The Algebra of ELECTRONICS

Solving Circuit Problems

by CHESTER H. PAGE

THE ALGEBRA OF ELECTRONICS

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To

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PREFACE

In recent years, there has been a renaissance of our forefathers' do-it-yourself philosophy. Fortunately, this trend is found in the intellectual realm, as well as in craftwork. Thousands of persons have learned algebra and calculus by home study of suitable texts.

This book is addressed to these people, with their healthy intellectual curiosity. It is addressed equally to electronic technicians, and TV and radio servicemen, who have a practical knowledge of circuits and wish to acquire understanding.

The book starts with direct current, to introduce the basic concepts without confusing detail. Networks of resistance are discussed topologically, in terms of trees, branches, links, and loops. Mesh and nodal analysis are presented as special cases, for which the network equations can be written by inspection, in a form that continues to be valid for the general AC case. This leads into the study of determinants and the solution of simultaneous equations. Practical solution methods are emphasized.

The fourth chapter treats of general properties of networks, and their representation as T-networks, Π-networks, and "black boxes."

The transition to alternating current problems is made via chapters on capacitance and inductance, developed from fundamentals. Simple tuned circuits follow, and lead into the concept of impedance, and its various representations in terms of phase angles and complex numbers. The arithmetic and algebra of complex numbers is treated in detail.

A major chapter on general AC networks elaborates the treatment of mutual inductance, and clarifies the question of the algebraic sign of mutual inductance in multi-coil assemblies. The behavior of air-core and iron-core transformers is thoroughly explained, and various equivalents are analyzed.

Chapter X is devoted to the analysis of specific circuits, such as double-tuned interstage, FM discriminator, bridged-T, twin-T, and an RC ladder used in phase shift oscillators.

Chapter XI discusses impedance matching sections and the various phenomena associated with matching, mismatching, filtering, and insertion loss.

The study of diodes as nonlinear elements leads to triodes, and their linearized approximation in terms of amplification factor and mutual conductance. Amplifiers are treated as specific circuits involving thermionic tubes and transistors, and finally as general active "black boxes." Because the practical limits to amplification depend upon noise, a chapter is devoted to this subject.

The text concludes with a study of modulation, demodulation, and distortion, explained in terms of frequency components and Fourier series. Problems, hints, and answers, close the book.

The basic topological concepts of electric networks in the early chapters follow the philosophy of Professor Ernst Guillemin, who has written several excellent books for a more advanced audience. All of us who are interested in either network research or teaching owe a debt of gratitude to Professor Guillemin for his unceasing output of ideas and enthusiasm.

C.H.P.

Silver Spring, Maryland
September, 1958

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Chapter I

DIRECT CURRENT

1-1 Voltage. It is a familiar fact of electrostatics that unlike charges (positive and negative) attract each other. A force must be applied to move them apart; work must be done against the force that tends to pull the charges together. This work becomes potential energy associated with the separated charges, just as the work done in lifting a weight becomes potential energy.

Consider two conducting bodies, say metal plates, not quite in contact. Let one plate carry the charge Q and the other, $-Q$. (These charges can be obtained by various means. One way is to rub a glass rod with silk, and then "wipe" the charge off the rod onto the metal plate.) We now separate the plates, applying the necessary force by way of insulating handles. The work we do becomes electrical potential energy, and we say that the "potential difference" between the plates has increased. Conversely, if we have charged separated plates, we can let their electrical attraction pull them together, and do work for us, such as lifting a weight. The total amount of work they can do, by going completely together, is the potential energy of the system. For a given separation, this energy is greater, the greater the charge on the plates. The potential energy per unit of charge (W/Q) is called the "potential difference," or *p.d.*, between the plates, and is measured in *volts*. The charge Q is measured in *coulombs*, and the energy in *joules*. (The practical unit of charge, the coulomb, is approximately the charge on 6×10^{18} electrons. The joule is more familiar as the *watt-second*; 1 kilowatt-hour equals 3.6 million joules.)

The potential energy of our pair of separated plates can be reduced by letting them get closer together; it will also be reduced if the insulating handles are imperfect, and some of the charge "leaks" from one plate to the other, urged to do so by the attractive force between the unlike charges. In this case, the used-up potential energy shows up as heat (thermal energy). This will be treated in detail a little later. In either case, loss of potential energy means a lowering of the p.d. between the plates. Now if the plates are connected to the terminals of a battery or a generator, the p.d. will be held constant, even though we have used some energy. The used energy was, of course, supplied by the battery. But to keep the p.d. constant, additional separated charge must have been supplied to the plates. This is obvious when the energy loss was due to charge loss; the case of energy loss due to motion of the plates will be discussed in a later chapter (V). Thus the battery or generator has some sort of internal "force" that tends to push positive charges out one terminal, and negative out the other, to supply positive and negative charges to the plates. Such an electrical separating force is called an *electromotive force* (emf) and is measured by the potential difference it maintains between the terminals, hence it is measured in volts. We see, then, that the term "voltage" is used for both electromotive force (a cause), and potential difference (a result) even though these quantities are logically different. In fact, if we short-circuit a dry cell by connecting its terminals together with a good conductor, we do not affect the emf of the cell, but we can no longer have a p.d. between the terminals. Indeed, after a short while, we will no longer have a cell!

In problems involving electric currents in equilibrium with their driving forces, as we are throughout this book, it is best to think of an emf as a *source* voltage, and a p.d. as a *resulting* voltage across any device which is not a source.

1-2 Current. Electrostatics is the study of electricity when the charges are essentially at rest. Most practical usages of electricity involve the flow of charges through a conductor, analogous to the flow of water through a pipe. In the water analog, the flow is measured by the quantity (gallons) passing a given point in a unit of time (minute). In a river, an open pipe supplied by nature, this flow of water is called a *current*, and is measured in gallons per minute, or millions of gallons per hour, or some other convenient combination of quantity and time. By analogy, the flow of electric charge is called electric *current*, and is measured in *coulombs per second*. For convenience, this unit of current has been given a name of its own: *ampere*. Thus a current of ten amperes means the flow of ten coulombs of charge each second. (Note that "current" is the "flow of charge." It is logically redundant to say "a current flows through a wire.")

When the flow of charge is steady, the current is $I = Q/t$. When the flow is not steady, the instantaneous current is the instantaneous rate of charge flow and is given by the time derivative: $I = dQ/dt$.

Recall the discussion on potential energy (W) and potential difference (V). Since potential difference is the energy per unit charge, $V = W/Q$, we can write $W = VQ$ for the work done by the charge Q in "falling" through a potential difference V. When the p.d. is constant, we can differentiate with respect to time and find

$$P = dW/dt = V\,dQ/dt = VI$$

since *power* is the rate of doing work. (Power is measured in *watts*, or joules per second.) This equation, $P = VI$, is one of the basic relations in the study of electricity.

Another basic relation was discovered by G. S. Ohm in 1827. Ohm found experimentally that if the voltage across a wire (the engineer's way of saying "the potential difference between the two ends of a wire") was increased, the current through the wire increased proportionately. That is, the ratio V/I is constant, for a given piece of wire. This ratio was given the name *resistance* ($R = V/I$) and is measured in *volts per ampere*. The unit of resistance, one volt per ampere, has been named the *ohm* to honor this pioneer electrician.

Ohm also found that if a second wire, identical with the first, was connected to offer the current an additional path, the current was doubled. That is, the two wires connected side-by-side, or in parallel, each passed as much current as the first wire by itself. This implies that the currents in the alternative paths can be computed independently, and added to find the total current.

He also found that if the two wires were connected in *series* (end-to-end) so that they carried the same current, the necessary voltage was doubled. That is, the voltage across an end-to-end set of wires is the sum of the individual voltages. In hindsight, these findings seem obvious, for a four-foot length of wire is the same thing whether we consider it as one four-foot length, two two-foot lengths, or four one-foot lengths, etc. Similarly, a fat wire can be conceived as a bundle of thin wires side-by-side.

Extension of the above experiments and logic showed that the resistance of a conductor is proportional to its length, and inversely proportional to its cross-sectional area, $R \propto l/A$. The constant of proportionality (ρ) that makes this relation an equation, $R = \rho l/A$, is a characteristic of the material of which the conductor is made, and is called the *resistivity*. The resistivity of a material varies with temperature, but is independent of the shape and size of the conductor. Since $\rho = RA/l$, its unit is (ohms) times (square centimeters) divided by (centimeters), which simplifies to (ohms) times (centimeters), or ohm-cm. For good conductors, such as the metals, the resistivity is only a few millionths of an ohm-cm, so is commonly listed in handbooks in microhm-cm.

The combination of the power equation ($P = VI$) and Ohm's law ($V = RI$) yields by algebraic substitution, $P = I^2R = V^2/R$ as alternate ways of computing power.

Example.

An electric lamp that draws 100 watts at 120 volts has a resistance $R = V^2/P = 144$ ohms, and draws a current of $I = V/R = 0.833$ ampere, or $I = P/V = 0.833$ ampere.

1-3 Resistors in Combination. A brief digression on "things" and "representation" of things is in order at this point. We have seen that resistance is an abstraction; it is the ratio of a voltage to a current. A "device" which is used because it has resistance, is called a *resistor*. The schematic symbol —ΛΛΛ— used in *wiring* diagrams represents a resistor, and connecting lines represent actual wires. In this case the diagram is a conventional "picture" of an actual assembly of concrete "things," such as resistors and batteries. On the other hand *circuit* diagrams also represent abstractions, such as combinations of voltage and resistance. Real resistors can overheat, or burn up, or have peculiar unexpected properties,

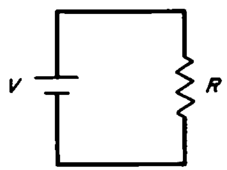


FIG. 1.1.

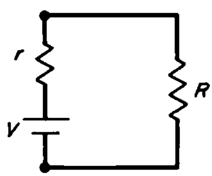


FIG. 1.2.

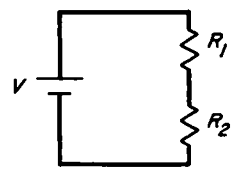


FIG. 1.3.

whereas the abstract resistance of a theoretical diagram is a well-behaved mathematical quantity. The circuit diagrams, or networks, in textbooks represent abstract concepts of resistance, voltage, etc. Circuit theory is an intriguing mathematical game, whose answers are always perfect. If the idealized "mathematical circuit" turns out to be a reasonable representation of the properties of a real device, then the answers of the game will also be a reasonable approximation to what the real device will do.

The mathematics of circuit theory is perfect and exact; the engineer's big problem is to make sure that his mathematical model truly represents the device he is building. Stray wiring capacitance, lead inductance, and leaky insulation do not show on a *wiring diagram*, but must be included in an *abstract circuit diagram* if the engineer wants his analysis to give him good results. For example, Fig. 1.1 as a wiring diagram represents a resistor connected across a battery. Every practical man knows that a battery cannot deliver an infinite current; if the resistance is made too small, the voltage across it will not be V, but will be less. For our purposes, however, Fig. 1.1 is an abstract diagram, and the voltage across the resistance is V no matter how much current is drawn. For "reasonable" currents, the two interpretations of the figure are equivalent. For "large" currents, we shall see later that the voltage symbol —|⊢ by itself is not sufficient to represent a real battery. In fact, a real battery has *internal resistance*, it behaves like a series combination of a perfect battery and a resistor, as in Fig. 1.2. This representation would never be used in a *wiring diagram*, for some technician would be sure to follow it literally and install a resistor r!

Ohm's law tells what happens when a single resistor is connected across a voltage source, as in Fig. 1.1: $I = V/R$. How do we find the current when two known resistors appear in series across a voltage, as in Fig. 1.3? If R_1 and R_2 were pipes carrying water, we would not hesitate to say that the same water flows through both pipes; i.e., they carry the same current. In the electrical case, this conclusion is still true. The argument by analogy does not *prove* the electrical case, it merely *suggests* it. The proof, however, follows the same lines for both water and electric charge. We assume that water cannot suddenly appear or disappear; it must all be accounted for. If the flow through R_1 is different from that through R_2, it can be due only to a leak at the connection, hence a third "pipe" should appear in the diagram. Similarly, electric charge is conserved, and cannot appear, disappear, or pile up at a connection. A current can split at a "fork in the road," but where there is only one path, as in Fig. 1.3, the current must be the same at all points in the circuit.

Now by Ohm's law, the voltage across R_1 is

$$V_1 = IR_1$$

and that across R_2 is

$$V_2 = IR_2$$

where the same symbol I appears in each equation, because it represents the same current in both cases. The total voltage, or p.d., between the upper terminal of R_1 and the lower terminal of R_2 is the sum

$$V = V_1 + V_2 = I(R_1 + R_2)$$

as is suggested by our old friend, the water flow analog, with p.d. analogous to water pressure. Since $V = IR$, it is apparent that the net resistance of the series combination is

$$R = R_1 + R_2$$

This result can be deduced rigorously by appealing to the conservation of energy. The power dissipated in the resistances is

$$P_1 = I^2R_1, \quad P_2 = I^2R_2$$

and the total power supplied by the voltage source is therefore

$$P = P_1 + P_2 = I^2(R_1 + R_2) = I^2R$$

The argument applies to any number of resistances in series:

$$R = R_1 + R_2 + R_3 \cdots$$

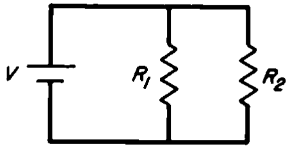


FIG. 1.4.

Similarly, if we connect two resistances in parallel, i.e., *across the same voltage*, as in Fig. 1.4, the respective currents are

$$I_1 = V/R_1 \quad \text{and} \quad I_2 = V/R_2$$

Our fundamental hypothesis on the conservation of charge requires the total current to be

$$I = I_1 + I_2 = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V/R$$

so that the total current is the same as would be drawn by a single resistance R computed from

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \quad \text{or} \quad R = \frac{R_1R_2}{R_1 + R_2}$$

Example.

Two resistances of 50 ohms and 100 ohms yield a *series* resistance of 150 ohms; and a *parallel* resistance of 33⅓ ohms.

Again, our formula can be extended to any number of resistances in parallel:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

For more than two resistances in parallel, the simplest computation is to use the equation as shown: add the reciprocals of the resistances, and take the reciprocal of the sum, i.e.,

$$R = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \cdots}$$

The formula corresponding to $R_1R_2/(R_1 + R_2)$ is not convenient for more than two resistances.

Example.

The parallel combination of 6, 4, 3, and 2 ohms has the resistance R given by

$$\frac{1}{R} = \frac{1}{6} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{2}{12} + \frac{3}{12} + \frac{4}{12} + \frac{6}{12} = \frac{15}{12}$$

so that $R = 12/15 = 0.8$ ohm.

A *series-parallel* combination, such as shown in Fig. 1.5, requires a piecemeal analysis using both formulas. The series combination of 10 and 20

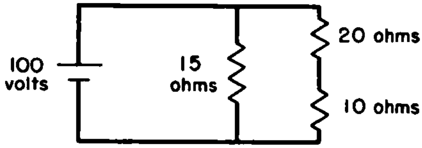


Fig. 1.5.

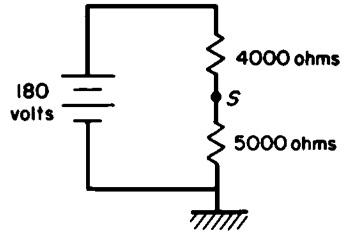


Fig. 1.6.

is 30 ohms; this 30 ohms in parallel with 15 ohms yields a net resistance of $30 \times 15/45 = 10$ ohms. The battery current is therefore 10 amperes, of which $100/15 = 6\frac{2}{3}$ is in the 15-ohm branch, and the remaining $3\frac{1}{3}$ is in the 30-ohm branch. This $3\frac{1}{3}$ amperes produces a *voltage drop* (p.d.) of $66\frac{2}{3}$ volts across the 20-ohm resistance, and $33\frac{1}{3}$ volts across the 10 ohms. Note that the 10–20 series combination divides the supply voltage in that ratio.

This voltage-dividing property of resistors in series is often used in radio receivers, where there is a supply of, say, 180 volts for the plate of a tube and, say, 100 volts is wanted for the screen grid. If we connect a voltage divider as in Fig. 1.6, we will have 100 volts across the lower resistor, or between point S and ground. The 100 volts at S is, however, the *no-load*, or *open-circuit voltage* (OCV). If the screen draws, say, 4 ma (0.004 ampere) at 100 volts, the extra 4-ma current through 4000 ohms would produce an additional voltage drop of 16 volts. Instead of 100 volts at S, we would have only 84 volts; but at 84 volts the screen would draw less than 4 ma. If we know the screen current at 84 volts, we can recompute the actual voltage at S; repeating this procedure would give a set of successive approximations that would finally steady down to the correct answer—but what a lot of work! In any case, we don't really want to know what voltage will appear at S; we want to know what resistances to use in the voltage divider so that we will have 100 volts at S, with a current drain of 4 ma.

Since 4 ma at 100 volts represents a load of 25,000 ohms, our complete circuit is as in Fig. 1.7. If we wish to keep $R_2 = 5000$, the total current is $100/5000 + 0.004$, or 24 ma. This current must produce an 80-volt drop across R_1, or $R_1 = 80/0.024 = 3333\frac{1}{3}$ ohms.

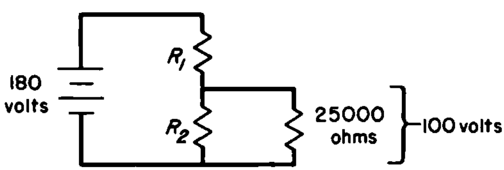


Fig. 1.7.

Problem.

Leave R_1 at 4000 ohms, and compute the approximate value of R_2. (Answer: 6250)

Let us now investigate the general behavior of a voltage divider (Fig. 1.8) by using algebra (instead of arithmetic, which is used for particular situations). Let the load resistance be R_L, the voltage across the load, V_L, and the current through the load, I_L. We are interested in the way the

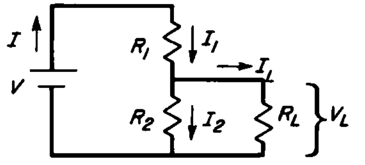


FIG. 1.8.

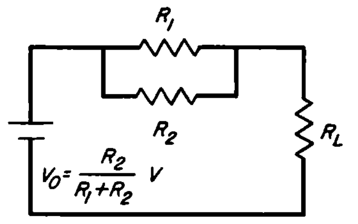


FIG. 1.9.

output voltage varies with load. Since all current must be accounted for, we have $I = I_1 = I_2 + I_L$. We have also

$$V_L = I_L R_L = I_2 R_2$$

$$V = V_L + I_1 R_1$$

These last two equations give

$$I_1 = (V - V_L)/R_1 \quad \text{and} \quad I_2 = V_L/R_2$$

which, substituted into the first equation, yields

$$\frac{V}{R_1} - I_L = V_L\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_L \frac{R_1 + R_2}{R_1 R_2}$$

and, finally,

$$V_L = \frac{R_2}{R_1 + R_2} V - \frac{R_1 R_2}{R_1 + R_2} I_L$$

But this last equation also describes the behavior of the circuit of Fig. 1.9, and furthermore, the voltage V_o of Fig. 1.9 is the OCV of the original voltage divider! The implication of this result is, that if we have two boxes containing the alternate arrangements of Fig. 1.10, there is *no external experiment that can distinguish one box from the other.*

The OCV is obviously the same for the two boxes. The short-circuit currents are readily computed to be the same, also.

$$I_S = V/R_1 = V_o/R_0$$

For the purposes of *circuit theory*, then, the two boxes are *externally* equivalent; as *wiring diagrams* they would differ, for one battery would run down without a load. Even with a perfect battery, the boxes are not *internally* equivalent, for the total power dissipation differs. But to all *external* appearances, the boxes behave identically.

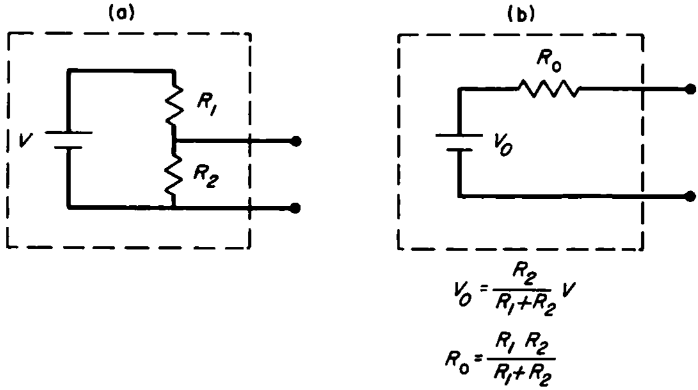


FIG. 1.10.

This example is a particular illustration of Thevenin's theorem. The general case will be discussed in later chapters.

The power delivered to a load by a voltage divider exhibits an interesting property. The load current is

$$I_L = V_o/(R_0 + R_L)$$

so the load power is

$$P = I_L^2 R_L = V_o^2 R_L/(R_0 + R_L)^2$$

The load power goes to zero as R_L goes to zero, by virtue of the R_L in the numerator. As R_L increases without limit, the power again goes to zero by virtue of the square in the denominator. The value of R_L for maximum power can be found by differentiation. Now

$$\frac{dP}{dR_L} = V_o^2 \frac{(R_0 + R_L) - 2R_L}{(R_0 + R_L)^2}$$

which vanishes, indicating the maximum of P, when $R_L = R_0$, and giving

$$P_{\max} = V_o^2/4R_0$$

This maximum power is called the "available power." Note that it is obtained by matching resistances, i.e., by making the load resistance equal to the *internal resistance* of the source.

All real sources (batteries, generators, amplifiers, etc.) are imperfect—they have internal resistance and are equivalent to the "black box" of Fig. 1.10b. In the ideal case of linear behavior, the internal resistance is constant (independent of current) and can be found experimentally as the ratio: (open-circuit voltage) ÷ (short-circuit current).

1-4 Graphical Description of Source and Load. Let us reconsider our previous problem of a load R_2 connected across the source Fig. 1.10b. We shall take the output voltage (V) and current (I) as variables for description of the source and load behavior. For any current I the output, or terminal voltage, is

$$V = V_o - R_0 I \tag{1-1}$$

whereas the load resistance specifies the relation

$$V = R_L I \tag{1-2}$$

These equations must *both* be satisfied, and by the same pair of values V, I. The solution of these simultaneous equations can be found by substituting one in the other, yielding

$$I = V_o/(R_0 + R_L) \tag{1-3}$$

$$V = V_o R_L/(R_0 + R_L) \tag{1-4}$$

This problem can also be solved graphically. Equation (1-1) says that the allowable pairs of V, I are represented by the points of the straight line of Fig. 1.11. The line is constructed by connecting its extreme points: (a) open-circuit voltage and zero current and (b) zero voltage and short-circuit current. The slope of the line is $-R_0$. Since Fig. 1.11 shows the

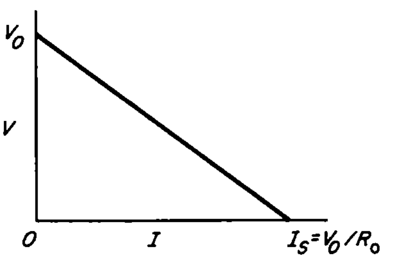


FIG. 1.11.

