed on modern lines; the unit pole is not mentioned, and the magnetic-field strength H is introduced as another name and symbol for the ampere-turns per cm.

The system of units employed is called the practical system, but it is not the m.k.s. system, and the name of Giorgi is not mentioned. The centimetre is maintained as the unit of length. The units of B, H, $\mathcal{E}$ and D are 1 V sec/cm², 1 A/cm, 1 V/cm and 1 C/cm² respectively, but the gauss and maxwell are also employed. The rationalized system is used and the permeability of space is given as

$$\mu_0 = 1.256 \times 10^{-8}\,\text{H/cm} = 1.256 \times 10^{-8}\,\text{Vsec/A cm}$$
$$= 1.256\,\text{G cm/A} = 1.256\,\text{M/A cm}$$

where G and M stand for gauss and maxwell.

An outstanding feature of the book is the great number of numerical examples which are worked out in detail.

This is a textbook that can be unreservedly recommended to either student or teacher with a knowledge of German. G. W. O. H.

Electronic Valves

Book II, Data and Circuits of Receiver and Amplifier Valves. Pp. 409 + xii. Price 21s.

Book III, Data and Circuits of Modern Receiver and Amplifier Valves (1st Supplement). Pp. 213 + x. Price 12s. 6d.

Book IV, Application of the Electronic Valve in Radio Receivers and Amplifiers. Pp. 416 + xxiv with 256 illustrations. Price 35s.

Cleaver-Hume Press Ltd., 42a, South Audley St., London, W.1.

The first book of this series, "Fundamentals of Radio Valve Technique," was reviewed in *Wireless Engineer* for December 1949, p. 413. The series has the general title "Electronic Valves," and is published by N. V. Philips, Gloeilampenfabrieken, Eindhoven, Holland.

Books II and III chiefly comprise details of the characteristics of Philips' valves. In the former nearly three-quarters of the book is devoted to this and most of the rest to examples of typical apparatus embodying the valves. Circuit diagrams with values of components

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and general technical descriptions are given. The valves included are mainly the E, C and K series. Book III is similar but covers the more recent E20, U20, D20 and U series of Philips' Valves.

Book IV is in the nature of a wireless-receiver text book and is of general application. It covers the receiver from the aerial input to the detector; a.f. amplification is to be treated in Book V which has not yet appeared. Many matters, too, which some may consider should have been included in Book IV, such as noise, are not treated here but are to be covered in a future Book VI.

Book IV opens with a general discussion of the single-tuned circuit and then goes on to deal with the properties of coupled circuits. Amplification is then treated, first at radio and then at intermediate frequency and included in this section is a detailed account of aerial-coupling methods.

The frequency-changer follows and, as part of it, the oscillator is treated in great detail. A very detailed analysis is given of the operation of a squegging oscillator. It forms one of the best published accounts of this and includes a method of predicting whether or not a given oscillator will squeg.

The determination of circuit values for superheterodyne tracking is covered and cross modulation and modulation distortion are very fully treated. There is also a section on superheterodyne whistles. The detector is well explained and the effect of the ratio of the a.c. to d.c. loads of a diode is covered, not only as regards distortion but as regards the effect on the selectivity of the preceding tuned circuit and the modulation depth of the signal developed across it. It is too often overlooked in text books that a low ratio of a.c./d.c. loads so affects the input tuned circuit that the modulation depth of the signal developed across it is reduced and that, in its turn, this reduces the amount of distortion introduced in the detector itself by the low ratio of the loads. This is far from new, of course, but it is not always recognized and it is refreshing to find it fully treated. It is shown in one example that in a particular case the effect results in the critical modulation depth (for the appearance of distortion) of the incoming signal being 81% instead of 66% only.

The treatment throughout is very thorough but it is done almost entirely from the viewpoint of the designer of broadcast receivers. Most emphasis is given to medium- and long-waveband problems and short-wave matters are discussed chiefly from the point of view of the designer who wants to include a single s.w. band of wide coverage in what is primarily a medium- and long-wave set.

There is a good deal of mathematics in the book, although not of a very highbrow nature. The choice of symbols is sometimes strange to British eyes and the elaborate subscripts employed sometimes make the equations unnecessarily difficult to follow. The language too, might be improved. The book is a translation and, although it is a good one, the reader cannot be but aware of it.

W. T. C.

Telecommunications and Equipment in Germany during the period 1939-1945.

British Intelligence Objectives Sub-committee Surveys, Report No. 29. Pp. 55. H.M. Stationery Office, York House, Kingsway, London, W.C.2. Price 1s. 6d.

This is a survey of a large number of B.I.O.S., C.I.O.S., F.I.A.T. and J.I.O.A. reports which are listed individually as a bibliography. It is concerned mainly with line telephony and telegraphy, but there are a few pages devoted to research (mainly on materials) and about six pages to radio, including broadcasting, wire broadcasting and television.

The Williamson Amplifier

By D. T. N. Williamson. Pp. 28 + vi with 31 illustrations. Iliffe & Sons Ltd., Dorset House, Stamford St., London, S.E.1. Price 3s. 6d. (postage 2d.).

A collection of articles on a high-quality amplifier reprinted from *Wireless World*.

CORRESPONDENCE

Differential Amplifiers

Sir,—In the April issue Dr. D. H. Parnum introduces the term 'transmission factor' as an overall measure of performance for differential amplifiers fed at the input with a high ratio of in-phase interference/anti-phase signal. Perhaps 'differential transmission factor' is a better term, and one not in conflict with other types of communication circuits?

There is still a place for 'discrimination ratio' as a design factor, for it represents the calculable degenerative discrimination assuming perfect balance. The several imperfections in balance cannot be predicted with any certainty, but they can be compensated for individually as Dr. Parnum has shown, and this possibility was pointed out in the passage from my article* which he has quoted.

The commercial amplifier I described was designed for the use of medical technicians, with a 'factor of safety,' or margin of performance, in order that the number of adjustments be kept to a minimum. This was done by employing degenerative circuits in preference to balanced circuits throughout. The literature on sensitive and d.c. amplifiers contains very many interesting and ingenious balancing circuits, but few have found any permanent place in electronic engineering because they have been called upon to work at the limit of their performance.

Dr. Parnum assumes that the tolerances of unbalance will be additive, and predicts a 'differential transmission factor' of 400, where I found the measured value in my Fig. 10(a) to be at least 3,000. I think this is because the single zero-adjusting resistance does, in fact, tend to equalize the amplification factors of the two valves as well. In the passage he quotes I should have referred to an 'effective' discrimination factor of 100,000, but I now prefer the new term discussed above.

Denis L. Johnston.

Aldenham, Hertfordshire.

* *Wireless Engineer*, Aug., Sept., Oct. 1947.

SYMPOSIUM ON INFORMATION THEORY

A symposium on Information Theory is to be held in the rooms of the Royal Society from September 26th–29th inclusive. The programme will discuss, in particular, the recent work of Dr. C. Shannon of the Bell Telephone Laboratories, N.J., who will himself present two papers, and it will be of interest to mathematicians, physicists, physiologists and communication engineers. Those wishing to attend are asked to write to Prof. Willis Jackson, Electrical Engineering Dept., Imperial College, London, S.W.7.

CORRECTION

An error occurred in the caption to Fig. 6 in the paper "Secondary Emission Valve" by G. Diemer and J. L. H. Jonker in the May issue, p. 137. The dotted-line curves are both for the secondary-emission triode under discussion and the solid-line for a different valve with greater spacing between the anode and the secondary emitter.

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