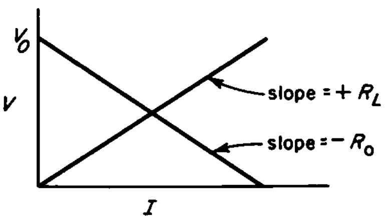


FIG. 1.12.

terminal voltage for a specified current, or vice versa, it is a graphical description of the electrical properties of the source. The load can similarly be described by plotting Eq. (1-2); superposing the load and source lines on the same graph gives Fig. 1.12; the intersection satisfies both equations and is the point specified by Eqs. (1-3) and (1-4).

This graphical technique will be very valuable later on, when we consider nonlinear devices, i.e., those whose "resistance" varies with the current.

1-5 Current Generators. Electrical sources are usually thought of as voltage sources because our first sources (batteries, dynamos) delivered a terminal voltage that was not sensitive to the current drawn. That is, these common sources have low internal resistance. (When we say "low," we mean relative to the resistance of the loads ordinarily used with these sources.) On the other hand, some of our modern devices such as photocells, pentodes, and radioactive batteries, have such high internal resistance that the output *current* is substantially independent of the load resistance; the load is a reasonable approximation to a short-circuit, relative to the internal resistance. The load current is

$$I = V_o/(R_0 + R_L) \doteq \begin{cases} V_o/R_L, & R_0 \ll R_L \\ V_o/R_0, & R_0 \gg R_L \end{cases}$$

and the terminal voltage is

$$V = V_oR_L/(R_0 + R_L) \doteq \begin{cases} V_o, & R_0 \ll R_L \\ V_oR_L/R_0, & R_0 \gg R_L \end{cases}$$

In terms of the short-circuit current I_0, we can write

$$I \doteq \begin{cases} V_o/R_L, & R_0 \ll R_L \\ I_0, & R_0 \gg R_L \end{cases}$$

$$V \doteq \begin{cases} V_o, & R_0 \ll R_L \\ I_0R_L, & R_0 \gg R_L \end{cases}$$

It is apparent that for $R_0 \ll R_L$, the source is more conveniently described as a source of voltage V_o. Similarly, for $R_0 \gg R_L$, the source is more conveniently described as a source of *current* I_0. The two extreme idealized cases are *constant-voltage* and *constant-current* sources.

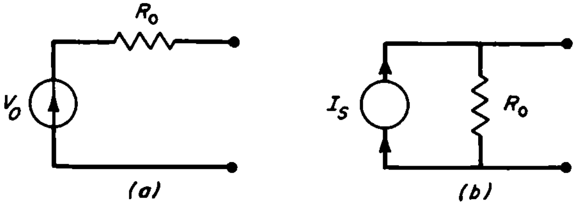


FIG. 1.13.

In fact, an actual source such as described graphically by Fig. 1.11, can be represented schematically by either Fig. 1.13a or Fig. 1.13b; the ideal source in Fig. 1.13a is a perfect *constant-voltage* generator, that in Fig. 1.13b a perfect *constant-current* generator. Since $V_o = I_SR_0$, it is readily verified that Fig. 1.13a and b indicate the same values of open-circuit voltage and of short-circuit current, and in fact both indicate the same behavior for all loads, as shown by Fig. 1.11.

The choice of representations offered in Fig. 1.13 is purely a matter of convenience; they are equally "real."

Since an ideal voltage generator exhibits a terminal voltage that is *independent* of current, it offers no reaction to a current supplied by another generator, i.e., it is a zero-resistance device and is equivalent to a short-circuit as far as all other sources are concerned. This is the reason for the equivalence of the two "black boxes" of Fig. 1.10; any experiment "looking" into the terminals will "see" resistances R_1 and R_2 in parallel, since the source V appears to be a short-circuit as far as resistance effects are concerned.

Similarly, since the current through an ideal current generator is unaffected by voltage, no additional current will be produced by an external generator, and a constant current source acts like an infinite resistance, or open circuit. Hence experiments "looking" into the terminals of Fig. 1.13 will "see" a total resistance R_0. The generator symbols adopted here exhibit these properties of "active short-circuit" and "active open circuit."

1-6 Conductance. The above "duality" between voltage and current sources suggests that we take another look at Ohm's law. We used it in the form $V = RI$, but we can also write it as $I = GV$, where we have put $G = 1/R$ for convenience. The quantity G is called *conductance* (the larger G, the better the element "conducts").

For "elements" in parallel, we found their net effect to be that of an element having the resistance R given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

If we describe these elements by their conductances ($G_1 = 1/R_1$, etc.), we have for the conductance of the parallel combination:

$$G = G_1 + G_2 + G_3 \cdots$$

Conversely, elements in series combine to yield a resistance

$$R = R_1 + R_2 + R_3 \cdots$$

or a conductance given by

$$\frac{1}{G} = \frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3} + \cdots$$

1-7 Nonlinear Resistance. Many electrical and electronic devices do not obey Ohm's law. Consider a wire-wound resistor which is overloaded, so that it heats up. Unless made of special alloy wire, its resistance will increase with temperature. Then the more current through the resistor, the hotter it will get and the higher will be its "resistance." Such a device can readily be described graphically by its V-I characteristic (Fig. 1.14). An ordinary tungsten-filament incandescent lamp is a familiar example of this type of resistor. The V-I curve of Fig. 1.14 is valid only for thermal equilibrium. If we change I (or V) quickly, the operating point (pair of values of V and I) will not be on the curve, but will drift to the curve as the filament comes to its new temperature equilibrium. Special "lamps" have been developed having a characteristic such as shown in

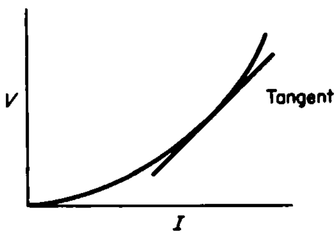


FIG. 1.14.

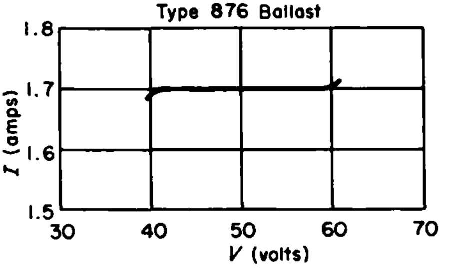


FIG. 1.15.

Fig. 1.15. They are used as "ballast" resistance to absorb variations in supply voltage. Since their temperature change is sluggish, they will not follow a rapid variation, such as the alternations of a 60-cps line supply, but will follow the slow variations of line voltage throughout the day. Hence such devices can be used to stabilize a load current against line voltage changes, in either ac or dc systems.

There are, however, devices which have unique curves of the type of Fig. 1.14. That is, the curve is valid for any rate of change of operating point, up to thousands of cycles per second. Such nonlinear devices are called *varistors* and are commercially available with a wide range of characteristics. Generally speaking, the voltage drop is an algebraic power of the current: $V = I^a$. The exponent a can be made as high as 5 or 6. This behavior is exhibited throughout a wide operating range; the V-I curve is customarily plotted on log-log graph paper so as to be a straight line.

Even though a varistor does not have a resistance in Ohm's sense (constant ratio V/I), small variations in I give the operating point a small excursion, and the path of the operating point can be considered to be a small part of a tangent to the V-I curve (Fig. 1.14). Thus for small variations the operating point simulates the behavior of a constant resistance:

$$R = dV/dI = \text{slope of tangent}$$

This *incremental resistance* is also called the *dynamic resistance* (at the given operating point). This concept will occur again in the study of thermionic tubes.

The current-regulating behavior of ballast lamps and varistors can be shown graphically. In Fig. 1.16, resistance R_0 represents the useful load, and B a voltage-dropping element, either a resistor or a varistor. We can consider R_0 to be internal resistance of a voltage source applied to B. If B is an ohmic resistance, the operating current can be found from a diagram similar to Fig. 1.12. If now the supply voltage V varies, say between V_1 and V_2, the current will range between the two intersections of Fig. 1.17.

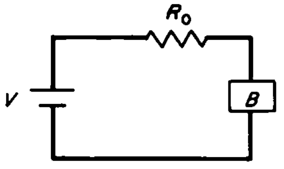


Fig. 1.16.

If, on the other hand, B is a varistor with V increasing more rapidly than the first power of I, the load line will be concave upward, as in Fig. 1.18. The variation of operating current for the curved load line is obviously less

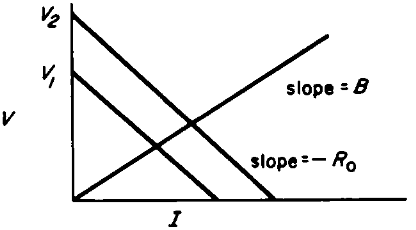


Fig. 1.17.

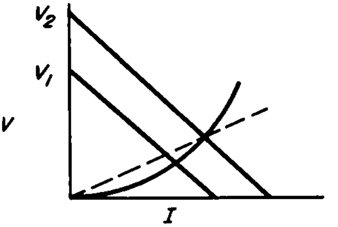


Fig. 1.18.

than for the (dotted) pure resistance load line. The range of voltage drop across the ballast device is greater, i.e., more of the V_2-V_1 supply variation appears across the ballast, and less across the useful load.

Chapter II

GENERAL DIRECT CURRENT NETWORKS

The most interesting and useful circuits are not built of simple series and parallel combinations. Any network (the general name for an interconnected set of electrical elements, such as resistors) that is not made up of series-parallel sets in series-parallel arrangements is called a "bridge" network. This is because a "bridge" network is composed of series-parallel subnetworks with additional elements making "bridges" across various gaps. The simplest example is that of Fig. 2.1, where M is the bridging

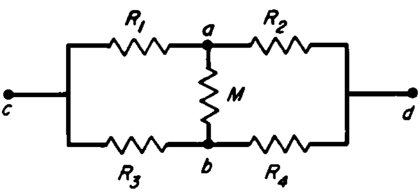


FIG. 2.1.

element. Removal of M reduces the network to a series-parallel arrangement.

Let the element M be removed, and a voltage source (V) connected between the points c and d. Then the current in the top line is

$$I_t = V/(R_1 + R_2)$$

and the p.d. between a and c is

$$I_tR_1 = VR_1/(R_1 + R_2)$$

Similarly, the p.d. between b and c is $VR_3/(R_3 + R_4)$. If these two p.d.'s are equal, the p.d. between a and b will be zero; if M is now connected, it will carry no current and have no effect on the network, since it will have zero voltage across it. This is true even if M is a complete short-circuit. Thus even a sensitive meter (voltmeter, ammeter, galvanometer) connected as element M will carry no current and "see" no voltage, so will give a zero reading. The condition for this zero p.d. is the equality of the *a-c* and *b-c* p.d.'s or

$$\frac{VR_1}{R_1 + R_2} = \frac{VR_3}{R_3 + R_4}$$

hence $R_1R_4 = R_2R_3$ or as ratios:

$$R_1/R_2 = R_3/R_4$$

$$R_1/R_3 = R_2/R_4$$

This condition is independent of the applied voltage V, and is known as the condition of *balance* of this particular bridge (Wheatstone bridge).

Now let one of the resistances, say R_1, be an unknown which it is desired to measure. If R_2 and R_4 are known, and R_3 is a calibrated variable resistor, then R_3 can be adjusted for balance as indicated by a zero reading on a meter M. The resistance R_1 is then computed from the equation for balance.

A familiar laboratory form of this bridge is the *slide-wire bridge* (Fig. 2.2). The unknown resistance is X; R_2 is a known fixed resistance. Since R_3 and R_4 enter the equation only as a ratio, their actual values need not be known. The series combination of R_3 and R_4 can be a uniform wire, say one meter long. The connection point b is provided by a sliding contact. The resistance ratio R_3/R_4 is equal to the ratio L_3/L_4 of the corresponding lengths of the slide wire, and these numbers can be read directly from a meter stick mounted alongside the wire.

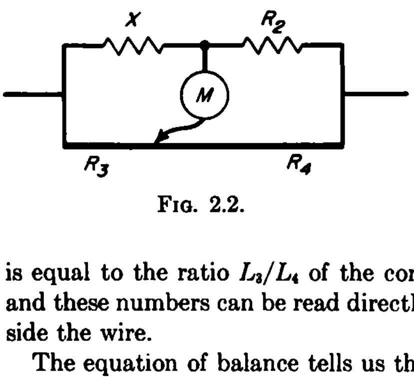


FIG. 2.2.

The equation of balance tells us the value of the unknown resistance X *if the bridge is balanced*, but to balance the bridge, we must adjust it for zero current through M. This raises the question of sensitivity: How much can the bridge be unbalanced before M will give a detectable reading? How much current will M pass if the ratio R_3/R_4 is off by (say) one percent? We can easily compute the OCV (M removed) between a and b if V is known and the unbalance is known. For example, let $X = R_2$ so that $R_3 = R_4$ for balance. Then if $R_3 \neq R_4$, the OCV between a and b is

$$\text{OCV} = \frac{VR_1}{R_1 + R_2} - \frac{VR_3}{R_3 + R_4} = \frac{V}{2} - \frac{V}{1 + R_4/R_3}$$

$$= V\frac{R_4/R_3 - 1}{2(R_4/R_3 + 1)}$$

The current through M will *not* be this OCV divided by the resistance of the meter, for the meter "sees" this OCV as a source having an internal resistance that depends on R_1, R_2, R_3 and R_4. Examining the sensitivity of the bridge requires first that we learn how to compute the current through M; in general, the current in any resistance of a network.

2-1 Network Topology. The most fundamental relations of a network are determined solely by the pattern of its connection, without regard to the values or even the kinds of elements. So far we have discussed only resistance; later on we shall consider inductance and capacitance as well, but this will not affect our present discussion of the behavior of a network.

Our starting point is the *graph* of a network: a set of lines and junctions representing elements and connections. For example, the networks of Fig. 2.1 and Fig. 2.2 can be represented by the graph of Fig. 2.3.

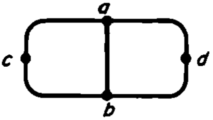


FIG. 2.3.

With a voltage source connected between c and d, the graph can take any of the forms of Fig. 2.4. These graphs are all *topologically* equivalent; they represent the same *junctions* (or nodes) and *branches* connected in the same way. Geometrical shape has no significance. The complicated behavior of a network is associated with the multiplicity of closed paths. A bookworm starting at a could follow the path *acbda*, returning to his starting point; or he could choose *acba*, *adba*, or *acda*.

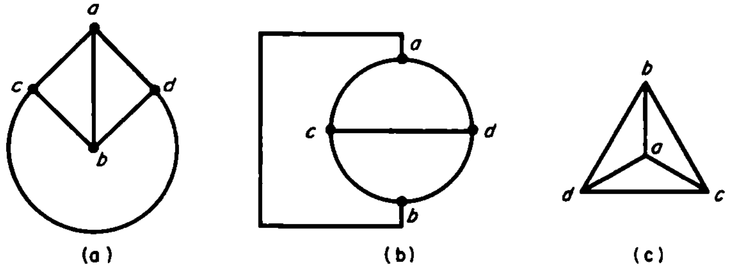


FIG. 2.4.

To simplify the properties of a graph, we look for an underlying structure having *all* the nodes but *no* closed paths. Such a skeleton is called a *tree*, and is constructed by removing branches from a graph until there are no closed paths. Some possible trees of Fig. 2.4a are shown in Fig. 2.5. The branches remaining in a given tree are called *tree-branches*; the branches that were removed are *links*. Restoration of any one link by itself (into a given tree) produces one, and only one, closed path. These closed paths associated with the links are called *loops* and will be of importance later.

2-2 Electrical Variables. Relating the topology, or structure, of a network to its electrical behavior requires the introduction of electrical variables, such as voltage or currents. Since the voltage across any branch is related to the current through the branch by Ohm's law, we can describe the network performance completely in terms of all the branch currents, or all the branch voltages. We shall first let the branch currents be the variables, and discuss the voltage viewpoint later.

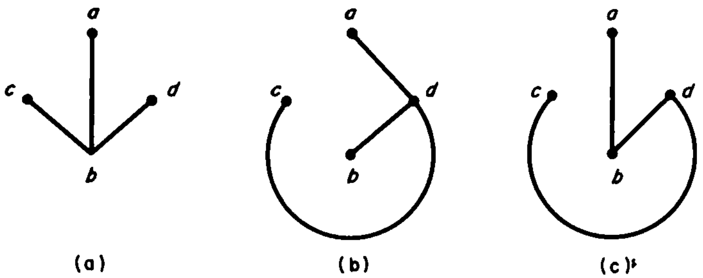


FIG. 2.5.

Let the network graph have B branches. Not all B branch currents can be independent, since at each node (junction) we have a constraining condition by virtue of the conservation of charge; i.e., charge can be neither created nor destroyed. We shall now see that currents in the links are independent and that all other branch currents can be expressed in terms of the link currents.

We constructed a *tree* by removing *link* branches to open all closed paths. The tree connects all the original nodes, say N in number. If we remove a "terminal" node and the branch that connects it, we have a smaller tree. Continuing this process of simultaneously removing one node and one tree-branch, we eventually arrive at a single node, analogous to the root of the tree. Thus the number of tree-branches is $N - 1$, one less than the total number of nodes. This is true for each of the trees that can be associated with a given graph. Since we started with B branches, and removed L links to form the tree, the number of links satisfies the equation

$$B = L + (N - 1)$$

With these L links removed, there are no closed paths, hence all branch currents are zero. This means that if the L link currents are specified to be zero, the remaining currents are determined. Hence no more than L branch currents can be independent (i.e., arbitrarily specified). That the L link currents *are* all independent is seen by restoring one link. This produces a closed circuit not involving any other link, hence its current can be specified without regard to other link currents. Thus for a graph of B branches and N nodes, there are $L = B - N + 1$ arbitrarily specifiable link currents, and $N - 1$ remaining branch currents that are uniquely determined. We shall soon see how to express these $N - 1$ branch currents as simple sums of the link currents.

2-3 Loop Currents. If we indicate links by dotted lines, and tree branches by solid lines, the graph of Fig. 2.4 can be represented as in Fig. 2.6, for one choice of tree.

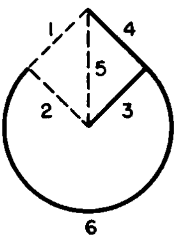


FIG. 2.6.

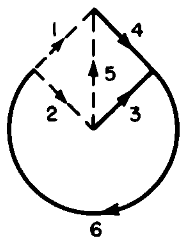


FIG. 2.7.

Allowing link 1 to be present, and no other links, we have a closed path, or *loop*, comprising branches 1,4, and 6. Link 2, by itself, closes the loop 2,3,6; and link 5 closes 3,4,5. Since we are to be concerned with currents, we shall find it convenient to associate a "sense" with each branch, to be able to assign "forward" and "backward" or positive and negative to the flow through a branch. For example, assign the arbitrary "positive direction arrows" to the various paths as in Fig. 2.7. Then our three loops can be more explicitly described as $+1$, $+4$, $+6$; $+2$, $+3$, $+6$; and $+5$, $+4$, -3. (We can read $(+)$ as "forward"; "minus" as "backward.") A simple scheme for describing the relations between the loops and branches is given by the table:

Loop No. \ Branch No.	1	2	3	4	5	6
1	1	0	0	1	0	1
2	0	1	1	0	0	1
5	0	0	−1	1	1	0

A zero in the table indicates that the corresponding branch is not in the loop. "Plus one" indicates a forward traversal of a branch in a given loop, and "minus one" a backward traversal.

Since each link occurs in one, and only one, loop, we have numbered the loops to correspond to their links. We now see that in numbering the branches of the graph it would have been better to number all the links first, so that the loop numbers would have been consecutive. The "awkward" numbering used here, however, emphasizes the correspondence between loops and links; consecutive loop numbers generate a deceptively simple looking table.

Since a current through link 1, in the absence of other links, must "circulate" around loop 1, it is convenient to think in terms of circulatory *loop currents* instead of in terms of link currents. Since the current in any loop is identically the current through the link associated with that loop, we can use one symbol for both. That is, we shall use i_2 to represent either the current through link 2, or the current in loop 2; they are identical. What we have gained from the loop concept is the association of links, loops, and branches as shown by the table. We can now state, by *inspection of the columns*, that branch 4 carries the current of loop 1 and the current of loop 5, and no other. Thus the forward current through branch 4 is $i_1 + i_5$; that through branch 3, $i_2 - i_5$.

To recapitulate: we formed the table a *row* at a time, by closing one link at a time, and tracing the resulting loop. By reading the table a *column* at a time, we can express the branch currents in terms of loop currents or link currents.

For notational distinction, we shall let j stand for a branch current (*any* branch) and restrict i to link or loop currents. The current relations indicated by the table can then be written:

$$j_1 = i_1 \qquad j_4 = i_1 + i_5$$

$$j_2 = i_2 \qquad j_5 = i_5$$

$$j_3 = i_2 - i_5 \qquad j_6 = i_1 + i_2$$

A more elaborate network is represented by Fig. 2.8. The corresponding loop-branch relation table is:

Loop \ Branch	1	2	3	4	5	6	7	8
1	1	0	0	0	−1	1	1	0
2	0	1	0	0	−1	1	0	0
3	0	0	1	0	0	0	1	−1
4	0	0	0	1	0	1	1	−1

This is constructed a row at a time, by tracing a loop, entering 1 or −1 in the appropriate branch column for each branch traversed in order, and finally filling the empty spaces with zeros.

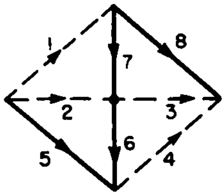


FIG. 2.8.

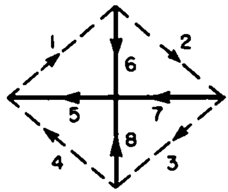


FIG. 2.9.

The same network can be analyzed with the more convenient tree shown in Fig. 2.9.

Problem. Construct the loop-branch table for Fig. 2.9 and show that the branch currents are given by

$$\dot{j}_1 = \dot{i}_1 \qquad \dot{j}_5 = \dot{i}_1 - \dot{i}_4$$
$$\dot{j}_2 = \dot{i}_2 \qquad \dot{j}_6 = \dot{i}_1 - \dot{i}_2$$
$$\dot{j}_3 = \dot{i}_3 \qquad \dot{j}_7 = \dot{i}_2 - \dot{i}_3$$
$$\dot{j}_4 = \dot{i}_4 \qquad \dot{j}_8 = \dot{i}_3 - \dot{i}_4$$

2-4 Mesh Currents. The loops of Fig. 2.9 are particularly simple. Each loop is the periphery of one of the "holes" in the "net." By analogy with a fish net, such a "hole" is called a *mesh*. In this case then, each loop current is associated with a single mesh, and the loop currents can be represented pictorially as in Fig. 2.10.

A *planar* network is one that can be drawn in a plane without any line crossovers that are not connections (nodes). Such a circuit can be made by printed circuit techniques without using the back side of the support. The network of Fig. 2.11, for example, is nonplanar; it cannot be drawn on paper without a nonconnected crossover. A possible tree for Fig. 2.11 is shown in Fig. 2.12, along with its loop-branch table, zeros omitted. Such a network can be analyzed on a loop basis, but the term "mesh" would have no meaning.

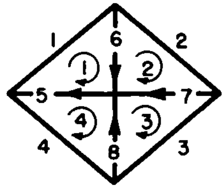


FIG. 2.10.

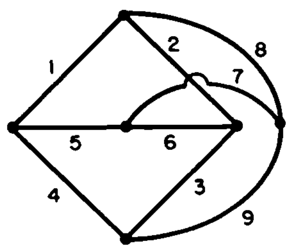


FIG. 2.11.

In a planar network, the choice of meshes for the loops is particularly convenient. Even without constructing a table, we can inspect Fig. 2.10 and immediately write:

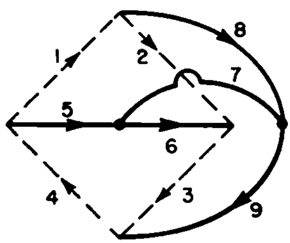
$$j_5 = i_1 - i_4$$

$$j_6 = i_1 - i_2$$

$$j_7 = i_2 - i_3$$

$$j_8 = i_3 - i_4$$

Note that the positive sense in the links was so chosen that all mesh currents circulate in the same sense; clockwise in this case. The easy way to



Loop \ Branch	1	2	3	4	5	6	7	8	9
1	1				−1		−1	1	
2		1				−1	1	−1	
3			1			1	−1		−1
4				1	1		1		1

FIG. 2.12.

do this is to draw the graph *without* direction arrows in the branches, and then to *assign* clockwise mesh currents. These define the positive sense in the links; the sense in the remaining branches (tree-branches) is arbitrary.

2-5 Branch Voltages. The discussion thus far has related to only the topological properties of networks and currents. To apply our new knowledge we must add a consideration of voltage. After all, the study of electrical circuits is just the study of the relations between voltages and currents as affected by the values (e.g., resistance) of the elements and the topology of their interconnections.

Let one of the branches of Fig. 2.10, say branch 6, contain resistance R and nothing else (i.e., no battery or generator). If the current through branch 6 is positive (i.e., has the sense of the arrow), Ohm's law tells us that the voltage across branch 6 is $V_6 = j_6R$, and that the "top" of branch 6 is positive with respect to the "bottom" of branch 6. Hence the potential at the "bottom" of branch 6 is *lower* than that at the "top," and the arrow points in the direction of *lower* potential—the direction of *positive voltage drop*. If the current is negative, i.e., directed against the arrow, the voltage drop is reversed, or opposite to the direction of the arrow. We conclude that the algebraic relation $V_6 = j_6R$ yields not only the magnitude of the voltage drop across R, but also its *sign*, if we adopt the convention that the arrow indicates the direction of positive *drop*.

If we now insert a battery of voltage E_6 with the polarity shown in Fig. 2.13, a charge traveling in the direction of the arrow still undergoes a potential *drop* on traversing the resistance but experiences a potential *rise* on passing through the battery. The net *drop* becomes $V_6 = j_6R - E_6$. Our choice of polarity of the battery inserted an emf in the direction of the arrow, i.e., a *positive* emf, using the arrow for the direction of positive emf. In general, then, the voltage drop across any branch is

$$V = jR - E$$

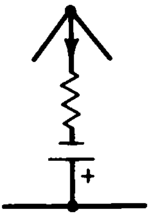


FIG. 2.13.

with the arrow indicating the positive sense of V, j, and E.

2-6 Kirchhoff's Voltage Law. If an imaginary charge is forced to travel once around any closed path in a network, it finds itself at its starting point, hence at its original potential. The net effect of all the voltage drops (positive and negative) it has experienced is zero. Algebraically, the sum of the voltage drops around *any* closed path is zero. This is Kirchhoff's voltage law. Thus for Fig. 2.10,

$$V_1 + V_6 + V_5 = 0$$
$$V_2 + V_7 - V_6 = 0$$
$$V_1 + V_2 + V_3 + V_4 = 0$$
$$V_1 + V_2 + V_7 + V_5 = 0$$
etc.

Note that the fourth of these equations is redundant; it is the sum of the first two equations. The corresponding closed path is the "sum" of the first two paths. To use Kirchhoff's voltage law efficiently, we should write down only those equations which are independent closed paths.

We have previously seen, however, that the independent closed paths are the loops belonging to the links. Thus we need to write one Kirchhoff equation for each loop—no more and no fewer. For loop 1 we have

$$V_1 + V_6 + V_5 = 0$$

which becomes, by the previous section,

$$j_1R_1 - E_1 + j_6R_6 - E_6 + j_5R_5 - E_5 = 0$$

or

$$j_1R_1 + j_6R_6 + j_5R_5 = E_1 + E_6 + E_5$$

This last expression can be interpreted as stating that the sum of the (passive) resistive drops around a loop equals the sum of the (active) emf's around that loop.

There is no need to retrace all the loops in order to write an equation of this form for each loop. We have already learned to summarize the loop-branch relations in tabular form. For Fig. 2.9, the table is, as you found in doing the last problem:

Loop \ Branch	1	2	3	4	5	6	7	8
1	1	0	0	0	1	1	0	0
2	0	1	0	0	0	−1	1	0
3	0	0	1	0	0	0	−1	1
4	0	0	0	1	−1	0	0	−1

Compare the entries in the first row with the coefficients of the terms j_nR_n and E_n in the voltage equation. They are the same. This is because the relations between loops and branches is indifferent to whether we are tracing voltage drops or currents around a loop. The relations are topological. Hence by inspection of the table we write all four loop voltage equations:

$$
\begin{aligned}
j_1R_1 + j_5R_5 + j_6R_6 &= E_1 + E_5 + E_6 \\
j_2R_2 - j_6R_6 + j_7R_7 &= E_2 - E_6 + E_7 \\
j_3R_3 - j_7R_7 + j_8R_8 &= E_3 - E_7 + E_8 \\
j_4R_4 - j_5R_5 - j_8R_8 &= E_4 - E_5 - E_8
\end{aligned}
$$

We have four equations for eight unknown branch currents. The branch emf's are assumed known, being the given voltages that excite the currents we wish to find. Recall that only the loop currents are independent; the branch currents were expressed in terms of the loop currents by using the columns of the table. These relations were written out in the last problem. Substituting these expressions for the j's into our loop voltage equations yields

$$
\begin{aligned}
(R_1 + R_5 + R_6)i_1 \quad - R_6i_2 \quad\quad - R_5i_4 &= E_1 + E_5 + E_6 \\
-R_6i_1 + (R_2 + R_6 + R_7)i_2 - R_7i_3 \quad\quad &= E_2 - E_6 + E_7 \\
-R_7i_2 + (R_3 + R_7 + R_8)i_3 - R_8i_4 &= E_3 - E_7 + E_8 \\
-R_5i_1 \quad\quad - R_8i_3 + (R_4 + R_5 + R_8)i_4 &= E_4 - E_5 - E_8
\end{aligned}
$$

The coefficients of the unknown i_1, i_2, i_3, i_4 form a symmetrical array:

$$
\begin{matrix}
(R_1 + R_5 + R_6) & -R_6 & 0 & -R_5 \\
-R_6 & (R_2 + R_6 + R_7) & -R_7 & 0 \\
0 & -R_7 & (R_3 + R_7 + R_8) & -R_8 \\
-R_5 & 0 & -R_8 & (R_4 + R_5 + R_8)
\end{matrix}
$$

which will, in the next chapter, be called the *determinant* of this set of equations. The property of the array that all its diagonal terms are positive, and all other terms negative if non-zero, results from our choice of loop currents as mesh currents all oriented in the same sense, as shown in Fig. 2.10.

In fact, the neat form of this array when the loops are taken as the meshes makes it possible to write this array from inspection of the graph Fig. 2.10. Each diagonal term is the total resistance around the corresponding mesh. Each off-diagonal term is the negative of the resistance that is common to (1) the mesh corresponding to the row of the array and to (2) the mesh corresponding to the column of the array.

The right-hand sides of the equations can also be written by inspection of the graph, for each right-hand side is the *total forward emf* encountered around the mesh in the direction of the circulation arrow.

2-7 Recapitulation. Arbitrary reference direction arrows are assigned to the branches of a graph, and a convenient choice is made of links and tree-branches. By inspection, we write a table of loop-branch relationships. The rows of this table then show us how to write Kirchhoff voltage equations for each loop, in terms of branch currents and resistances. The columns express the branch currents in terms of loop or link currents. Substitution of these relations into the Kirchhoff equations gives us L simultaneous equations for the L unknown loop currents. Methods of solving simultaneous equations will be discussed in the next chapter.

2-8 Mesh Analysis. We have also seen that for a network of meshes (i.e., a planar network) we can write the final simultaneous equations by inspection.

Example.

The circuit of Fig. 2.14 is assigned clockwise mesh currents as shown. The resistance values are indicated in ohms, the voltage sources in volts. *By inspection*, we write:

$$\begin{aligned} 6i_1 \quad -i_2 \quad -3i_3 &= 0 \\ -i_1 + \quad 5i_2 \quad -2i_3 &= 2 \\ -3i_1 \quad -2i_2 + \quad 7i_3 &= 1 \end{aligned}$$

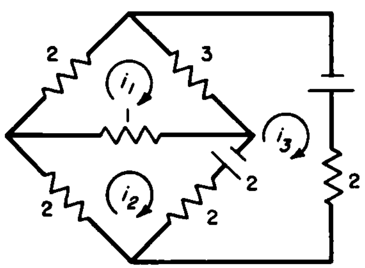


FIG. 2.14.

as the equations for the unknown mesh currents. If the current of primary interest is a branch current such as that through the 1-ohm bridge resistance, we see by inspection that this resistance carries a current $i_1 - i_2$ toward the left, or $i_2 - i_1$ toward the right. Hence after i_1, i_2, i_3 have been found, any branch current can also be found very simply.

2-9 Voltage Sources. We have seen that the voltage sources enter into the final equations only in terms of the total emf around each loop. Since each independent loop contains one, and only one, link branch, the equations would be unaffected if all voltage sources were removed from tree branches and replaced by suitable sources in the links. The suitable link source is an emf equal to the sum of all emf's in the associated loop.

We have taken all voltage sources as *series* elements of branches (of Fig. 2.14). Suppose we consider a voltage source to be connected across a resistance branch, as in Fig. 2.15, which shows part of a larger network. The voltage across R is now fixed at V, so the current through R is $I = V/R$, regardless of the remainder of the network. Hence one "unknown" is already found and effectively removed from the problem. The network for finding the remaining currents is shown in Fig. 2.16a which is equivalent to Fig. 2.16b, since all the battery does is to provide a fixed p.d. between the points b,c,d (at a common potential) and the point a. The net result of connecting the voltage source across R has been to simplify the network and convert the source into several *series* sources. Hence we need to consider only series sources of voltage in our general discussion.

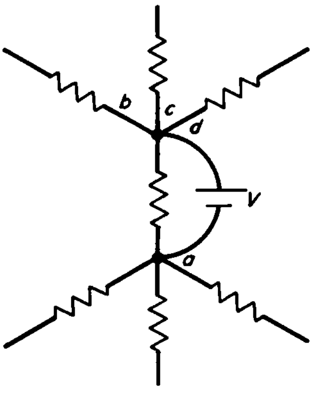


FIG. 2.15.

2-10 Voltage Variables. We have seen that a graph of B branches and N nodes could be analyzed in various ways as a tree of $N - 1$ tree-branches plus L connecting links. The link currents were independent and

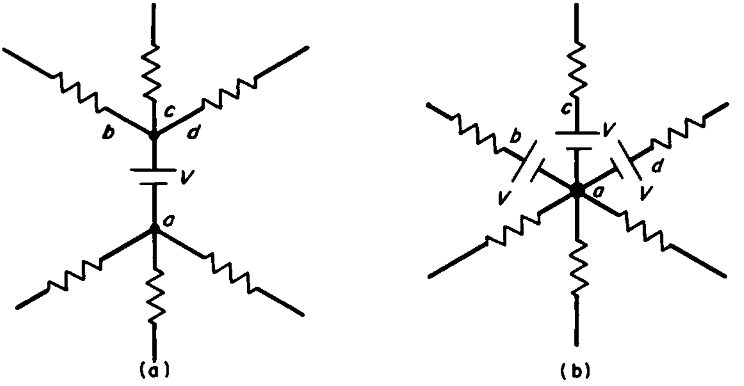


FIG. 2.16.

become the unknown variables of L simultaneous equations. We now turn back and reconsider the use of independent voltages as unknown variables. These can be taken as tree-branch voltage drops. Since there are no closed paths in a tree, its branch voltages are independent; furthermore, there are no other independent voltages, because the tree connects all nodes and thereby determines all node voltages in terms of tree-branch voltages. We can therefore anticipate the result of this approach to be

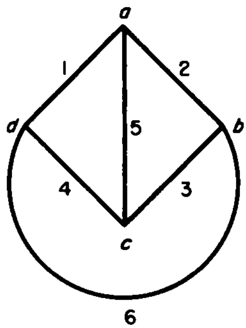


FIG. 2.17.

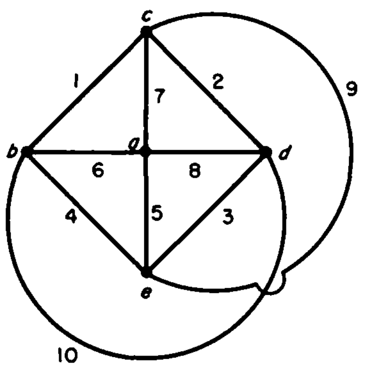


FIG. 2.18.

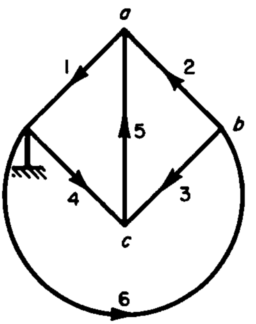


FIG. 2.19.

$N - 1$ simultaneous equations for $N - 1$ unknown voltages. For graphs having $N - 1 < L$, this will simplify the problem. An "indifferent" network is shown in Fig. 2.17; $B = 6$, $N = 4$, so $N - 1 = 3$ and $L = 3$ leading to three equations for either approach. On the other hand, Fig. 2.18 has $B = 10$, $N = 5$, making $N - 1 = 4$ while $L = 6$.

The potential difference between any two nodes is readily found as a simple sum of tree-branch voltages by tracing the path from one of the nodes to the other through the tree. Since a tree has no closed paths, this node-to-node path is unique. A particularly convenient set of voltage variables is formed of the p.d. between a reference node and each of the others in turn. This reference node is usually considered as "ground," so that any p.d. with respect to it is "voltage-to-ground," or simply "node voltage." Thus the voltages of Fig. 2.19 can equally well be described by $V_a = 1$, $V_b = -2$, $V_c = 3$, $V_d = 6$, $V_e = 1$. Since the node voltages (with respect to ground) are uniquely determined by the tree-branch voltages, and vice versa, they also comprise an independent set of voltages. *We shall therefore adopt the $N - 1$ node voltages as our independent voltage variables.*

Our next step in the analysis is to construct a node-voltage vs. branch-voltage table, analogous to our previous loop-branch table. Inspection of Fig. 2.19 (arrows arbitrary) shows that the voltage drop across branch 2, *in the direction of the arrow*, is $V_b - V_a$, where V_b and V_a are respectively the voltages at nodes b and a. Similarly, the drop across branch 1 is V_a and that across branch 4 is $-V_c$, since the potential of the reference node (ground) is taken to be zero. We can tabulate this information for all branches, a *column* at a time, as follows:

Node \ Branch	1	2	3	4	5	6
a	1	−1			−1	
b		1	1			−1
c			−1	−1	1	

After the ± 1's are entered as needed in each column, we can fill in the remaining space with zeros for completeness.

Comparison of the *rows* of the table with the graph shows that +1 indicates an arrow *leaving* a node; −1 indicates an arrow approaching a node. This property gives us an easy way of constructing the table by *rows*.

Example.

The graph of Fig. 2.20 possesses the following node-branch table (each row written directly by inspection).

Node \ Branch	1	2	3	4	5	6	7	8	9	10
a					−1	1	1	−1		
b	1			−1	1				1	
c	−1	1				−1				1
d		−1	1				−1		−1	

2-11 Node and Branch Currents. The net current *away* from a node is conveniently found from the node-branch table, since each row shows which branch arrows (branch currents) *leave* a given node. For node b Fig. 2.19, the second row of the table gives $j_2 + j_3 - j_6$ as the current leaving b. We know from the conservation of charge that the net current leaving any node must be zero. We shall ignore this for the time being, and write $j_b = j_2 + j_3 - j_6$ without also stating that it vanishes.

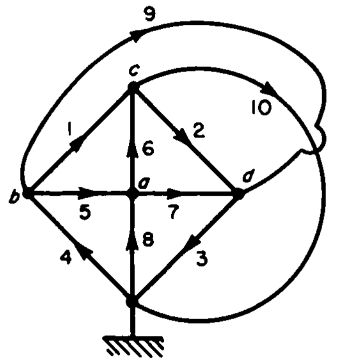


FIG. 2.20.

Now by Ohm's law, each branch current is related to the branch voltage drop by $I = GV$. We use conductance (G) instead of resistance for convenience. Then

$$j_2 = G_2V_2, \text{ etc., so that}$$
$$j_b = V_2G_2 + V_3G_3 - V_6G_6$$

and we have one such expression for each row of the table.

The *columns* of the table express the branch voltages in terms of node voltages:

$$V_1 = V_a \qquad V_4 = -V_c$$
$$V_2 = -V_a + V_b \qquad V_5 = -V_a + V_c$$
$$V_3 = V_b - V_c \qquad V_6 = -V_b$$

Substituting these into the row expressions for node currents gives

$$j_a = V_a(G_1 + G_2 + G_5) - V_bG_2 - V_cG_5$$
$$j_b = -V_aG_2 + V_b(G_2 + G_3 + G_6) - V_cG_3$$
$$j_c = -V_aG_5 - V_bG_3 + V_c(G_3 + G_4 + G_5)$$

2-12 Kirchhoff's Current Law. Kirchhoff's current law is another way of stating that charge is conserved. It states that the total current to or from a node is zero. Hence our three node currents above, j_a, j_b, and j_c, must each be zero. If we set each to zero, we have three equations for V_a, V_b, and V_c. In the present case, the only solution is $V_a = 0$, $V_b = 0$, $V_c = 0$, because we have not included any sources in the network. The simplest sources to include are current sources; if we add current sources as in Fig. 2.21, Kirchhoff's law says that $j_a = I_1 + I_2$, $j_b = -I_2$, $j_c = 0$. In words, the network current leaving a node must equal the external current supplied to that node. With these sources, then, the final circuit equations become

$$(G_1 + G_2 + G_5)V_a - G_2V_b - G_5V_c = I_1 + I_2$$
$$-G_2V_a + (G_2 + G_3 + G_6)V_b - G_3V_c = -I_2$$
$$-G_5V_a \quad -G_3V_b + (G_3 + G_4 + G_5)V_c = 0$$

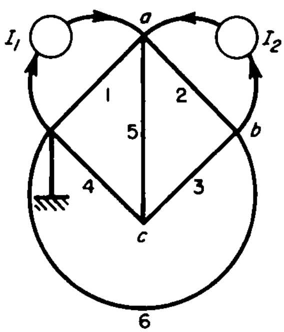


Fig. 2.21.

We note a similarity between the determinant of the coefficients of V_a, V_b, V_c and the determinant found in mesh analysis. The diagonal terms are positive, all non-zero off-diagonal terms are negative, and the determinant is symmetrical.

The present nodal analysis determinant resembles that of mesh analysis in still another way: it can be written directly from inspection of the network without construction of a table. If we label both rows and columns of the determinant with the names of the nodes, the array of coefficients is:

	a	b	c
a	$G_1 + G_2 + G_5$	$-G_2$	$-G_5$
b	$-G_2$	$G_2 + G_3 + G_6$	$-G_3$
c	$-G_5$	$-G_3$	$G_3 + G_4 + G_5$

Comparing this array with Fig. 2.21, we see that the entry in position (a, a) is the sum of all conductances connected to node a, etc., down the diagonal. An off-diagonal term, such as that in position (a, b) and (b, a), is given by the negative of the conductance in the branch between nodes a and b.

2-13 Voltage Sources. If the sources specified in the network to be

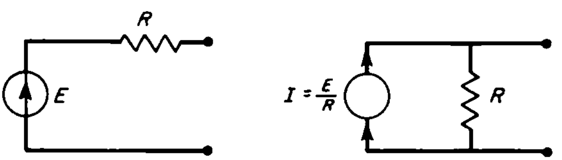


Fig. 2.22.

analyzed are voltage sources, we must convert them to equivalent current sources for setting up the circuit equations. This is, however, very simple. In Chapter I, we saw that the two sources of Fig. 2.22 gave the same terminal voltage and output current under all conditions. Thus any branch containing a series voltage source can be represented as a branch of the same resistance subject to a known parallel current source. Nodal analysis can then be used, and the circuit equations will give the correct values of the node voltages.

Although the two sources of Fig. 2.22 are *externally* equivalent under all conditions, they are not internally equivalent; the current through R is not the same in the two cases. Compare the sources under open-circuit and short-circuit, for example. Because of this effect, conversion of voltage sources to current sources will give the correct node voltages, but we must be careful if we compute branch currents. The current through any branch not containing a voltage source is found with no difficulty. For example, if in a part of a network (Fig. 2.23) we find (by solving the simultaneous equations)

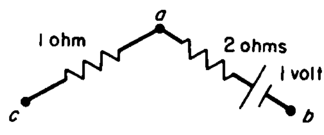


FIG. 2.23.

$$V_a = 3,\ V_b = 2,\ V_c = 1,$$

then the current from a to c is 2 amperes, but there is no current in the $a\ b$ branch, since there is no voltage across the 2-ohm resistance.

Example.

The circuit of Fig. 2.24 has resistance values indicated in ohms, sources in volts. Let d be the reference node and convert the 39-volt source to a 7.8-ampere source. Label the branch *conductances.* By inspection, the nodal analysis equations are

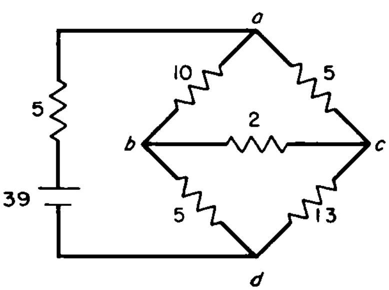


FIG. 2.24.

$$\begin{aligned} 0.5V_a \quad -0.1V_b \quad -0.2V_c &= 7.8 \\ -0.1V_a + 0.8V_b \quad -0.5V_c &= 0 \\ -0.2V_a \quad -0.5V_b + \tfrac{101}{130}V_c &= 0 \end{aligned}$$

These equations are satisfied by

$$V_a = 23,\ V_b = 11,\ V_c = 13$$

as can be verified by substitution. The current in each passive branch is readily found from the voltage across that branch. For the current in the active branch, we note that the voltage across the 5-ohm resistance is $39 - V_a = 16$ volts; the current through the voltage source and its series resistance is therefore 3.2 amperes.

This 3.2-ampere current could also be found as the current from the equivalent current source minus the current through its internal parallel conductance of 0.2 ohm—i.e., the *output* current of the generator. Since $V_a = 23$, the current in the internal conductance, which has this 23 volts across it, is 0.2 × 23 = 4.6, and 7.8 amperes − 4.6 amperes = 3.2 amperes as before.

This "backtracking" to find the actual *output* current of the real source seems a bit clumsy. The same problem occurs in mesh analysis when the real generator is a current source. In this case, the current source is rep-

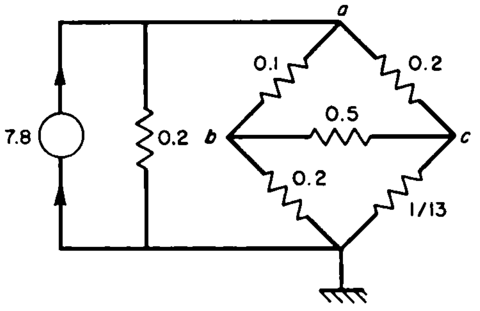


FIG. 2.25.

resented by its externally equivalent voltage source to allow writing the mesh equations by inspection. "Backtracking" to the real configuration is then needed to find the *terminal voltage.*

In general, any network containing either voltage or current sources, or both, can be analyzed by either loop analysis or nodal analysis without any "backtracking." But to do this requires using the full procedure of writing the loop-branch or node-branch table, the Kirchhoff equations, and substituting for the dependent variables in terms of the independent variables *and* the sources. Except in complicated artificial cases, the advantage of being able to write the circuit equations by inspection outweighs the disadvantage of "backtracking."

Chapter III

SIMULTANEOUS EQUATIONS

In the preceding chapter, we have learned how to analyze a network by both mesh and nodal analysis. The result in either case was a set of n simultaneous equations for n unknown voltages or currents. We must now learn the theory and practice of solving these simultaneous equations.

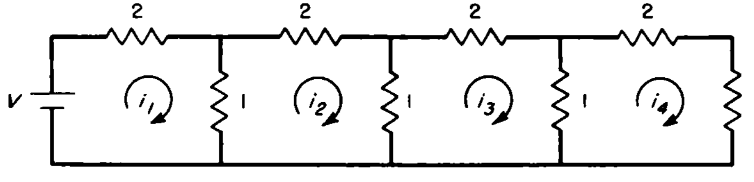


FIG. 3.1.

Consider the "ladder" network of Fig. 3.1; we can write the mesh equations by inspection.

$$3i_1 - i_2 = V$$
$$-i_1 + 4i_2 - i_3 = 0$$
$$-i_2 + 4i_3 - i_4 = 0$$
$$-i_3 + 4i_4 = 0$$

The last equation can be solved for $i_3 = 4i_4$; this relation substituted into the third equation gives $i_2 = 15i_4$; the second equation gives in turn $i_1 = 56i_4$. Substituting into the first equation gives $153i_4 = V$, hence $i_4 = V/153$, $i_3 = 4i_4 = 4V/153$, $i_2 = 15V/153$ and $i_1 = 56V/153$. The straightforward solution of these equations resulted from an easily solved bottom equation; simple substitutions allowed us to work back up through the list. This convenient structure can be forced upon a set of equations.

3-1 Triangularization. Consider th^ general set of equations

$$
\begin{aligned}
&(1)\quad a_{11}V_1 + a_{12}V_2 + a_{13}V_3 = A_1\\
&(2)\quad a_{21}V_1 + a_{22}V_2 + a_{23}V_3 = A_2\\
&(3)\quad a_{31}V_1 + a_{32}V_2 + a_{33}V_3 = A_3
\end{aligned}
\tag{3-1}
$$

where the a's and A's are given numbers, the V's are to be found. If we solve (1) for V_1 (in terms of V_2 and V_3) and substitute into (2) and (3), these become equations in V_2 and V_3 only; V_1 has been eliminated. The new equation (2) can be solved for V_2 in terms of V_3 and substituted into the new (3). The original set of equations will thus be changed to a set of the form

$$
\begin{aligned}
&(1')\quad b_{11}V_1 + b_{12}V_2 + b_{13}V_3 = B_1\\
&(2')\quad \qquad\quad\; b_{22}V_2 + b_{23}V_3 = B_2\\
&(3')\quad \qquad\qquad\qquad\; b_{33}V_3 = B_3
\end{aligned}
\tag{3-2}
$$

This *triangular* set of equations is readily solved: (3′) gives V_3 immediately; substitution of V_3 into (2′) gives V_2; substitution of both into (1′) gives V_1.

The scheme just described involved cumbersome manipulations for successive elimination of V_1 and V_2. A little thought yields a direct means of achieving this elimination. We multiply Eq. (1) by $-a_{21}/a_{11}$ (i.e., multiply each term of the equation) and add the result to (2), yielding:

$$(1)\quad a_{11}V_1 + a_{12}V_2 + a_{13}V_3 = A_1$$

$$(2')\quad 0 + \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)V_2 + \left(a_{23} - \frac{a_{21}}{a_{11}}a_{13}\right)V_3 = A_2 - \frac{a_{21}}{a_{11}}A_1$$

Using $-\dfrac{a_{31}}{a_{11}}$ as a multiplier of (1) and adding to (3) eliminates V_1 from (3):

$$(3')\quad 0 + \left(a_{32} - \frac{a_{31}}{a_{11}}a_{12}\right)V_2 + \left(a_{33} - \frac{a_{31}}{a_{11}}a_{13}\right)V_3 = A_3 - \frac{a_{31}}{a_{11}}A_1$$

Equations (2′) and (3′) comprise a set of two equations for two unknowns. We eliminate V_2 from (3′) by the same procedure we used to eliminate V_1 in the larger set.

This procedure would lead to involved algebraic expressions if we tried to carry it out with the *literal* coefficients to find a general formula for the solution, but applied to any *numerical* problem it is very simple. Let us try it on

$$
\begin{aligned}
&(1)\quad 2V_1 - V_2 + 3V_3 = 4\\
&(2)\quad V_1 + 3V_2 + 2V_3 = 0\\
&(3)\quad 2V_1 + V_2 + V_3 = 1
\end{aligned}
\tag{3-3}
$$

Elimination of V_1 gives

$$\begin{aligned}
&(1) \quad 2V_1 - V_2 + 3V_3 = 4 \\
&(2') \quad 3.5V_2 + 0.5V_3 = -2 \\
&(3') \quad 2V_2 - 2V_3 = -3
\end{aligned} \tag{3-4}$$

The next step gives

$$\begin{aligned}
&(1) \quad 2V_1 - V_2 + 3V_3 = 4 \\
&(2') \quad 3.5V_2 + 0.5V_3 = -2 \\
&(3'') \quad -2.2857V_3 = -1.8572
\end{aligned} \tag{3-5}$$

Then

$$V_3 = \frac{1.8572}{2.2857} \doteq 0.813$$

$$V_2 = (-2 - 0.5V_3)/3.5 \doteq -0.69$$

$$V_1 = (4 + V_2 - 3V_3)/2 \doteq 0.44$$

We could have avoided the appearance of the decimals by multiplying (2′) by 4, and (3′) by 7:

$$\begin{aligned}
&(1) \quad 2V_1 - V_2 + 3V_3 = 4 \\
&(2') \quad 14V_2 + 2V_3 = -8 \\
&(3') \quad 14V_2 - 14V_3 = -21
\end{aligned}$$

making

$$(3'') \quad -16V_3 = -13$$

The decimals will occur, of course, as soon as we evaluate V_3, V_2, and V_1 unless we wish to get involved with complicated fractions. This manipulation to avoid decimals and simplify the appearance of the final triangular set of equations is only useful in "book" problems, where the coefficients are simple integers. In real problems, the coefficients are not so simple, and we might just as well get out the slide rule to start with.

You have probably already noticed that writing these successive sets of equations involved superfluous work in writing the variables V_1, V_2, and V_3. The actual manipulation involved only the numerical coefficients. The whole problem can be done on the array of numbers that implies the Eq. (3-3):

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ 1 & 3 & 2 & 0 \\ 2 & 1 & 1 & 1 \end{vmatrix}$$

Subtracting ½ the first row from the second, and subtracting the first row from the third, yields

$$\begin{array}{cccc} 2 & -1 & 3 & 4 \\ & 3.5 & 0.5 & -2 \\ & 2 & 2 & -3 \end{array}$$

as a shorthand representation of Eq. (3-4). The final step gives

$$\begin{array}{cccc} 2 & -1 & 3 & 4 \\ & 3.5 & 0.5 & -2 \\ & & -2.286 & -1.857 \end{array}$$

as the representation of the final triangular set of Eq. (3-5).

This systematic elimination and triangularization is the best method of solving actual numerical problems.

There are occasions in which the above procedure can be somewhat simplified. For example, let the given equations be

$$\begin{aligned} 3V_1 + 2V_2 + 0 - V_4 &= 7 \\ 2V_1 - V_2 + V_3 + V_4 &= 3 \\ V_1 + V_2 + 0 - V_4 &= 0 \\ -V_1 + 2V_2 + 0 + 3V_4 &= 0 \end{aligned}$$

We can obviously simplify the work by rearranging the positions of the terms:

$$\begin{aligned} V_3 + 2V_1 - V_2 + V_4 &= 3 \\ 3V_1 + 2V_2 - V_4 &= 7 \\ V_1 + V_2 - V_4 &= 0 \\ -V_1 + 2V_2 + 3V_4 &= 0 \end{aligned}$$

The arithmetic of triangularizing can be further simplified by interchanging the second and fourth equations, giving the numerical array

$$\begin{vmatrix} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 3 & 2 & -1 & 7 \end{vmatrix}$$

Adding the second row to the third, and 3 times the second row to the fourth yields

$$\begin{vmatrix} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 2 & 3 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 8 & 8 & 7 \end{vmatrix}$$

Again, the last artificial manipulation (interchanging the second and fourth rows) depended upon the particularly simple numbers present. The original rearrangement to take advantage of the zero coefficients, however, did not depend on simplicity of the non-zero coefficients.

3-2 Determinants. Although the direct "brute-force" elimination scheme just described is the best for any *numerical* problem, a general method using *determinants* is the one generally taught in the schools and should be understood. The determinant method has the advantage that it can conveniently be written in general algebraic form and displays (to the trained observer!) some of the properties of the equations. It will also be used in the next chapter to deduce some general properties of networks.

Consider the pair of equations

$$a_{11}x + a_{12}y = V_1$$
$$a_{21}x + a_{22}y = V_2$$

Solving these by direct elimination of either x or y gives

$$(a_{11}a_{22} - a_{21}a_{12})x = a_{22}V_1 - a_{12}V_2$$
$$(a_{11}a_{22} - a_{21}a_{12})y = a_{11}V_2 - a_{21}V_1$$

The coefficient of x and y, $(a_{11}a_{22} - a_{21}a_{12})$ is written in "shorthand" notation as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

This symbol, or the function of its contents for which it stands, is called a *determinant* of the second order. By definition, the *value* of the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $(ad - bc)$. Note that the solutions of x and y can be written in the determinant notation as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} x = \begin{vmatrix} V_1 & a_{12} \\ V_2 & a_{22} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} y = \begin{vmatrix} a_{11} & V_1 \\ a_{21} & V_2 \end{vmatrix}$$

Determinants of higher order, such as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

are to be *defined* in such a way that the solutions of

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= V_1 \\ a_{21}x + a_{22}y + a_{23}z &= V_2 \\ a_{31}x + a_{32}y + a_{33}z &= V_3 \end{aligned} \tag{3-6}$$

are given by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} x = \begin{vmatrix} V_1 & a_{12} & a_{13} \\ V_2 & a_{22} & a_{23} \\ V_3 & a_{32} & a_{33} \end{vmatrix} \tag{3-7}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} y = \begin{vmatrix} a_{11} & V_1 & a_{13} \\ a_{21} & V_2 & a_{23} \\ a_{31} & V_3 & a_{33} \end{vmatrix} \tag{3-8}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} z = \begin{vmatrix} a_{11} & a_{12} & V_1 \\ a_{21} & a_{22} & V_2 \\ a_{31} & a_{32} & V_3 \end{vmatrix} \tag{3-9}$$

Determinants were invented for use with simultaneous equations; a study of the properties of determinants is a study of the properties of simultaneous equations in a compact notation.

3-3 Expansion of Third Order Determinant. We shall find an expansion formula for our third order determinant by brute-force methods and then guess at a generalization of the formula. The generalization will then be shown to be correct.

The three simultaneous equations (3-6) can be solved for x by triangularizing. Instead of eliminating z from the second and third, and then y from the third, we ask ourselves how we can perform both these eliminations at the same time. Multiplying the first equation by A, the second by B, the third by C, and adding give

$$(Aa_{11} + Ba_{21} + Ca_{31})x + (Aa_{12} + Ba_{22} + Ca_{32})y + (Aa_{13} + Ba_{23} + Ca_{33})z = AV_1 + BV_2 + CV_3$$

If this is to be the bottom equation of a triangular set, two of the coefficients must vanish, say those of y and z.

$$\begin{aligned} Aa_{12} + Ba_{22} + Ca_{32} &= 0 \\ Aa_{13} + Ba_{23} + Ca_{33} &= 0 \end{aligned}$$

Assuming C known for the time being, these equations can be solved for

A and B in terms of C, yielding

$$A = C\frac{a_{22}\,a_{33} - a_{32}\,a_{23}}{a_{12}\,a_{23} - a_{13}\,a_{22}}$$

$$B = C\frac{a_{13}\,a_{32} - a_{12}\,a_{33}}{a_{12}\,a_{23} - a_{13}\,a_{22}}$$

We now let $C = (a_{12}\,a_{23} - a_{13}\,a_{22})$ to eliminate the fractions, and find that

$$A = a_{22}\,a_{33} - a_{32}\,a_{23} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$B = a_{13}\,a_{32} - a_{12}\,a_{33} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C = a_{12}\,a_{23} - a_{13}\,a_{22} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

are a set of values for A, B, C that eliminate y and z. The resulting equation for x is:

$$\left\{a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}\right\} x \tag{3-10}$$

$$V_1\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - V_2\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + V_3\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

You will note that using the determinant notation for the values of A, B, C not only simplifies the writing of this result, but shows the nature of the expressions involved. The determinants for A, B, C are composed of elements of the third order determinant of all the coefficients. In fact, if we erase the first column and the first row of the third order determinant, the remaining terms are those in the determinant for A; similarly, erasing the first column and second row leaves the elements of B; removing the first column and third row leaves the elements of C.

3-4 Expansion of a Determinant by Minors. Since we wish Eq. (3-10) to mean the same thing as Eq. (3-7), we must interpret the expression in braces in the former equation to be the *definition* of the value of the determinant on the left of the latter equation. Let us see how we can express this as a recipe for expanding the large determinant.

Consider a general element a_{ij} in the large determinant at the intersection of the ith row and the jth column.

Erase the *entire* ith row and the entire jth column. The remaining elements are all those that belong to neither the row of a_{ij} nor the column of a_{ij}. The determinant of these remaining elements is called the *minor* of a_{ij}; we shall label it M_{ij}. Thus M_{23}, the minor of a_{23} in the following fourth order determinant,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

is the third order determinant

$$\begin{matrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{matrix}$$

With this definition of minors, we can write the expression in braces in Eq. (3-10) in terms of the minors of the determinant on the left of Eq. (3-7) as

$$a_{11} M_{11} - a_{21} M_{21} + a_{31} M_{31}$$

Thus we have developed an expansion of our third order determinant in terms of the elements of the first column and their minors.

Similarly, if we had eliminated x and z from the equations, or x and y, we would have had respectively the developments

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{12} M_{12} + a_{22} M_{22} - a_{32} M_{32}$$

$$= a_{13} M_{13} - a_{23} M_{23} + a_{33} M_{33}$$

We conclude that a determinant can be expanded in terms of the elements and minors of any column—provided we know what algebraic signs to attach to the quantities in the expansion. Examination of the three expansions shows that when the *sum of the row number and the column number is even*, the product of the element and its minor occurs with the $(+)$ sign; when this sum is odd, the $(-)$ sign appears. This association of $(+)$ and $(-)$ with element position is shown schematically for a fourth order determinant.

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

For any size determinant, the upper left corner holds (+); from there on, (+) and (−) alternate in both rows and columns.

Example.

Evaluate $D = \begin{vmatrix} 1 & 4 & 5 \\ 7 & 2 & 6 \\ 8 & 9 & 3 \end{vmatrix}$ by elements of the first column, and by elements of the second column.

$$D = 1\begin{vmatrix} 2 & 6 \\ 9 & 3 \end{vmatrix} - 7\begin{vmatrix} 4 & 5 \\ 9 & 3 \end{vmatrix} + 8\begin{vmatrix} 4 & 5 \\ 2 & 6 \end{vmatrix}$$

$$= (6 - 54) - 7\,(12 - 45) + 8\,(24 - 10)$$

$$= 295$$

$$D = -4\begin{vmatrix} 7 & 6 \\ 8 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 5 \\ 8 & 3 \end{vmatrix} - 9\begin{vmatrix} 1 & 5 \\ 7 & 6 \end{vmatrix}$$

$$= -4\,(21 - 48) + 2\,(3 - 40) - 9\,(6 - 35)$$

$$= 295$$

3-5 Complete Expansion. If the determinant to be expanded is of say fifth order, then its minors are of fourth order, and must in turn be expanded by minors, etc. This is a lengthy process! We note, however, that in an expansion term, say $a_{13} M_{13}$, the minor contains elements from *every* row and column *not* occupied by a_{13}. If a literal determinant is expanded out completely, it will be found to consist of sums (±) of products of terms with one factor from each row and one from each column. This can be illustrated by the expansion of a third order determinant.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

For an nth order determinant, each term contains n factors, and there are $n! = n(n-1)(n-2)\ldots 2\cdot 1$ such products in the expansion. A fourth order determinant is the sum of 24 product terms, and a fifth order determinant has 120 terms! The expansion is certainly very cumbersome. Its value lies in the theorems it helps demonstrate.

If the factors in each product are kept in the order of their row position, as in the expansion above, the column-indicating subscripts take on all permutations of their order. The sign of each product depends upon the number of inversions (from natural order) in the order of the subscripts: (+) for an even number of inversions, (−) for an odd number. The sequence 132 is an odd inversion: only one digit (3) precedes a smaller one. 231 is even, for both 2 and 3 precede 1, while 321 is odd, for 3 precedes *both* 1 and 2, and 2 precedes 1.

If the factors in the products are arranged in the natural order of the columns, e.g., $a_1c_2b_3$, then the row order takes on all permutations and the sign is (+) for an even inversion, (−) for an odd. The expansion does not actually distinguish between rows and columns. If we change rows into corresponding columns, and vice versa, we will not affect the value of a determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

From this we immediately conclude that our expansion by minors will work equally well by rows or columns, thus

$$D = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} = a_{11} M_{11} - a_{21} M_{21} + a_{31} M_{31}.$$

We can also deduce several basic theorems quite simply.

THEOREM 1: *If all the elements of any row (or column) are 0, the value of the determinant is zero.*

Proof: Each product will contain 0 as a factor.

THEOREM 2: *Multiplication of all the elements of any one row (or column) by a constant c multiplies the value of the determinant by c.*

Proof: An additional factor c is put into each product term.

THEOREM 3: *If two rows (or columns) are interchanged, the determinant changes sign.*

Proof: With the product factors arranged in the natural order of the columns, the interchange of two rows interchanges two labels. This changes the number of inversions by an odd number, changing even to odd and vice versa. Thus each product changes sign.

THEOREM 4: *If two rows (or columns) are identical, the determinant is equal to zero.*

Proof: Interchanging two *identical* rows does not change the value of a determinant,

$$\text{hence } D = -D,$$

$$\text{therefore } D = 0.$$

THEOREM 5: *If the elements of one row (or column) are proportional to those of another row (or column), the determinant is equal to zero.*

Proof: Apply theorems 2 and 4.

In the remaining theorems, third order determinants are used for illustrative purposes. The arguments and theorems are obviously valid for any order.

THEOREM 6: This deals with the addition of two determinants, and is difficult to express verbally. Consider two determinants that are identical except for the elements of a single corresponding row (or column):

$$D_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

The theorem states that

$$D_1 + D_2 = D = \begin{vmatrix} (a_{11} + b_{11}) & a_{12} & a_{13} \\ (a_{21} + b_{21}) & a_{22} & a_{23} \\ (a_{31} + b_{31}) & a_{32} & a_{33} \end{vmatrix}$$

Proof: The minors of the elements of the first column are the same for D_1, D_2, and D. Hence, expanding by minors,

$$\begin{aligned} D &= (a_{11} + b_{11})\, M_{11} - (a_{21} + b_{21})\, M_{21} + \ldots \\ &= a_{11}\, M_{11} - a_{21}\, M_{21} + \ldots \\ &+ b_{11}\, M_{11} - b_{21}\, M_{21} + \ldots \\ &= D_1 + D_2 \end{aligned}$$

THEOREM 7: *Adding a constant multiple of one row (or column) to another row (or column) does not change the value of a determinant.*

Proof: $$\begin{vmatrix} (a_{11} + ca_{12}) & a_{12} & a_{13} \\ (a_{21} + ca_{22}) & a_{22} & a_{23} \\ (a_{31} + ca_{32}) & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} ca_{12} & a_{12} & a_{13} \\ ca_{22} & a_{22} & a_{23} \\ ca_{32} & a_{32} & a_{33} \end{vmatrix}$$

by Theorem 6; and the last determinant vanishes by Theorem 5.

THEOREM 8: *The value of a* triangular *determinant is the product of its diagonal terms,* i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}\, a_{22}\, a_{33}$$

Proof: Expanding the determinant in products containing one, and only one, term from each row and column, the product of the diagonal terms is the only one which does not contain zero as a factor.

3-6 Cofactors. A cofactor is a minor with the appropriate sign included. The cofactor of the element a_{ij} is conveniently called A_{ij}, and is equal to the minor M_{ij} if $(i + j)$ is even, and equal to *minus* M_{ij} if $(i + j)$ is odd. Thus our earlier expansion of a determinant by minors of the first column is more readily written in terms of cofactors:

$$D = \begin{vmatrix} a_{11}\, a_{12}\, a_{13} \ldots a_{1n} \\ a_{21}\, a_{22}\, a_{23} \quad - \\ a_{31}\, a_{32}\, a_{33} \quad - \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \ldots\ldots a_{nn} \end{vmatrix} = a_{11}\, A_{11} + a_{21}\, A_{21} + a_{31}\, A_{31} + \ldots + a_{n1}\, A_{n1}$$

Since we can expand by *any* row or column, we have also:

$$D = a_{12}\, A_{12} + a_{22}\, A_{22} + a_{32}\, A_{32} + \ldots + a_{n2}\, A_{n2}$$

$$D = a_{21}\, A_{21} + a_{22}\, A_{22} + a_{23}\, A_{23} + \ldots + a_{2n}\, A_{2n}$$

This cofactor expansion is known as the Laplace expansion of a determinant.

If we attempt a "false" expansion by associating the elements of one row with the cofactors of a *different* row, say

$$a_{11}\, A_{21} + a_{12}\, A_{22} + a_{13}\, A_{23} + \ldots,$$

we find an interesting result. Since the cofactors A_{2i} do *not* depend on the elements of the second row (being formed by erasing the second row), and the elements a_{1i} of the first row obviously do not depend on those of the second row, we see that the above *false expansion does not depend on the elements of the second row.* Hence it must have the same value for any choice of second row elements. Now let the second row be identical with the first row; then the above expansion is the *correct* expansion by elements of the second row. Now any determinant with two rows identical vanishes by Theorem 4; the false expansion must give zero. Thus we have the following:

THEOREM: *The false expansion comprising the elements of any row (or column) in association with the cofactors of a* different *row (or column) has the value* zero.

3-7 Cramer's Rule. Consider the simultaneous equations:

$$\begin{aligned} a_{11} V_1 + a_{12} V_2 + \ldots + a_{1n} V_n &= I_1 \\ a_{21} V_1 + a_{22} V_2 + \ldots + a_{2n} V_n &= I_2 \\ &\vdots \\ a_{n1} V_1 + a_{n2} V_2 + \ldots + a_{nn} V_n &= I_n \end{aligned} \tag{3-11}$$

Multiply the first equation by A_{11}, the cofactor of a_{11}; the second by A_{21}, etc., and add the equations. We find

$$\begin{aligned} &(a_{11} A_{11} + a_{21} A_{21} + \ldots + a_{n1} A_{n1}) V_1 \\ +\ &(a_{12} A_{11} + a_{22} A_{21} + \ldots + a_{n2} A_{n1}) V_2 \\ +\ &\ldots = I_1 A_{11} + I_2 A_{21} + \ldots + I_n A_{n1} \end{aligned}$$

The coefficient of V_1 is the Laplace expansion of the determinant of the coefficients A. The coefficients of all other V_i are false expansions and therefore vanish. The terms on the right form the Laplace expansion of a determinant having $I_1, I_2, \ldots I_n$ for elements of the first column. Thus we have the solution

$$V_1 = \frac{\begin{vmatrix} I_1 & a_{12} & a_{13} & \ldots & a_{1n} \\ I_2 & a_{22} & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ I_n & a_{n2} & \ldots & & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \ldots & & a_{1n} \\ a_{21} & a_{22} & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ a_{n1} & a_{n2} & \ldots & & a_{nn} \end{vmatrix}} \tag{3-12}$$

and for V_j, the determinant in the numerator has the elements of the jth column replaced by $I_1, I_2, \ldots I_n$.

If we write this result in the form

$$V_j = \frac{A_{j1}I_1 + A_{j2}I_2 + A_{j3}I_3 + \ldots + A_{jn}I_n}{A}$$

we note that if I_1 alone differs from zero

$$V_j = A_{j1}I_1/A$$

and that in general if I_k alone differs from zero,

$$V_j = A_{jk}I_k/A$$

so that the total solution V_j is the sum of the solutions for each I_k acting separately. This is the *principle of superposition:* the network response is the sum of the responses obtained for all sources acting separately.

The solution of simultaneous equations in terms of determinants by Cramer's rule is very elegant, and it is useful in theoretical discussions. For numerical examples, however, we are still faced with the problem of evaluating the determinants. The easiest way of doing this involves the same operations as were used to solve the equations by triangularizing them. Successive applications of Theorem 7 are used to put the determinant in triangular form; Theorem 8 then gives its value.

Example.

Evaluate

$$\begin{vmatrix} 1 & 3 & 5 & 7 \\ -2 & 4 & -6 & 8 \\ 1 & -5 & 6 & 7 \\ 3 & 4 & 2 & 9 \end{vmatrix}$$

Add twice the first row to the second, subtract the first row from the third, and subtract 3 times the first from the fourth:

$$\begin{vmatrix} 1 & 3 & 5 & 7 \\ 0 & 10 & 4 & 22 \\ 0 & -8 & 1 & 0 \\ 0 & -5 & -13 & -12 \end{vmatrix}$$

Continuing the triangularizing:

$$\begin{vmatrix} 1 & 3 & 5 & 7 \\ 0 & 10 & 4 & 22 \\ 0 & 0 & 4.2 & 17.6 \\ 0 & 0 & -11 & -1 \end{vmatrix}$$

and finally

$$\begin{vmatrix} 1 & 3 & 5 & 7 \\ 0 & 10 & 4 & 22 \\ 0 & 0 & 4.2 & 17.6 \\ 0 & 0 & 0 & 45.1 \end{vmatrix} = 1 \times 10 \times 4.2 \times 45.1 \doteq 1894$$

Chapter IV

GENERAL NETWORK PROPERTIES

In this chapter, we shall use Cramer's rule for the solution of simultaneous equations, and Laplace's expansion of a determinant by cofactors to derive some useful theorems about linear networks.

4-1 Thevenin's Theorem. Consider a network, one branch of which comprises the resistance R in series with the voltage E_1, and the other branches of which are unspecified. Put all these other branches into a "black box" with two terminals for connecting the first branch, as in Fig. 4.1. Choose this "external" branch as link 1; let there also be a voltage source in the box, say E_j in the jth loop. Any number of "internal" sources can be allowed without affecting our final result, but the restriction to a single internal source simplifies the algebra.

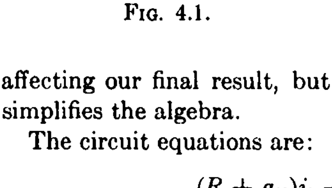


FIG. 4.1.

The circuit equations are:

$$
\begin{aligned}
(R + a_{11})i_1 + a_{12}i_2 + \ldots + a_{1n}i_n &= E_1 \\
a_{21}i_1 + a_{22}i_2 + \ldots + a_{2n}i_n &= 0 \\
a_{j1}i_1 + a_{j2}i_2 + \ldots + a_{jn}i_n &= E_j \\
a_{n1}i_1 + a_{n2}i_2 + \ldots + a_{nn}i_n &= 0
\end{aligned}
\qquad (4\text{-}1)
$$

where the a_{ij}'s are unknown, being determined by the contents of the box.

Solving for i_1 by Cramer's rule, we have

$$i_1 = \frac{\begin{vmatrix} E_1 & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ E_j & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} (R + a_{11}) & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}$$

By Theorem 6, Chapter III, the determinant in the denominator can be written as

$$\begin{vmatrix} R & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

These two determinants, together with the one in the numerator, are identical except for their first columns. Hence the cofactors of the first column are the same for each determinant. Expanding by cofactors

$$i_1 = \frac{E_1 A_{11} + E_j A_{j1}}{R A_{11} + A} = \frac{E_1 + E_j A_{j1}/A_{11}}{R + A/A_{11}} \tag{4-2}$$

First consider $E_1 = 0$; the terminal voltage of the box is

$$R i_1 = \frac{R E_j A_{j1}/A_{11}}{R + A/A_{11}}$$

Let $R \to \infty$ (i.e., become indefinitely large) to find the OCV (open-circuit voltage) at the terminals. The term A/A_{11} in the denominator becomes negligible compared to R, and we have

$$V_o = E_j A_{j1}/A_{11}$$

so we can write $i_1 = \dfrac{V_o}{R + A/A_{11}}$. The box "looks like" a source whose

OCV is V_o and whose internal resistance is A/A_{11} (Fig. 4.2). Next consider the case $E_j = 0$, $E_1 \neq 0$, giving $i_1 = \dfrac{E_1}{R + A/A_{11}}$, which describes the circuit of Fig. 4.3. Thus when the source in the box is "dead," the box appears to be a resistance $R_i = A/A_{11}$; this is called the *input resistance* of the network in the box. When there is a "live" source in the box, the

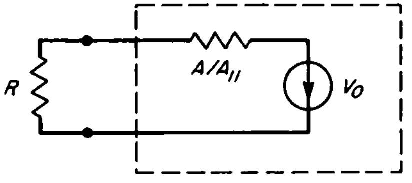


FIG. 4.2

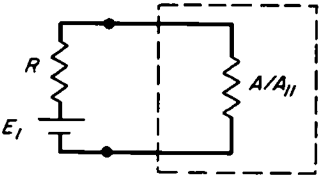


FIG. 4.3.

box is equivalent to a source having (1) an internal resistance equal to the input resistance R_i and (2) an emf equal to the OCV at the box terminals. This result is known as *Thevenin's theorem.*

If we short the terminals ($R = 0$), the short-circuit current is, from Eq. (4-2),

$$I_s = \frac{E_j A_{j1}/A_{11}}{A/A_{11}}$$

so that the terminal voltage can be written

$$V_1 = Ri_1 = \frac{R(A/A_{11})I_s}{R + A/A_{11}}$$

and the box is also equivalent to the current source of Fig. 4.4. This is known as *Norton's theorem.*

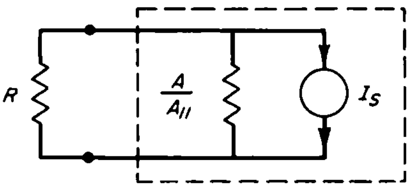


FIG. 4.4.

4-2 Wheatstone Bridge. Let us reconsider the Wheatstone bridge (Fig. 4.5). We have used M for the resistance of the balance-indicating meter, and S for the internal resistance of the source. Figures 4.5a and b look different, but are identical circuits. If we wish to find the meter current, Fig. 4.5b is convenient for choosing loops (meshes) such that M carries only one unknown current. With mesh currents as indicated, the circuit equations are

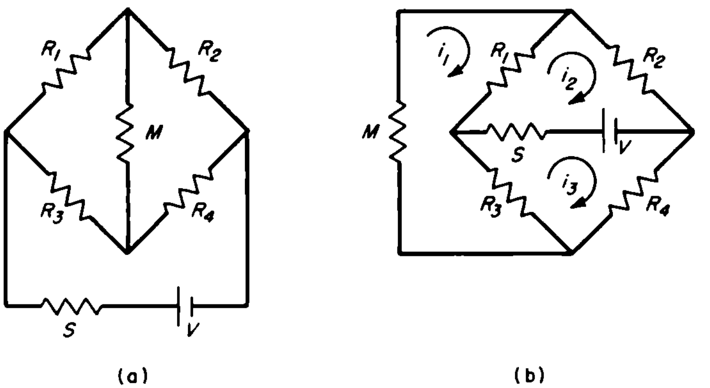


FIG. 4.5.

$$\begin{aligned} (M + R_1 + R_3)i_1 - R_1 i_2 - R_3 i_3 &= 0 \\ -R_1 i_1 + (R_1 + R_2 + S)i_2 - S i_3 &= V \\ -R_3 i_1 - S i_2 + (R_3 + R_4 + S)i_3 &= -V \end{aligned} \qquad (4\text{-}3)$$

In this form, M plays the role of the external resistance R of the preceding section. Borrowing this earlier analysis, Eq. (4-2), and noting that we have *two* internal sources (according to the equations), we have

$$i_1 = \frac{VA_{21}/A_{11} - VA_{31}/A_{11}}{M + A/A_{11}} = \frac{V(A_{21} - A_{31})/A_{11}}{M + A/A_{11}}$$

where

$$A = \begin{vmatrix} R_1 + R_3 & -R_1 & -R_3 \\ -R_1 & R_1 + R_2 + S & -S \\ -R_3 & -S & R_3 + R_4 + S \end{vmatrix}$$

$$A_{11} = \begin{vmatrix} R_1 + R_2 + S & -S \\ -S & R_3 + R_4 + S \end{vmatrix}$$

$$A_{21} = -\begin{vmatrix} -R_1 & -R_3 \\ -S & R_3 + R_4 + S \end{vmatrix}$$

$$A_{31} = \begin{vmatrix} -R_1 & -R_3 \\ R_1 + R_2 + S & -S \end{vmatrix}$$

Now $A_{21} - A_{31} = [R_1(R_3 + R_4 + S) + R_3 S] - [R_1 S + R_3(R_1 + R_2 + S)]$
$= R_1 R_4 - R_2 R_3$

so that $i_1 = 0$, if $R_1R_4 - R_2R_3 = 0$, or $R_1/R_2 = R_3/R_4$, our previously determined condition of balance. In terms of our black box discussion, the meter sees a source whose OCV is

$$V_o = V(A_{21} - A_{31})/A_{11} = V(R_1R_4 - R_2R_3)/A_{11}$$

and whose internal resistance is

$$R_i = A/A_{11}$$

The sensitivity of the bridge for detecting a small unbalance can be expressed in terms of the OCV produced by a small unbalance and the internal resistance of the equivalent source seen by the meter. Although the OCV is naturally sensitive to the precise degree of balance, the internal resistance is not, and can be taken as the internal resistance *at balance*. The problem is not of sufficiently general interest to warrant writing out the lengthy algebra.

In the practical case where S is negligible in comparison with the other resistances, we take $S = 0$ and find:

$$A_{11} = (R_1 + R_2)(R_3 + R_4)$$
$$A = R_1R_2(R_3 + R_4) + R_3R_4(R_1 + R_2)$$

so that

$$R_i = \frac{A}{A_{11}} = \frac{R_1R_2}{R_1 + R_2} + \frac{R_3R_4}{R_3 + R_4}$$

which is shown in Fig. 4.6. This result for $S = 0$ can be found by in-

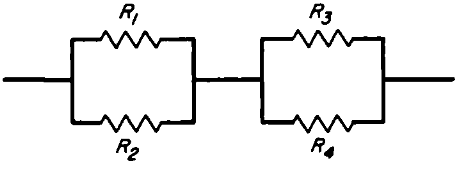


FIG. 4.6.

spection of Fig. 4.5b, since $S = 0$, $V = 0$ is a direct connection across the middle of the bridge. The OCV also simplifies for $S = 0$ and becomes

$$V_o = V\left\{\frac{R_1}{R_1 + R_2} - \frac{R_3}{R_3 + R_4}\right\}$$

which can also be found from inspection, more conveniently from Fig. 4.5a.

4-3 Conjugacy. If the meter and the battery are interchanged, we find that the condition for balance is unaffected. Interchange the M and S branches in Fig. 4.5a. Rotate the figure through 90°, and it becomes

Fig. 4.5b with the substitution of S for M, M for S, R_3 for R_1, R_1 for R_2, R_2 for R_4, and R_4 for R_3. Making these substitutions in the original equation for balance, $R_1R_4 - R_2R_3 = 0$, we find $R_3R_2 - R_1R_4 = 0$, which is the same. *If the bridge is balanced, it is still balanced with the meter and battery interchanged.* This is true independently of the resistances of the meter and the battery. Such a pair of branches in a network is sometimes called a *conjugate pair*, or the branches are *conjugate branches.*

4-4 Two-Port Networks. Thevenin's theorem applied to the external behavior of a network at a single pair of terminals, or a single *port* of access. Most of our useful theorems about networks relate to *two-port* networks; there are two "input" terminals and two "output" terminals. The branches connected to these terminals (*external* branches) are often considered to be a source and a load.

Initially, we shall describe the properties of the "black box" in terms of the unspecified internal elements. Later we shall see the relation between its behavior and certain parameters that can be measured without opening the box.

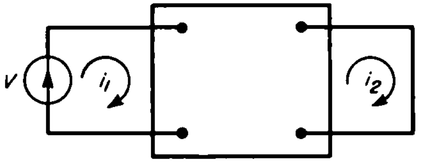


FIG. 4.7.

Let the box of Fig. 4.7 be defined by the mesh resistances a_{ij} appearing in Eq. (4-4).

$$\begin{aligned} a_{11}i_1 + a_{12}i_2 + \ldots + a_{1n}i_n &= V \\ a_{21}i_1 + a_{22}i_2 + \ldots + a_{2n}i_n &= 0 \\ &\vdots \\ a_{n1}i_1 + a_{n2}i_2 + \ldots + a_{nn}i_n &= 0 \end{aligned} \tag{4-4}$$

By Cramer's rule:

$$i_2 = \frac{VA_{12}}{A}$$

To be more explicit, we rewrite this as

$$i_2 = \left(\frac{VA_{12}}{A}\right)_{E_1 = V,\, E_2 = 0}$$

where E_1 is the emf in the first branch, and E_2 is the emf in the second branch. If we now shift the voltage source V to the *second* branch and solve for the current in the *first* branch, we find

$$i_1 = \left(\frac{VA_{21}}{A}\right)_{E_1=0,\, E_2=V}$$

Recall that the determinant $|a_{ij}|$ is symmetrical, i.e., $a_{ij} = a_{ji}$, when the current loops are used for the closed paths of Kirchhoff's voltage law. The symmetry of the determinant induces a similar relation between cofactors, namely $A_{ij} = A_{ji}$, hence $A_{21} = A_{12}$, and we have the result:

$$(i_1)_{E_1=0,\, E_2=V} = (i_2)_{E_1=V,\, E_2=0}$$

The implication is shown in Fig. 4.8, where a perfect (i.e., resistanceless) source and a perfect (also resistanceless) ammeter are implied. The gist

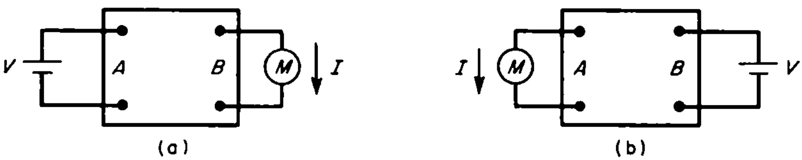


FIG. 4.8.

of the theorem is, that if V is the same in Fig. 4.8a and b, so also is I. A and B are any pair of terminals on any network of resistances not containing sources. (The theorem remains true when the box contains also inductance and capacitance. There is, however, an element known as a *gyrator* for which the theorem is not true, for the determinant cannot be made symmetrical. *Gyrators* are realizable in microwave "plumbing," but practical gyrators are not yet available for ordinary circuits.)

The theorem need not be written in terms of the *same* voltage (V) applied at one port or the other. For with any voltage e_1 applied in the first mesh, and zero voltage in the second (i.e., the second port is short-circuited), the solution of Eq. (4-4) is

$$i_2 = e_1 A_{12}/A \text{ for } e_2 = 0$$

i.e.,

$$(i_2/e_1)_{e_2=0} = \frac{A_{12}}{A}$$

Similarly, applying any e_2 to the second port, with the first port shorted:

$$(i_1/e_2)_{e_1=0} = \frac{A_{21}}{A} = \frac{A_{12}}{A}$$

The reciprocity theorem can therefore be written

$$(i_1/e_2)_{e_1=0} = (i_2/e_1)_{e_2=0} \tag{4-5}$$

Recalling that a voltage source is essentially an "active short-circuit," we see that Eq. (4-5) expresses the reciprocal property of a network that is *shorted* at both ports.

A similar relation for the networks *open* at both ports can be derived from the nodal analysis equations; in this case the network is excited by current sources, and the responses are open circuit voltages. The analysis is entirely parallel to the preceding, so will not be given. The result is:

$$(e_1/i_2)_{i_1=0} = (e_2/i_1)_{i_2=0} \tag{4-6}$$

4-5 Equivalent T. Let us now consider resistances and sources connected to the terminals, as in Fig. 4.9. Assume the box to contain no

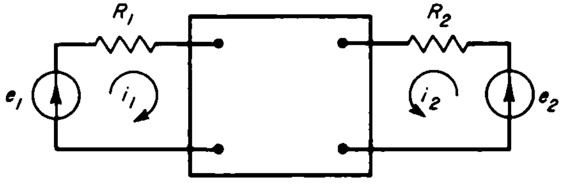


FIG. 4.9.

sources. The circuit equations (4-7) are more amenable to analysis if the known resistances R_1 and R_2 are transposed to the right-hand side, as in Eq. (4-8).

$$\begin{aligned}
(R_1 + a_{11})i_1 + a_{12}i_2 + \cdots + a_{1n}i_n &= e_1 \\
a_{21}i_1 + a_{22}i_2 + \cdots + a_{2n}i_n &= e_2 \\
&\vdots \\
a_{n1}i_1 + a_{n2}i_2 + \cdots + a_{nn}i_n &= 0
\end{aligned} \tag{4-7}$$

$$\begin{aligned}
a_{11}i_1 + \cdots + a_{1n}i_n &= e_1 - R_1i_1 \equiv E_1 \\
a_{21}i_1 + \cdots + a_{2n}i_n &= e_2 - R_2i_2 \equiv E_2 \\
&\vdots \\
a_{n1}i_1 + \cdots + a_{nn}i_n &= 0
\end{aligned} \tag{4-8}$$

In this form we can solve for the currents in terms of E_1 and E_2 and properties of the determinant $|a_{ij}|$:

$$\begin{aligned}
Ai_1 &= E_1A_{11} + E_2A_{21} \\
Ai_2 &= E_1A_{12} + E_2A_{22}
\end{aligned} \tag{4-9}$$

We next substitute the expressions for E_1 and E_2 in terms of the known e_1, e_2 and the unknown i_1 and i_2 (from Eq. 4-8), finding

$$\begin{aligned} Ai_1 &= A_{11}(e_1 - R_1i_1) + A_{21}(e_2 - R_2i_2) \\ Ai_2 &= A_{12}(e_1 - R_1i_1) + A_{22}(e_2 - R_2i_2) \end{aligned} \tag{4-10}$$

which simplify by transposition to two simultaneous equations for i_1 and i_2:

$$\begin{aligned} (A + R_1A_{11})i_1 + R_2A_{21}i_2 &= A_{11}e_1 + A_{21}e_2 \\ R_1A_{12}i_1 + (A + R_2A_{22})i_2 &= A_{12}e_1 + A_{22}e_2 \end{aligned} \tag{4-11}$$

These equations are in a form that lets us change the external voltages and resistances readily; all effects of the hidden network in the box are summarized by A and its particular cofactors appearing in Eq. (4-11). These equations are more suitable for our purposes if we reduce the right-hand sides to e_1 and e_2 respectively. Multiply the first by A_{22}, the second by $-A_{21}$, and add to eliminate e_2; and use a similar process to eliminate e_1, obtaining

$$\begin{aligned} &[A_{22}(A + R_1A_{11}) - A_{21}R_1A_{12}]i_1 + [A_{22}R_2A_{21} - A_{21}(A + R_2A_{22})]i_2 \\ &\qquad = (A_{11}A_{22} - A_{21}A_{12})e_1 \\ &[A_{11}R_1A_{12} - A_{12}(A + R_1A_{11})]i_1 + [A_{11}(A + R_2A_{22}) - A_{12}R_2A_{21}]i_2 \\ &\qquad = (A_{11}A_{22} - A_{21}A_{12})e_2 \end{aligned} \tag{4-12}$$

Some of the terms drop out, and the equations can be written as

$$\begin{aligned} \left(R_1 + \frac{AA_{22}}{D}\right)i_1 - \frac{AA_{21}}{D}i_2 &= e_1 \\ -\frac{AA_{12}}{D}i_1 + \left(R_2 + \frac{AA_{11}}{D}\right)i_2 &= e_2 \end{aligned} \tag{4-13}$$

with

$$D \equiv A_{11}A_{22} - A_{12}A_{21}$$

Leaving Eq. (4-13) for the time being, let us consider the behavior of

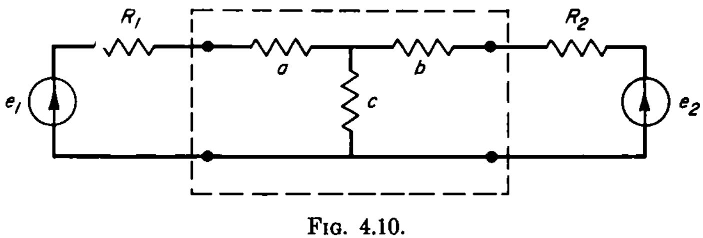


FIG. 4.10.

the specific box of Fig. 4.10. This particular box, connected to the *same resistances and voltages* as was our general box, gives the equations

$$\begin{aligned} (R_1 + a + c)i_1 - ci_2 &= e_1 \\ - ci_1 + (R_2 + b + c)i_2 &= e_2 \end{aligned} \tag{4-14}$$

Comparing Eq. (4-14) with Eq. (4-13), we see that they describe the *same external behavior* if

$$\begin{aligned} a + c &= AA_{22}/D \\ c &= AA_{21}/D = AA_{12}/D \\ b + c &= AA_{11}/D \end{aligned} \tag{4-15}$$

Since the determinant $|a_{ij}|$ can always be set up in a symmetrical form (we exclude gyrators from consideration, as we shall throughout), we can make $A_{12} = A_{21}$. Then solving Eq. (4-15) for a, b, c, we have

$$\begin{aligned} a &= A(A_{22} - A_{12})/D \\ b &= A(A_{11} - A_{12})/D \\ c &= AA_{12}/D \end{aligned} \tag{4-16}$$

with $$D \equiv A_{11}A_{22} - A_{12}^2$$

This shows that for *any* network in the box, there is a simple *T-network* that is externally equivalent. (The embarrassment of having $D = 0$ cannot occur except for degenerate cases, since $D = 0$ implies that the ratio i_1/i_2 is independent of the external R_1, R_2, e_1, and e_2.)

4-6 Two-Port Description. Referring back to Eqs. (4-8) and (4-9), we see that the behavior of the "box" can be expressed in terms of its *terminal* voltages E_1 and E_2 without reference to the external resistances. Figure 4.11 indicates the variables together with their reference directions. It is, of course, understood that if current I_1 enters the upper left terminal, then a current I_1 leaves the lower left terminal. The currents are determined by the voltages as shown in Eq. (4-17), which is Eq. (4-9) rewritten.

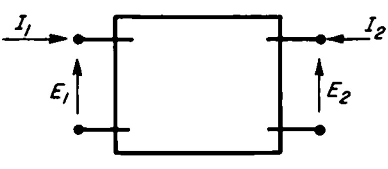


Fig. 4.11.

$$\begin{aligned} I_1 &= \left(\frac{A_{11}}{A}\right) E_1 + \left(\frac{A_{21}}{A}\right) E_2 \\ I_2 &= \left(\frac{A_{12}}{A}\right) E_1 + \left(\frac{A_{22}}{A}\right) E_2 \end{aligned} \tag{4-17}$$

Conversely, we can solve these and express the terminal voltages in terms of the currents:

$$E_1 = \left(\frac{AA_{22}}{D}\right) I_1 - \left(\frac{AA_{21}}{D}\right) I_2$$
$$E_2 = -\left(\frac{AA_{12}}{D}\right) I_1 + \left(\frac{AA_{11}}{D}\right) I_2 \tag{4-18}$$

These sets of relations can conveniently be abbreviated by writing

$$I_1 = y_{11}E_1 + y_{12}E_2$$
$$I_2 = y_{21}E_1 + y_{22}E_2 \tag{4-19}$$

and

$$E_1 = z_{11}I_1 + z_{12}I_2$$
$$E_2 = z_{21}I_1 + z_{22}I_2 \tag{4-20}$$

We can also describe the network by how the variables at one port depend upon those at the other ports, using the so-called *general circuit parameters*, A, B, C, and D:

$$E_1 = AE_2 - BI_2$$
$$I_1 = CE_2 - DI_2 \tag{4-21}$$

The twelve coefficients used in these three representations can be evaluated in terms of the determinant $|a_{ij}|$ and its cofactors (as in Eq. 4-17), in terms of the resistances of the equivalent T, or of the equivalent Π, or in terms of *directly measurable external properties* of the network.

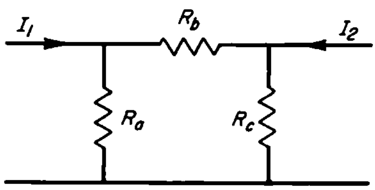


FIG. 4.12.

4-7 Equivalent Π Network. Equations (4-17) and (4-19) can be interpreted as describing the simple Π-section of Fig. 4.12. The nodal analysis of Chapter II showed that we could write, by inspection of the figure:

$$I_1 = \left(\frac{1}{R_a} + \frac{1}{R_b}\right) E_1 - \frac{1}{R_b} E_2$$
$$I_2 = -\frac{1}{R_b} E_1 + \left(\frac{1}{R_a} + \frac{1}{R_c}\right) E_2$$

Comparison with Eq. (4-17) yields

$$R_b = -\frac{A}{A_{12}} \text{ (for } A_{12} = A_{21})$$
$$R_a = \frac{A}{A_{11} + A_{12}} \tag{4-22}$$
$$R_c = \frac{A}{A_{22} + A_{12}}$$

The necessary condition $A_{12} = A_{21}$ is satisfied for passive reciprocal networks. (When $A_{12} = A_{21}$, there is no *reciprocal* Π equivalent as Fig. 4.12. We shall encounter this problem later when we study *amplifiers*.)

4-8 Transformation between T and II. The equivalent networks, Fig. 4.13, can readily be related (note the "dual" positions of the cor-

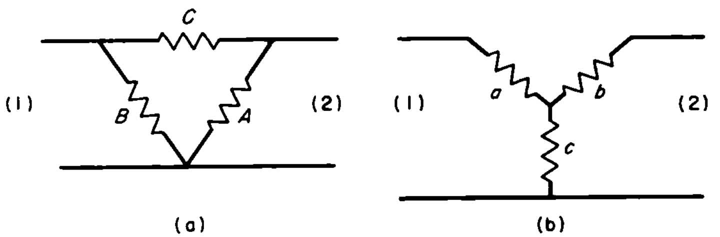


FIG. 4.13.

responding resistances). To make the short-circuit input resistances equal, we must have

$$a + \frac{bc}{b + c} = \frac{BC}{B + C} \tag{4-23}$$

The two open circuit voltage transfer ratios are:

$$\frac{V_{20}}{V_1} = \frac{c}{c + a} = \frac{A}{A + C} \tag{4-24}$$

$$\frac{V_{10}}{V_2} = \frac{c}{c + b} = \frac{B}{B + C} \tag{4-25}$$

From Eq. (4-24) we find, by inverting,

$$1 + \frac{a}{c} = 1 + \frac{C}{A}$$

hence $aA = cC$; similarly Eq. (4-25) yields $bB = cC$. Let this common value be

$$S = aA = bB = cC \tag{4-26}$$

Multiplying Eq. (4-23) by $(b + c)$ yields

$$ab + ac + bc = \frac{(b + c)BC}{B + C} = \frac{SB + SC}{B + C} = S \tag{4-27}$$

Multiplying Eq. (4-23) by $(B + C)$ yields

$$BC = a(B + C) + \frac{(B + C)bc}{b + c} = a(B + C) + S \tag{4-28}$$

Substituting for a from Eq. (4-26) yields

$$ABC = (A + B + C)S$$

giving the final result:

$$aA = bB = cC = ab + ac + bc = \frac{ABC}{A + B + C} \tag{4-29}$$

This set of relations allows ready computation of a, b, c from A, B, C, or vice versa. Equation (4-29) is also known as the Star-Delta, or Delta-Wye, transformation, from the geometrical resemblance of Fig. 4.13b to a "star" or letter "y," and of Fig. 4.13a to the Greek "delta."

4-9 Black-Box Variables. There are several elementary measurements that we can make on a two-port network. We excite the network at the first port, with either a voltage source or a current source—it doesn't matter. First, we open the second port (i.e., set $I_2 = 0$) and measure E_1, I_1, E_2. Second, we short the second port (i.e., set $E_2 = 0$) and measure E_1, I_1, I_2. From these measurements, we have all the properties of the unknown network in the box. Since this network has an equivalent T-network previously derived, there are only three independent parameters and the coefficients in Eq. (4-19), (4-20), and (4-21) are all interrelated and expressible in terms of the black-box measurements above. Excitation at the *second* port, with the short or open applied to the first port, would yield no new information.

We shall discuss Eqs. (4-20) and (4-21) in detail.

From Eq. (4-20), setting $I_2 = 0$, we have

$$z_{11} = (E_1/I_1)_{I_2=0}; \quad z_{21} = (E_2/I_1)_{I_2=0} \tag{4-30}$$

Setting $E_2 = 0$ makes the second equation of (4-20):

$$0 = z_{21}I_1 + z_{22}I_2$$

hence

$$(I_1/I_2)_{E_2=0} = -z_{22}/z_{21} \tag{4-31}$$

Substituting this expression for I_1 into the first equation of (4-20) yields

$$(E_1/I_2)_{E_2=0} = \left(-\frac{z_{11}z_{22}}{z_{21}} + z_{12}\right) = -\frac{z_{11}z_{22} - z_{12}z_{21}}{z_{21}} \tag{4-32}$$

$$= -|z|/z_{21}$$

If we apply the excitation at the second port, and open circuit the first port, we have

$$z_{12} = (E_1/I_2)_{I_1=0}$$

Comparing this with Eq. (4-30) and the reciprocity theorem Eq. (4-6), we see that $z_{12} = z_{21}$.

Let us now apply this same treatment to Eq. (4-21). Setting $I_2 = 0$ gives A and C; $E_2 = 0$ gives B and D:

$$\begin{aligned} A &= (E_1/E_2)_{I_2=0} & C &= (I_1/E_2)_{I_2=0} \\ B &= -(E_1/I_2)_{E_2=0} & D &= -(I_1/I_2)_{E_2=0} \end{aligned} \tag{4-33}$$

Comparing Eq. (4-33) with Eqs. (4-30), (4-31), and (4-32), we have the relations

$$\begin{aligned} A &= z_{11}/z_{21} \\ B &= |z|/z_{21} \\ C &= 1/z_{21} \\ D &= z_{22}/z_{21} \end{aligned} \tag{4-34}$$

Note that

$$\begin{vmatrix} A & -B \\ C & -D \end{vmatrix} = -AD + BC = -\frac{z_{11}z_{22}}{z_{21}^2} + \frac{z_{11}z_{22} - z_{21}z_{12}}{z_{21}^2}$$
$$= -z_{12}/z_{21} = -1$$

We can solve Eq. (4-21) for E_2, I_2 in terms of E_1, I_1:

$$E_2 = \frac{\begin{vmatrix} E_1 & -B \\ I_1 & -D \end{vmatrix}}{\begin{vmatrix} A & -B \\ C & -D \end{vmatrix}}$$

$$I_2 = \frac{\begin{vmatrix} A & E_1 \\ C & I_1 \end{vmatrix}}{\begin{vmatrix} A & -B \\ C & -D \end{vmatrix}}$$

and since we have just found that the determinant in the denominator has the value -1, these relations become

$$\begin{aligned} E_2 &= DE_1 - BI_1 \\ I_2 &= CE_1 - AI_1 \end{aligned} \tag{4-35}$$

4-10 Input and Output Resistance. To simplify the wording of our discussion, let us call the left-hand port of our network (Fig. 4.14) the

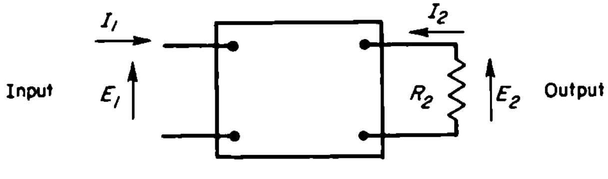


Fig. 4.14.

input port, and the other, the *output port*. For any output termination (or *load*) R_2, we wish to find the input resistance, E_1/I_1. To distinguish this input resistance from any external resistance R_1 that we may later connect, let us call the input resistance Z_1.

The presence of R_2 at the output forces the relation $E_2 = -R_2I_2$ by Ohm's law. The minus sign appears by virtue of our reference directions for positive voltage and current in Fig. 4.14 with E_2 a voltage *rise*. The simplest network equations to use here are Eq. (4-21), which immediately yield

$$E_1 = -(AR_2 + B)I_2$$
$$I_1 = -(CR_2 + D)I_2$$

so that

$$Z_1 = E_1/I_1 = \frac{AR_2 + B}{CR_2 + D} \qquad (4\text{-}36)$$

From Eq. (4-34), this can also be written, if desired, as

$$Z_1 = \frac{z_{11}R_2 + |z|}{R_2 + z_{22}}$$

Let us now consider the special cases $R_2 = 0$, and $R_2 = \infty$, giving respectively the "short-circuit input impedance"[1] (Z_{1sc}) and the "open circuit input impedance" (Z_{1oc}). From Eq. (4-36) we have

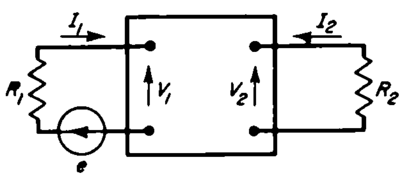


FIG. 4.15.

$$Z_{1sc} = B/D, \quad Z_{1oc} = A/C \qquad (4\text{-}37)$$

Let us now have external resistance at both ends (Fig. 4.15). (The input resistance Z_1 is not affected by R_1.) We now ask also for the *output resistance* Z_2 "seen" by the load R_2 (recall Thevenin's theorem).

Using Eq. (4-35) with $E_1 = -R_1I_1$ (letting $e = 0$), we find

$$Z_2 = \frac{DR_1 + B}{CR_1 + A} \qquad (4\text{-}38)$$

for the *output resistance*. Again the special cases $R_1 = 0$, $R_1 = \infty$ yield the "short-circuit output impedance" and the "open-circuit output impedance":

$$Z_{2sc} = B/A, \quad Z_{2oc} = D/C \qquad (4\text{-}39)$$

Note that

$$Z_{1sc}/Z_{1oc} = Z_{2sc}/Z_{2oc} \qquad (4\text{-}40)$$

[1] Here we use the term "impedance" instead of "resistance" to distinguish this *network property* from *external* loads. Later on, when we come to the study of alternating currents, we shall define the concept of *impedance* as a generalization of *resistance*.

From Eqs. (4-37) and (4-39) we could, if desired, express A, B, C, D (and therefore also z_{11}, z_{12}, z_{22}) in terms of the open-circuit and short-circuit input and output impedances.

4-11 Image Impedances. We now ask, what are the conditions in Fig. 4.15 for the source to be matched to *its* load Z_1, and at the same time for the load R_2 to be matched to the source resistance (Z_2) seen by the load? This relationship is important in practical applications, as will be seen shortly. We are familiar with the anthropomorphic colloquialism of having a source "see" a load, and vice versa. Pursuing the analogy, the network acts like a lens, so that the source does not "see" the load R_2, but its "image" Z_1; the load "sees" the image Z_2 of the source resistance. If the source and load both "see" matching resistances, i.e., $R_1 = Z_1$ and $R_2 = Z_2$, the network is said to be "matched on an image impedance basis." These image-matching conditions applied to Eqs. (4-36) and (4-38) require:

$$R_1 = \frac{AR_2 + B}{CR_2 + D} \quad \text{and} \quad R_2 = \frac{DR_1 + B}{CR_1 + A} \tag{4-41}$$

Clearing of fractions gives

$$CR_2R_1 + DR_1 = AR_2 + B$$

and

$$CR_1R_2 + AR_2 = DR_1 + B$$

Adding these two equations yields

$$CR_1R_2 = B$$

while subtracting one from the other yields

$$DR_1 = AR_2$$

From these we readily find

$$R_1^2 = \frac{AB}{CD}; \quad R_2^2 = \frac{BD}{AC}$$

so that the image impedances are uniquely determined by the network. We shall call these image impedances of the network

$$Z_{I1} \text{ and } Z_{I2}:$$

$$Z_{I1} = \sqrt{\frac{AB}{CD}}; \quad Z_{I2} = \sqrt{\frac{BD}{AC}}$$

From Eqs. (4-37) and (4-39) we can show the interesting relations:

$$\begin{aligned} Z_{I1} &= \sqrt{Z_{1sc}Z_{1oc}} \\ Z_{I2} &= \sqrt{Z_{2sc}Z_{2oc}} \end{aligned} \tag{4-42}$$

The image impedances can therefore be readily determined experimentally in terms of the open-circuit and short-circuit input impedances of the two ports.

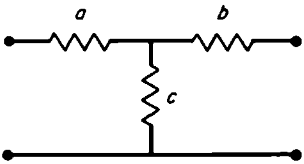


FIG. 4.16.

For the T-network of Fig. 4.16, we can write by inspection

$$Z_{1oc} = a + c, \quad Z_{1sc} = a + \frac{bc}{b + c}$$

$$Z_{2oc} = b + c, \quad Z_{2sc} = b + \frac{ac}{a + c}$$

Use of Eq. (4-42) gives

$$Z_{I1}^2 = (a + c)\left(a + \frac{bc}{b + c}\right)$$

$$Z_{I2}^2 = (b + c)\left(b + \frac{ac}{a + c}\right)$$

If the T is symmetrical ($a = b$), the image impedances are equal, and

$$Z_I^2 = a(a + 2c) \tag{4-43}$$

4-12 Attenuators. When we drive several loads from a single source, say several loudspeakers from a common amplifier, we often wish to adjust the power in one load without changing the power delivered to the other loads. This requires that the load presented to the amplifier remains constant—the speakers are to be connected via *attenuators* that present the

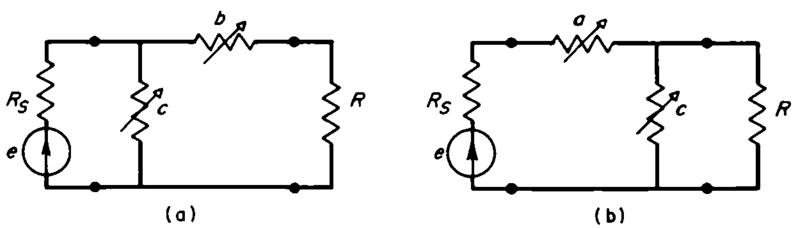


FIG. 4.17.

same input resistance for all volume settings. Thus if each speaker is connected to the amplifier through an *L-pad*, as in Fig. 4.17, the resistances in the pad shall vary in such a way as to keep the input resistance constant. In Fig. 4.17a, the input resistance is

$$R_i = \frac{c(b + R)}{c + b + R}$$

which can be solved for c:

$$c = \frac{(R + b)R_i}{R + b - R_i} \tag{4-44}$$

If c is made to depend upon b as in this equation, the input resistance remains constant. The output resistance, on the other hand, is

$$R_o = \frac{b(R_s + c)}{b + R_s + c}$$

which will *not* remain constant if c and b are related as in Eq. (4-44). Similar results hold for Fig. 4.17b.

Now the frequency response characteristic of a loudspeaker depends upon the output resistance of the driving source. This is basically because a loudspeaker can also be used as a microphone, and so used will generate a voltage, i.e., become a source. If we excite a speaker with a sharp "click," either mechanical or electrical, it will "ring." The nature and duration of its ringing will depend upon the load it sees as a generator, for its load will absorb energy from the "ringing." In fact, the reciprocity principle can be used to show that the frequency response characteristic of a loudspeaker driven from a source of given resistance is the same as its microphone characteristic when working into a load of that same resistance. In many cases, then, we want an attenuator that will preserve constant input *and* output resistance. In fact, we usually want both the source and the load to see matching resistance. This implies that the attenuator matches the source and load simultaneously, i.e., it is operated under image impedance matching conditions.

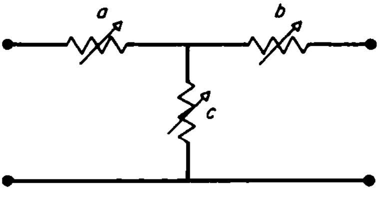


FIG. 4.18.

These conditions can be satisfied by using a *T-pad* attenuator (Fig. 4.18). For simplicity we consider the case where the source and load resistances are equal (R), so that the T-pad will be symmetrical and both image impedances will equal R. From Eq. (4-43) we have

$$R^2 = a(a + 2c); \quad \text{so that } c = \frac{R^2 - a^2}{2a} \tag{4-45}$$

is the relation between c and a to maintain proper input and output resistance. The extreme values for a and c are $a = 0$, $c = \infty$ for a direct connection of source to load; and $a = R$, $c = 0$ for zero power delivered to the load.

The T-pad can be improved by modifying it to a *bridged-T* (Fig. 4.19), using only two variable resistances. The equivalent T is shown in Fig. 4.20.

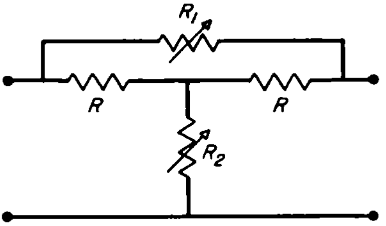


FIG. 4.19.

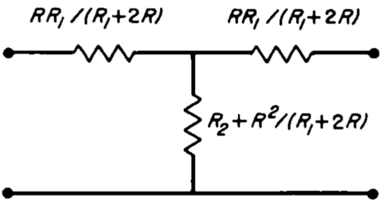


FIG. 4.20.

Problem.

Derive the equivalent T of Fig. 4.20 from the bridged-T of Fig. 4.19. Applying Eq. (4-45) to our equivalent T, we have

$$R^2 = a(a + 2c) = \frac{RR_1}{R_1 + 2R}\left(\frac{RR_1}{R_1 + 2R} + \frac{2R^2}{R_1 + 2R} + 2R_2\right)$$

$$= \frac{RR_1(R + 2R_2)}{R_1 + 2R}$$

making

$$R(R_1 + 2R) = R_1(R + 2R_2)$$

which gives $R_1R_2 = R^2$ as the desired relation between the variable resistances.

Chapter V

CAPACITANCE

5-1 Conductors and Dielectrics. A conductor, ideally, is a body that will not allow a difference of potential between any points of the body. The conductor contains a supply of mobile electrons; any potential difference makes these (negative) electrons move toward the higher (more positive) potential. This action will neutralize any charge distribution that is responsible for the original potential difference.

Although the electrons can move freely, in the sense of being able to move to any part of the conductor, their motion is hindered by "friction," and some electrical energy is converted to heat. This effect is, of course, the source of electrical resistance and the Joule heating, I^2R.

If, however, we subject a conductor to a simple electrostatic field, such as produced by the charges $\pm Q$ in Fig. 5.1, free electrons will be attracted

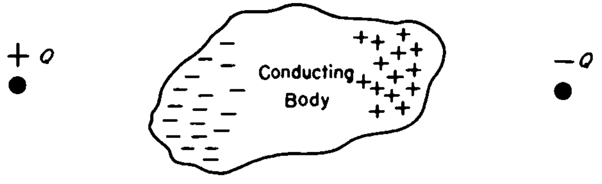


Fig. 5.1.

by the $+Q$ (and repelled by $-Q$) with a resulting redistribution that puts the conductor at the same potential throughout. Because there is no provision for maintaining a current, the resistivity of the body will not affect the equilibrium distribution of charge, but only the speed with which equilibrium is attained.

The equivalent separation of plus and minus charges, if in these positions in free space, would produce at all points now occupied by the body, an electric field exactly equal and opposite to that produced by $\pm Q$. This is just another way of saying that the *net* field is zero in all parts of the body, hence there is no potential difference in the body.

Substances lacking free electrons for conductivity are called insulators, or dielectrics. Their electrons are bound as parts of the atoms and molecules of the dielectric; they try to move in an applied field but behave as if tied with rubber bands. The stronger the applied field, the more the

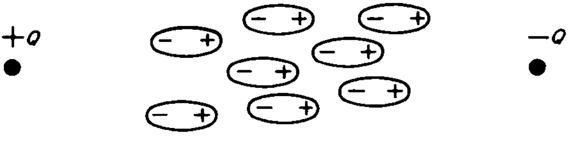


FIG. 5.2.

"band" stretches. This results in a lack of symmetry in the distribution of the positive charge (protons) and negative charge (electrons) of the atoms. This distorted or *polarized* atom is still neutral (has no net charge) but is what the physicists call a *dipole* (Fig. 5.2). The effect of these dipoles is to reduce the field, not to zero as in a conductor, but to some fraction $1/K$ of its free-space value. K is called the *dielectric constant* of the material.

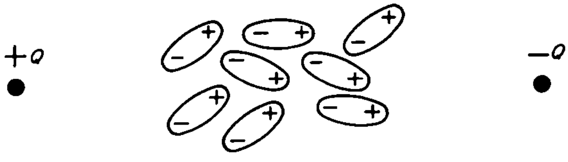


FIG. 5.3.

There are many substances in which the molecules have a permanent dipole nature. In the absence of an applied field, these dipoles have random orientations and give no net effect. Application of an external field tends to align the dipoles as in Fig. 5.3. The forces that hold the molecules together restrain this alignment; the degree of alignment achieved is proportional to the applied field.

5-2 Capacitance. Consider two conducting bodies carrying equal and opposite charges (Fig. 5.4). The potential difference between them, V, is their potential energy per unit charge. Both experiment and theory tell us that V is proportional to Q: if we double the charge, the resulting potential

FIG. 5.4.

difference is doubled. The constant ratio Q/V for any fixed pair of bodies is called the *capacitance* between them:

$$C = Q/V$$

For a given pair of bodies in given positions, the charge on them is therefore proportional to the p.d.

$$Q = CV \tag{5-1}$$

We can readily compute the work required to charge the bodies; at any stage of charging when the p.d. is v, the work required to increase the charge separation by dq is

$$dW = vdq \tag{5-2}$$

At all times we have $q = Cv$, so

$$dW = Cvdv \tag{5-3}$$

and charging from $v = 0$ up to $v = V$ (the final potential)

$$W = \int_0^V Cvdv = \tfrac{1}{2}CV^2 \tag{5-4}$$

This result can be written in various forms by using Eq. (5-1):

$$W = \tfrac{1}{2}CV^2 = \tfrac{1}{2}Q^2/C = \tfrac{1}{2}QV \tag{5-5}$$

If the space between the bodies were filled with a dielectric having dielectric constant K, the field in this region would be reduced by the factor K, and the potential difference for the same charge would therefore also be reduced by K:

$$\begin{aligned} V_1 &= V/K \\ C_1 &= Q/V_1 = KQ/V = KC \end{aligned} \tag{5-6}$$

Equations (5-6) show that the capacitance would be *increased* by the factor K. The use of high-K dielectrics makes it possible to build high-capacitance capacitors in a small volume. This is desirable, for example, when we wish to use the capacitor for energy storage, as in power supply filters or electrical photoflash equipment.

Physicists have derived formulas for the capacitance between bodies in terms of their size, shape, and separation. The most useful of these is for the parallel-plate capacitor of Fig. 5.5. For two plates of area A and separation d, the capacitance is

$$C = k\,KA/d \tag{5-7}$$

where K is the dielectric constant of the medium between the plates ($K = 1$ for air and vacuum), and k is a constant depending on the choice of units. For C in micromicrofarads ($\mu\mu$f), d in centimeters, A in square centimeters, $k = 0.09$.

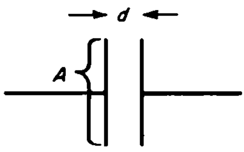


FIG. 5.5.

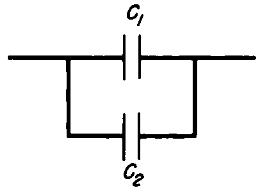


FIG. 5.6.

5-3 Capacitance Networks. As in the case of resistors, the basic combinations of capacitors are *series* and *parallel.* Capacitors in parallel (Fig. 5.6) are obviously subjected to the same potential difference; the total charge is given by

$$Q = Q_1 + Q_2 = C_1V + C_2V = (C_1 + C_2)V \tag{5-8}$$

making the combined capacitance

$$C = C_1 + C_2 \tag{5-9}$$

The result is easily generalized to any number of capacitors in parallel.

In the series connection (Fig. 5.7) the middle conductor can receive no net charge, so $Q_1 = Q_2$. The potential difference between the outermost leads is the sum of the p.d.'s on the individual capacitors, so

$$V = V_1 + V_2 = Q/C_1 + Q/C_2 = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = Q/C \tag{5-10}$$

making

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \tag{5-11}$$

The extension to any number of capacitors in series is apparent. Note that capacitors in *parallel* combine like resistors in *series*, and conversely. (*Capacitance* combines like *conductance.*)

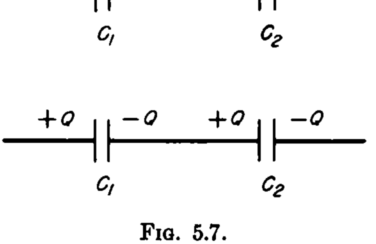


FIG. 5.7.

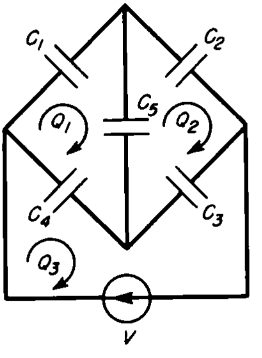


FIG. 5.8.

Any series-parallel combination of capacitors can be computed by repeated applications of Eqs. (5-9) and (5-11). More complicated networks are analyzed by the same techniques we have already used for resistor networks. Kirchhoff's law that the net current to a network node is zero implies that the net charge on a node is zero, for current is the time rate of flow of charge. (If there is no net current to a node *at any time*, no charge is accumulated.) For resistors, $V = RI$; for capacitors, $V = Q/C$; hence the network relations we derived using V, I, R are applicable if we substitute V, Q, $1/C$ (Fig. 5.8)

$$\begin{aligned} Q_1/C_1 + (Q_1 - Q_2)/C_5 + (Q_1 - Q_3)/C_4 &= 0 \\ (Q_2 - Q_1)/C_5 + Q_2/C_2 + (Q_2 - Q_3)/C_3 &= 0 \\ (Q_3 - Q_1)/C_4 + (Q_3 - Q_2)/C_3 &= V \end{aligned} \tag{5-12}$$

and the capacitance "seen" by V is $C = Q_3/V$

5-4 Interelectrode Capacitance. A typical important capacitance network is that consisting of the interelectrode capacitances of a vacuum tube. The simplest case is that of a triode (Fig. 5.9) having *grid-plate*, *grid-cathode*, and *plate-cathode* capacitances. Because these capacitances are internal, they cannot be disconnected from each other for separate measurement. We can, however, short-circuit any capacitance and measure the resulting parallel combination of the other two. This gives three measurements:

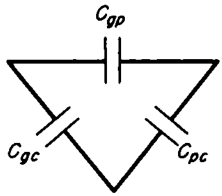


FIG. 5.9.

$$\begin{aligned} C_1 &= C_{gp} + C_{pc} \\ C_2 &= C_{gp} + C_{gc} \\ C_3 &= C_{gc} + C_{pc} \end{aligned} \tag{5-13}$$

These simultaneous equations are readily solved:

$$\begin{aligned} C_{pc} &= (C_1 - C_2 + C_3)/2 \\ C_{gp} &= (C_1 + C_2 - C_3)/2 \\ C_{gc} &= (-C_1 + C_2 + C_3)/2 \end{aligned} \tag{5-14}$$

Before proceeding to detailed consideration of the *pentode*, let us examine the capacitive behavior of its shield-grid (and suppressor grid). Consider a simple parallel-plate capacitor (Fig. 5.10) before and after adding a "floating" third plate. The capacitance of (*a*) is $1/d$ (for a suitable area). That of (*b*) is composed of two in series, $1/x$ and $1/(d - x)$. But the capacitance of the series combination is given by

$$1/C = 1/C_1 + 1/C_2 = x + (d - x) = d$$

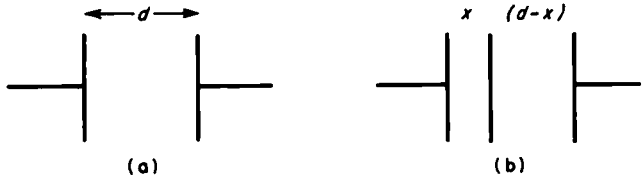


FIG. 5.10.

so (*a*) and (*b*) have the *same* capacitance. If the inserted plate were grounded, however, the outer plates would be shielded from each other, and the combination would consist of two independent capacitors with their common point grounded. This effect is because the net charge on the

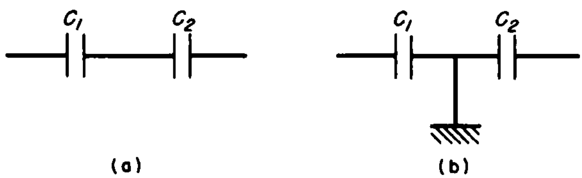


FIG. 5.11.

inserted plate is no longer required to be zero; charge can flow on or off via the ground wire, and the outer plates can be independently charged by arbitrary amounts (Fig. 5.11). If we now punch holes in the inserted plate, the outer plates can "see" each other—the shielding is incomplete. The equivalent circuit becomes that of Fig. 5.12, where the "bridging" capacitance is small.

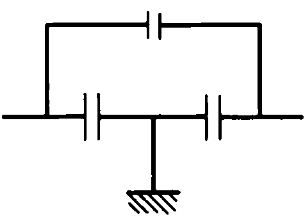


FIG. 5.12.

In a pentode, the suppressor grid (g_3) is connected directly to the cathode, and the shield-grid (g_2) is effectively connected to the cathode at signal frequency by a large bypass capacitor (Fig. 5.13). The network of capacitances (Fig. 5.14) reduces to a triangle, where C_{out} for example, is the sum of C_{pg_1}, C_{pc}, and C_{pg_3}. The values of C_{g_1p}, C_{in}, and C_{out} can be found as in the case of the triode. Measurements on a typical pentode yield:

$$\begin{aligned} C_{in} + C_{out} &= 12 \quad \text{pf}^1 \\ C_{gp} + C_{out} &= 2.03 \quad \text{pf} \\ C_{gp} + C_{in} &= 10.03 \quad \text{pf} \end{aligned} \tag{5-15}$$

[1] pf (picofarad) = $\mu\mu$f (micromicrofarad).

making

$$C_{out} = 2 \quad pf$$
$$C_{in} = 10 \quad pf \qquad (5\text{-}16)$$
$$C_{g_1p} = 0.03 \quad pf$$

C_{in} and C_{out} are shunt capacitances across the input and output circuits, respectively, while C_{g_1p} transmits output signal back to the input. The *feedback* is kept small by the extremely low value of C_{g_1p} achieved by the use of shielding grids.

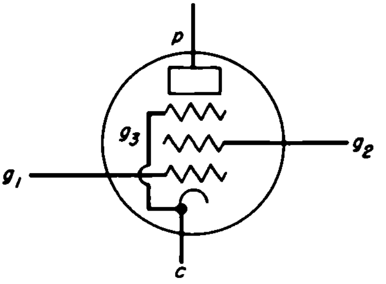


FIG. 5.13.

5-5 Charging and Discharging. Let a capacitor C_1 have a charge Q, and a second capacitor C_2 have no charge (Fig. 5.15). Connect the two together via resistance R. The charge Q will distribute itself between C_1

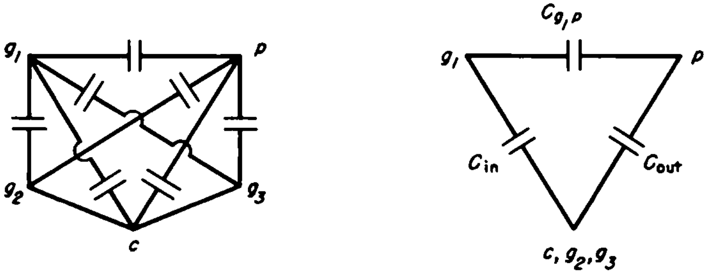


FIG. 5.14.

and C_2 so as to make $V_1 = V_2$, the equilibrium condition. The resulting potential and charge distribution must satisfy

$$Q_1 + Q_2 = Q \qquad (5\text{-}17)$$
$$V = V_1 = V_2$$

We know, however, that

$$V_1 = Q_1/C_1, \quad V_2 = Q_2/C_2$$

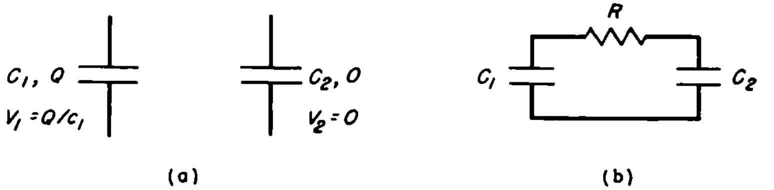


FIG. 5.15.

making

$$V = (Q - Q_2)/C_1 = Q_2/C_2 \tag{5-18}$$

which immediately yields

$$Q_2 = \frac{C_2}{C_1 + C_2} Q, \quad Q_1 = \frac{C_1}{C_1 + C_2} Q, \quad V = \frac{Q}{C_1 + C_2} \tag{5-19}$$

Note that the answer Eq. (5-19) is *independent* of R, and is in fact the same as would be obvious if we had made $R = 0$, so that the capacitors were directly in parallel.

The stored energy was originally

$$W_0 = \tfrac{1}{2}Q^2/C_1$$

and in the final state,

$$W = W_1 + W_2 = \tfrac{1}{2}Q^2/(C_1 + C_2)$$

There has been an energy loss of

$$\tfrac{1}{2}Q^2\left(\frac{1}{C_1} - \frac{1}{C_1 + C_2}\right) = \tfrac{1}{2}Q^2 \frac{C_2}{C_1(C_1 + C_2)} = \frac{C_2}{C_1 + C_2} W_0 \tag{5-20}$$

The fraction $C_2/(C_1 + C_2)$ of the original energy has been lost as heat in the resistor. It is surprising to find that this energy loss is *independent* of R. The detailed history of this charge-sharing process will be discussed later.

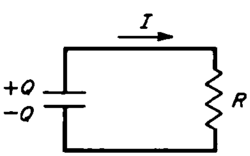


FIG. 5.16.

Suppose now we connect just a resistor R across a charged capacitor C, as in Fig. 5.16. Since current is the rate of flow of charge, we have

$$I = -\frac{dQ}{dt} = -C\frac{dV}{dt} \tag{5-21}$$

defining the current during discharge. By Ohm's law, we have $I = V/R$; substituting into Eq. (5-21) yields

$$\frac{dV}{dt} = -\frac{V}{RC} \tag{5-22}$$

as a *differential equation* describing the behavior of the potential during discharge. This differential equation has a simple solution (which can be verified by substitution):

$$V = V_0 e^{-t/RC} \tag{5-23}$$

where V_0 is the initial potential, i.e., the value of V at $t = 0$. At time $t = RC$, $V_1 = V_0/e$; at $t = 2RC$, $V_2 = V_1/e = V_0/e^2$. In *any* interval of time of duration RC, the voltage drops to $1/e$ of its value at the beginning of that interval. The time RC (ohms × farads = seconds) is called the

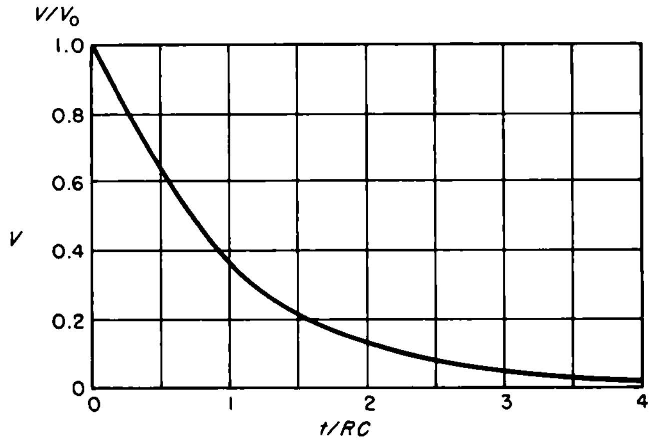


FIG. 5.17.

time constant of this circuit. ($1/e = 0.368 \doteq 37\%$). The discharge is shown graphically in Fig. 5.17.

Conversely, if we charge a capacitor (from zero charge) through a resistor as in Fig. 5.18, the equation is

$$E = RI + V = RC\frac{dV}{dt} + V \quad (5\text{-}24)$$

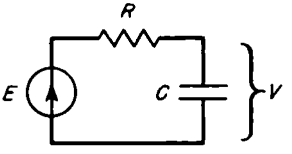


FIG. 5.18.

This yields

$$\frac{dV}{dt} = -\frac{V - E}{RC}$$

or

$$\frac{d(V - E)}{dt} = -\frac{(V - E)}{RC} \quad (5\text{-}25)$$

since E is constant. Equation (5-25) is of the same form as Eq. (5-22), with $(V - E)$ as the dependent variable, so has the solution

$$(V - E) = (V - E)_0 e^{-t/RC} \quad (5\text{-}26)$$

or

$$V = E - Ee^{-t/RC} = E(1 - e^{-t/RC}) \quad (5\text{-}27)$$

since $E_0 = E$, $V_0 = 0$. This result is shown graphically in Fig. 5.19.

Our charge-sharing problem (Fig. 5.15b) can now be analyzed in detail (Fig. 5.20)

$$I = -\frac{dQ_1}{dt} = +\frac{dQ_2}{dt} \quad (5\text{-}28)$$

$$I = (V_1 - V_2)/R$$

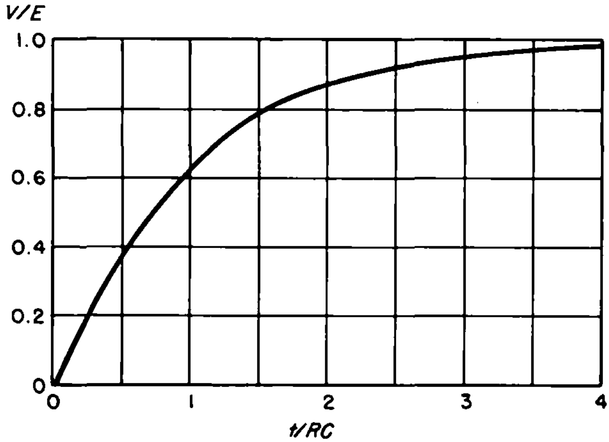


FIG. 5.19.

Eliminating I gives simultaneous equations for V_1 and V_2:

$$-C_1 \frac{dV_1}{dt} = C_2 \frac{dV_2}{dt}$$
$$-RC_1 \frac{dV_1}{dt} = V_1 - V_2 \tag{5-29}$$

The first of these can be integrated immediately:

$$-C_1V_1 = C_2V_2 + \text{constant}$$

and since $V_2 = 0$ at $t = 0$ (the start of the experiment), the constant of integration must be $-C_1V_{10}$, where V_{10} indicates the initial value of V_1. Therefore

$$V_2 = (V_{10} - V_1)C_1/C_2$$

which can be substituted into the second equation of (5-29), yielding

$$-RC_1 \frac{dV_1}{dt} = V_1(1 + C_1/C_2) - V_{10}C_1/C_2$$

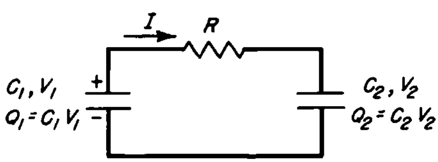


FIG. 5.20.

or

$$-\frac{RC_1C_2}{C_1 + C_2}\frac{dV_1}{dt} = V_1 - V_{10}C_1/(C_1 + C_2) \tag{5-30}$$

Equation (5-30) has a simple exponential solution

$$V_1 = \frac{C_1}{C_1 + C_2}V_{10} + \frac{C_2}{C_1 + C_2}V_{10}e^{-t/RC}$$
$$C \equiv \frac{C_1C_2}{C_1 + C_2} \tag{5-31}$$

which states that the initial value ($t = 0$) of V_1 is V_{10}, the final value ($t = \infty$) is $V_{10}C_1/(C_1 + C_2)$ and that the change from initial to final value is a simple exponential decay with time constant RC, where $C = C_1C_2/(C_1 + C_2)$. Note that C is the capacitance of the *series* combination of C_1 and C_2, which is exactly the combination seen by the resistance R.

5-6 Current Generator. Return to Fig. 5.18, with the solution (5-27) for the charging of a capacitor by a generator. For $RC > t$,

$$e^{-t/RC} = 1 - t/RC + \tfrac{1}{2}(t/RC)^2 - \cdots$$

so that Eq. (5-27) becomes

$$V = E(1 - e^{-t/RC}) \doteq \left(\frac{E}{R}\right)\frac{t}{C} - \frac{1}{2}\left(\frac{E}{R}\right)\frac{t^2}{RC^2} + \cdots \tag{5-32}$$

Let R and E increase indefinitely, keeping $E/R = I$ = constant, so that the voltage generator with internal resistance R becomes a current generator. All terms of Eq. (5-32) except the first go to zero, leaving:

$$V = It/C \tag{5-33}$$

for the charging by a constant current generator. This can obviously be rewritten as

$$Q = It$$

in agreement with a direct analysis of the constant current situation.

The result (5-33) indicates a voltage rise proportional to time. This relation is of great importance in high-speed electronic computers, and shows how the distributed capacitance of wiring limits the speed.

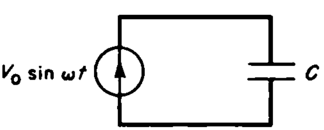


FIG. 5.21.

5-7 Sinusoidal Voltage. Let the capacitance C be subjected to a sinusoidal voltage, $V = V_0 \sin \omega t$. The basic relation $Q = CV$ yields, on differentiation,

$$I = C\,dV/dt = \omega CV_0 \cos \omega t \tag{5-34}$$

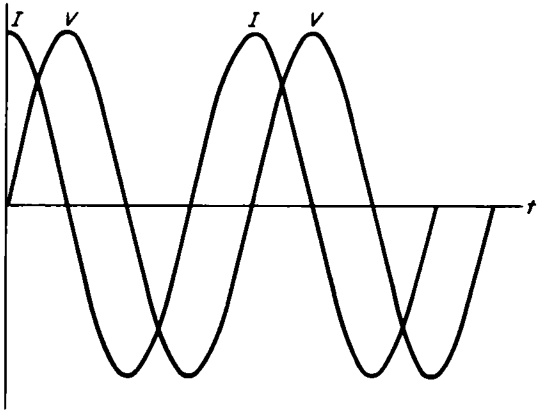


FIG. 5.22.

The amplitude of the current is ωC times that of the voltage, and the current is 90° out of phase with the voltage. The current is said to *lead* the voltage by 90°, or the voltage *lags* the current. Plots of current and voltage vs. time (Fig. 5.22) show the lead-lag relationship. (The current *leads* because the wave occurs *earlier*, i.e., at *smaller* values of t.)

Chapter VI

INDUCTANCE

6-1 Magnetic Flux. A current-carrying wire is surrounded by a magnetic field; the magnetic flux encircles the wire (Fig. 6.1).

If the wire forms a loop, say a circle (Fig. 6.2) all the flux goes through the loop in one sense, and returns via the outside. The circled dots indicate the "heads of the arrows" representing the flux, i.e., coming out of the paper. The circled crosses are the "feathers," where the flux goes into the paper. If the wire is wound into a cylindrical coil (solenoid) we have a

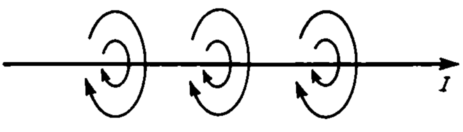


FIG. 6.1.

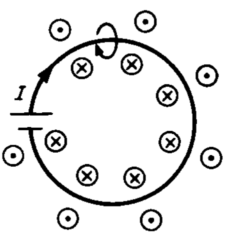


FIG. 6.2.

stack of circles carrying current in the same sense. Hence the flux passes through all the circles in the same sense. Between adjacent wires, the flux from one opposes and neutralizes that from its neighbor. Hence the flux around the solenoid is as in Fig. 6.3, neglecting "leakage" between turns near the ends. The flux comes out of one end of the solenoid, and goes back in the other. The magnetic field is the same as that of a bar magnet. The total flux is proportional to I and to the number of turns

$$\varphi \propto NI \tag{6-1}$$

The *flux-linkage* is the product of the flux and the number of turns linked.

$$\Phi = N\varphi \propto N^2 I \tag{6-2}$$

(Note that flux lines and wire turns are interlocked, or linked, closed curves

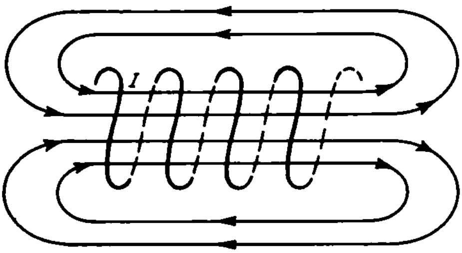


FIG. 6.3.

(Fig. 6.2)). The constant of proportionality in Eq. (6-2) depends upon the shape and size of the winding, and the permeability of the core material. These constants can be lumped together, and we define *the inductance* L by

$$\Phi = LI \tag{6-3}$$

Note that L is proportional to the square of the number of turns.

Michael Faraday, an Englishman, and Joseph Henry, an American, independently discovered (about 1830) that a *changing* flux through a coil produces an emf; there is a voltage (rise) *induced* in the coil. They found that the induced emf is proportional to the rate of change of flux:

$$e \propto -N\, d\varphi/dt$$

and, in mks units[1]

$$e = -N\, d\varphi/dt \tag{6-4}$$

which can be written

$$e = -\frac{d\Phi}{dt} = -L\frac{di}{dt}$$ [2]

if N is constant, as we shall assume throughout.

The voltage *drop* across the inductance is

$$v = -e = L\, di/dt \tag{6-5}$$

Note that

$$\Phi = LI \text{ is analogous to } Q = CV$$

$$v = \frac{d\Phi}{dt} \text{ is analogous to } i = \frac{dq}{dt}$$

6-2 Energy Storage. Increasing the current through an inductance from zero to some final value I requires work to be done against the induced

[1] e in volts, t in seconds, L in henries, i in amperes, φ in webers.

[2] We have changed to lower case e and i to emphasize that these quantities vary with time.

back-voltage. We have

$$dW = vidt = \left(L\frac{di}{dt}\right)idt = Lidi \tag{6-6}$$

so that

$$W = \int_0^I Lidi = \tfrac{1}{2}LI^2 \tag{6-7}$$

is the energy stored in the magnetic field. It is this stored energy that tries to maintain the current if the battery is removed or shorted.

6-3 Inductance Networks. Inductances in series (Fig. 6.4) must

FIG. 6.4.

carry the same current. The voltages across the coils, on the other hand, are additive. The total voltage across the combination is

$$v = v_1 + v_2 + v_3 = L_1\,di/dt + L_2\,di/dt + L_3\,di/dt$$
$$= (L_1 + L_2 + L_3)\,di/dt$$

so that the inductance of the series combination is

$$L = L_1 + L_2 + L_3 \tag{6-8}$$

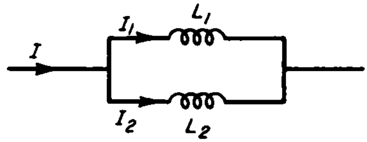


FIG. 6.5.

In the parallel case (Fig. 6.5) the currents add ($i = i_1 + i_2$), but the terminal voltages are common. Thus

$$di/dt = di_1/dt + di_2/dt$$
$$= v/L_1 + v/L_2 = v\left(\frac{1}{L_1} + \frac{1}{L_2}\right) = v/L$$

so that

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} \tag{6-9}$$

Inductances in series or parallel combine like resistances in series or parallel, respectively.

In a general network, we saw previously that the sum of the currents to a node must vanish *at all times*. Therefore the sum of the time derivatives of the currents to a node must also vanish. In a resistance network,

$$v = Ri, \Sigma i = 0 \text{ at a node}$$

whereas in an inductance network we have

$$v = Li', \Sigma i' = 0 (i' \equiv di/dt)$$

so we can use our earlier resistance network techniques by substituting di/dt for i and L for R

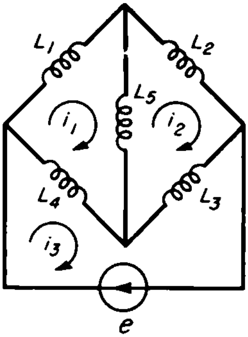


FIG. 6.6.

Example.

The network of Fig. 6.6 satisfies the following equations:

$$\begin{aligned} L_1 i'_1 + L_5(i'_1 - i'_2) + L_4(i'_1 - i'_3) &= 0 \\ L_5(i'_2 - i'_1) + L_2 i'_2 + L_3(i'_2 - i'_3) &= 0 \qquad (6\text{-}10) \\ L_4(i'_3 - i'_1) + L_3(i'_3 - i'_2) &= e \end{aligned}$$

and the inductance "seen" by e is $L = e/i'_3$.

6-4 "Charge" and "Discharge." Consider the establishment of equilibrium current through an inductance and resistance (Fig. 6.7). The source voltage E must equal the sum of the voltage drop in R and the drop (back-voltage) in L:

$$E = Ri + L\, di/dt$$

$$\frac{di}{dt} = -\frac{R}{L}i + \frac{E}{L} \qquad (6\text{-}11)$$

$$= -\frac{R}{L}\left(i - \frac{E}{R}\right)$$

or

$$\frac{d(i - E/R)}{dt} = -\frac{R}{L}(i - E/R) \qquad (6\text{-}12)$$

so that $(i - E/R)$ satisfies a differential equation of the same form as Eqs. (5-22) and (5-25). The solution is

$$(i - E/R) = ce^{-Rt/L}$$

At $t = 0$, $i = 0$, so that $c = -E/R$, and we have

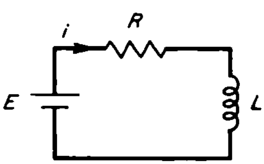


FIG. 6.7.

$$i = \frac{E}{R}(1 - e^{-Rt/L}) \qquad (6\text{-}13)$$

The plot of current vs. time looks like Fig. 5.19, and the equilibrium current is E/R. The time constant is L/R (*henries* divided by *ohms* equals *seconds*).

For R/L small, $e^{-Rt/L}$ can be expanded in a power series, yielding

$$i = \frac{E}{R}\left\{\frac{Rt}{L} - \frac{1}{2}\left(\frac{Rt}{L}\right)^2 + \cdots\right\}$$

$$\doteq \frac{Et}{L}$$

Thus a constant voltage will give a linearly rising current, hence a linearly increasing magnetic field. This is desired for the horizontal sweep on a TV picture tube.

For the "discharge" case, let the current in Fig. 6.7 be at its equilibrium value E/R and replace the voltage source by a short-circuit (Fig. 6.8). The differential equation is

$$L\frac{di}{dt} + Ri = 0$$

The solution is

$$i = I_0e^{-Rt/L}$$

with $I_0 = E/R$. The current decays exponentially from its initial value to zero, with the time constant L/R. If $R = 0$, the inductance maintains the current without change.

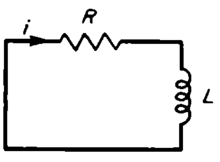


FIG. 6.8.

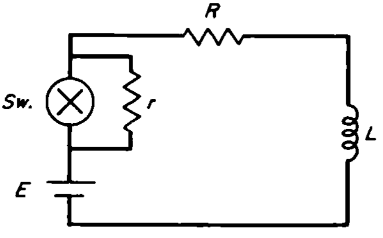


FIG. 6.9.

Consider next the case where the battery is removed by opening a switch, as in Fig. 6.9. The equation is

$$L\frac{di}{dt} + (R + r)i = 0 \tag{6-14}$$

with

$$I_0 = E/R \text{ as before, hence}$$

$$i = \frac{E}{R}e^{-(R+r)t/L}$$

and the voltage across the switch is

$$v_s = ri = \frac{r}{R}Ee^{-(R+r)t/L}$$

As r is made larger, the initial voltage on the switch rises correspondingly, since the initial current is still E/R. The stored energy in the inductance tries to maintain the current. As $r \to \infty$, the voltage across the switch $\to \infty$ theoretically. In practice, the voltage rises rapidly as the switch contact opens, and the switch gap breaks down. An arc is formed (the resistance r), and the switch points are damaged. If we try to make a nonarcing switch, say by using a triode as a switch, there will be a shunt capacitance which will limit the voltage rise.

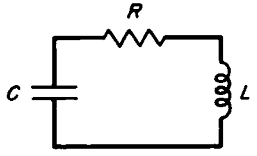


FIG. 6.10.

When a mechanical switch is used, as in automobile ignition systems, a capacitor is shunted across the switch to provide a path for the current that the inductance insists on maintaining. In this case we have the circuit of Fig. 6.10, and the equation:

$$q/C + Ri + L\,di/dt = 0 \tag{6-15}$$

Differentiating, we have

$$i/C + R\,di/dt + L\,d^2i/dt^2 = 0 \tag{6-16}$$

The initial conditions are $i = I_0$, $q = 0$, at $t = 0$.

For the present, let $R = 0$ for simplicity:

$$\frac{d^2i}{dt^2} = -\frac{i}{LC} \tag{6-17}$$

This has the solution

$$i = A \sin \omega t + B \cos \omega t, \quad \omega^2 \equiv 1/LC \tag{6-18}$$

with A and B arbitrary constants to be chosen to satisfy the initial conditions. At $t = 0$, $i = B = I_0$. For q, use Eq. (6-15):

$$\begin{aligned} q &= -LC\frac{di}{dt} = -\omega^2\frac{di}{dt} \\ &= -\omega^2 A \cos \omega t + \omega^2 B \sin \omega t \\ &= -\omega^2 A \text{ at } t = 0, \text{ requiring } A = 0. \end{aligned}$$

The final solution is therefore the oscillation:

$$i = I_0 \cos \omega t, \quad \omega = \sqrt{1/LC} \tag{6-19}$$

The voltage across the condenser, and therefore across the switch, is

$$q/C = -L\frac{di}{dt} = \omega L I_0 \sin \omega t = \sqrt{\frac{L}{C}}\, I_0 \sin \omega t \tag{6-20}$$

Note that the voltage amplitude is proportional to $\sqrt{L/C}$.

For $R \neq 0$, but not too large, the solution is an oscillation whose amplitude decays exponentially. This will be developed in the next chapter.

6-5 Mutual Inductance. Consider two coils so placed that a fraction (k_1) of the flux from the first coil links the second coil:

$$\varphi_2 = k_1\varphi_1 = k_1\frac{\Phi_1}{N_1} = \frac{k_1 L_1}{N_1}\, i_1 \tag{6-21}$$

and the fraction k_2 of the flux from the second coil links the first:

$$\phi_1 = k_2 L_2 i_2 / N_2 \tag{6-22}$$

The flux linkage in the second coil due to current in the first coil

$$\Phi_2 = N_2 \varphi_2 = \frac{N_2 k_1 L_1 i_1}{N_1} \equiv M i_1 \tag{6-23}$$

where the proportionality constant M is called the *mutual inductance* between the coils. It will be shown in Chapter IX that the relation is symmetrical, i.e.,

$$\Phi_1 = \frac{N_1 k_2 L_2 i_2}{N_2} \equiv M i_2 \tag{6-24}$$

is the flux linkage in the first coil due to current in the second coil. For current in both coils, the flux linkages are given by

$$\begin{aligned} \Phi_1 &= L_1 i_1 + M i_2 \\ \Phi_2 &= M i_1 + L_2 i_2 \end{aligned} \tag{6-25}$$

From Eqs. (6-23) and (6-24):

$$M = N_2 k_1 L_1 / N_1$$

and

$$M = N_1 k_2 L_2 / N_2$$

whence

$$M^2 = k_1 k_2 L_1 L_2$$

and

$$M = \sqrt{k_1 k_2}\sqrt{L_1 L_2} \equiv k\sqrt{L_1 L_2} \tag{6-26}$$

The constant $k = \sqrt{k_1 k_2}$ is called the *coefficient of coupling*. Since k_1 and k_2 are both fractions less than or equal to unity, we have $k \leq 1$. The case $k = 1$ is *unity coupling*; the two coils link the same flux.

From Eq. (6-25), the voltages are

$$\begin{aligned} v_1 &= L_1 i'_1 + M i'_2 \\ v_2 &= M i'_1 + L_2 i'_2 \end{aligned} \tag{6-27}$$

If only one coil carries current,

$$\begin{aligned} v_1 &= L_1 i'_1 \\ v_2 &= M i'_1 = k_1 (N_2/N_1) L_1 i'_1 \\ &= k_1 (N_2/N_1) v_1 \end{aligned} \tag{6-28}$$

For unity coupling, $k_1 = 1$, and the induced emf's are proportional to the numbers of turns.

In an automobile spark coil, a heavy winding of relatively few turns provides the inductance considered in Figs. 6.9 and 6.10. Interruption of the current develops a voltage peak of several hundred volts across this coil. Another winding of many turns of fine wire surrounds the same iron core; by Eq. (6-28), the emf induced in this is tens of thousands of volts. A similar arrangement is used in a TV fly-back transformer, to develop the accelerator voltage for the picture tube.

6-6 Sinusoidal Current. Let the inductance L carry the current

$$i = I_0 \sin \omega t$$

The applied voltage to produce this current must equal the induced back-voltage (provide the voltage drop):

$$v = Li' = \omega L I_0 \cos \omega t$$

This result corresponds to Eq. (5-34) with voltage and current interchanged. From the discussion following Eq. (5-34), we conclude that for inductance, the voltage leads the current by 90°, or the current lags the voltage.

If an inductance and capacitance are connected in series, they carry the *same* current. The voltage drop across the inductance *leads* the current by 90°; that across the capacitance *lags* the same current by 90°, hence the voltage drops are 180° out of phase or in *opposite* polarity. If their amplitudes are the same (because of suitable choice of ω, L, and C), the *net* voltage drop across the combination is *zero*. This is the case of series resonance to be discussed in the next chapter.

Chapter VII

SERIES-TUNED CIRCUITS

7-1 Natural Oscillations. In the preceding chapter, we saw that the circuit of Fig. 7.1 (Fig. 6.10) was subject to the differential equation

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0 \tag{7-1}$$

For $R = 0$, the solution was sinusoidal. For $R \neq 0$, we know from experience that there will be a *damped* sine wave solution; we solve Eq. (7-1) by trying the solution

$$i = Ae^{-at}\sin\omega t \tag{7-2}$$

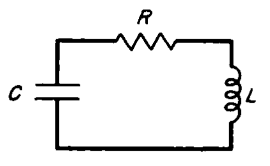


FIG. 7.1.

Since $di/dt = -aAe^{-at}\sin\omega t + \omega Ae^{-at}\cos\omega t$ and

$$\frac{d^2i}{dt^2} = a^2Ae^{-at}\sin\omega t - \omega aAe^{-at}\cos\omega t - \omega aAe^{-at}\cos\omega t - \omega^2Ae^{-at}\sin\omega t$$

substitution into Eq. (7-1) yields:

$$\{L(a^2 - \omega^2) - Ra + 1/C\}\, Ae^{-at}\sin\omega t + \{-2L\omega a + R\omega\}\, Ae^{-at}\cos\omega t = 0 \tag{7-3}$$

For Eq. (7-3) to be satisfied identically (i.e., for all t), the coefficients of $\sin\omega t$ and $\cos\omega t$ must both vanish. The coefficient of $\cos\omega t$ yields

$$a = R/2L \tag{7-4}$$

whereupon the coefficient of $\sin\omega t$ yields

$$L\left(\frac{R^2}{4L^2} - \omega^2\right) - \frac{R^2}{2L} + \frac{1}{C} = 0$$

or

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} \tag{7-5}$$

Note that the presence of resistance ($R \neq 0$) introduces the damping factor $e^{-Rt/2L}$ and reduces the *frequency* of the oscillation from

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

to

$$f_N = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{7-6}$$

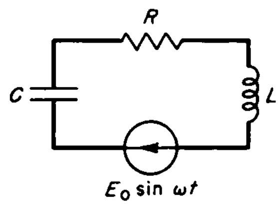


FIG. 7.2.

(The angular frequency ω is related to the frequency f by $\omega = 2\pi f$)

The oscillations of *natural frequency* f_N decay experimentally with a time constant $2L/R$.

Let us next excite this series circuit with a generator having the voltage $e = E_0 \sin \omega t$, where w is *arbitrary* (Fig. 7.2). Since the total voltage drop must now be $E_0 \sin \omega t$, Eq. (6-15) must be modified to read

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E_0 \sin \omega t \tag{7-7}$$

or, differentiating to remove q:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \omega E_0 \cos \omega t \tag{7-8}$$

The steady-state solution must have angular frequency ω, but its phase (lag or lead) is unknown. We try the solution

$$i = A \sin \omega t + B \cos \omega t \tag{7-9}$$

Differentiating and substituting, we find

$$\frac{di}{dt} = \omega A \cos \omega t - \omega B \sin \omega t$$

$$\frac{d^2i}{dt^2} = -\omega^2 A \sin \omega t - \omega^2 B \cos \omega t$$

$$\{-L\omega^2 A - R\omega B + A/C\} \sin \omega t + \{-L\omega^2 B + R\omega A + B/C - \omega E_0\} \cos \omega t = 0 \tag{7-10}$$

Again, the coefficients of $\sin \omega t$ and $\cos \omega t$ must vanish, yielding two equations for A and B (ω is given):

$$\begin{aligned}(\omega^2 L - 1/C)A + \omega R B &= 0 \\ \omega R A - (\omega^2 L - 1/C)B &= \omega E_0\end{aligned} \tag{7-11}$$

These yield

$$A = E_0 \frac{R}{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$$

$$B = -E_0 \frac{\omega L - \dfrac{1}{\omega C}}{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2} \tag{7-12}$$

The solution (7-9) can be written

$$\begin{aligned} i &= I_0 \sin(\omega t - \theta) \\ &= (I_0 \cos\theta) \sin \omega t - (I_0 \sin\theta) \cos \omega t \end{aligned}$$

so that

$$I_0 \cos\theta = A, \quad I_0 \sin\theta = -B$$

and

$$\begin{aligned} I_0^2 &= A^2 + B^2 \\ \tan\theta &= -B/A \end{aligned} \tag{7-13}$$

Using our results for A and B:

$$I_0^2 = \frac{E_0^2}{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$$

$$I_0 = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \tag{7-14}$$

$$\tan\theta = \left(\omega L - \frac{1}{\omega C}\right)/R$$

7-2 Series Resonance. If the applied voltage has the frequency satisfying $\omega^2 = 1/LC$, we have $\tan\theta = 0$ and $\theta = 0$, so that the current is *in phase* with the voltage, just as it is in a purely resistive circuit. In fact, this condition also yields $I_0 = E_0/R$, so the current is precisely the same as if the voltage were applied directly across R. This is the case of *phase resonance*.

It is also interesting to examine the *amplitude resonance*, i.e., the conditions for *maximum amplitude* of the current. Differentiating Eq. (7-14) with respect to ω yields

$$\frac{dI_0}{d\omega} = -\tfrac{1}{2}E_0 \left\{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right\}^{-3/2} 2\left(\omega L - \frac{1}{\omega C}\right)\left(L + \frac{1}{\omega^2 C}\right)$$

which vanishes for $\omega L = 1/\omega C$, the same frequency as for *phase resonance*. This common frequency is the frequency of *series resonance*, the frequency at which the current is in phase with the voltage and also has its maximum amplitude. We shall see later that for parallel-tuned circuits, phase resonance and amplitude resonance do not coincide.

7-3 Impedance. We have seen that a current through a resistance produces a voltage drop $v = Ri$ so that if

$$i = I_0 \sin \omega t$$

then

$$v = RI_0 \sin \omega t \equiv V_0 \sin \omega t, \; V_0 = RI_0$$

For capacitance,

$$i = I_0 \sin \omega t$$

$$v = -\frac{I_0}{\omega C} \cos \omega t \equiv V_0 \sin (\omega t - \pi/2), \; V_0 = I_0/\omega C$$

For inductance,

$$i = I_0 \sin \omega t$$

$$v = \omega L I_0 \cos \omega t \equiv V_0 \sin (\omega t + \pi/2), \quad V_0 = \omega L I_0$$

For the series RLC circuit, we had

$$v = V_0 \sin \omega t$$

$$i = I_0 \sin (\omega t - \theta), \quad \tan \theta = \left(\omega L - \frac{1}{\omega C}\right) / R$$

which can be written (by changing the origin of t) as

$$i = I_0 \sin \omega t$$

$$v = V_0 \sin (\omega t + \theta)$$

where

$$\tan \theta = \left(\omega L - \frac{1}{\omega C}\right) / R$$

$$V_0 = I_0 \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{7-15}$$

In all these cases, although v and i may be out of phase, $V_0 \propto I_0$. It is very convenient to generalize Ohm's law and the resistance concept to the form:

$$V_0 = ZI_0 \tag{7-16}$$

where Z is called *impedance* (actually, *absolute magnitude* of the impedance, for reasons given in the next chapter). The impedance is characterized not only by its magnitude Z (ohms), but also by the angle θ by which the voltage leads the current. This is sometimes indicated by writing

$$\mathbf{Z} = Z \angle \theta$$

For example, an impedance of $\mathbf{Z} = 10 \angle 30°$ carrying 2 amperes ($i = 2 \sin \omega t$) would exhibit a voltage drop

$$v = 20 \sin (\omega t + 30°)$$

When the phase angle of $\mathbf{Z}$ is zero, the impedance is a "pure" resistance; when the phase angle is $\pm\pi/2$, the impedance is called a pure *reactance.* For $\theta = +\pi/2$ (for inductance) the reactance is said to be *positive*; for $\theta = -\pi/2$ (capacitance), the reactance is *negative.*

If an impedance $Z_1 \angle \theta_1$ carries a current $I_0 \sin \omega t$, the voltage drop is $v_1 = Z_1 I_0 \sin (\omega t + \theta)$. If two impedances are in *series,* they carry the same current and the voltage drop is the sum of the separate voltage drops:

$$v = I_0 Z_1 \sin (\omega t + \theta_1) + I_0 Z_2 \sin (\omega t + \theta_2) \tag{7-17}$$

We wish to express this as

$$v = I_0 Z \sin (\omega t + \theta) \tag{7-18}$$

so that we can express the impedance $Z \angle \theta$ of the combination in terms of the impedances $Z_1 \angle \theta_1$ and $Z_2 \angle \theta_2$. Expanding $\sin (\omega t + \theta)$ in Eq. (7-18) gives:

$$v = I_0 \{Z \cos \theta \sin \omega t + Z \sin \theta \cos \omega t\} \tag{7-19}$$

while Eq. (7-17) gives

$$v = I_0 \{(Z_1 \cos \theta_1 + Z_2 \cos \theta_2) \sin \omega t + (Z_1 \sin \theta_1 + Z_2 \sin \theta_2) \cos \omega t\} \tag{7-20}$$

For Eqs. (7-19) and (7-20) to be identical, we must have:

$$Z \cos \theta = Z_1 \cos \theta_1 + Z_2 \cos \theta_2 \tag{7-21}$$

$$Z \sin \theta = Z_1 \sin \theta_1 + Z_2 \sin \theta_2 \tag{7-22}$$

These equations, (7-21) and (7-22), can be explained (and solved) graphically. The terms involving Z_1 are the horizontal and vertical com-

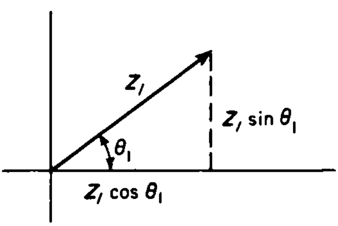


FIG. 7.3.

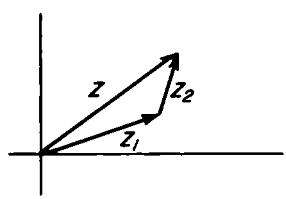


FIG. 7.4.

ponents of the vector Z_1 of Fig. 7.3. The construction of Fig. 7.4 yields a vector Z whose horizontal component is the sum of the horizontal components of Z_1 and Z_2, as required by Eq. (7-21). The vertical components similarly satisfy Eq. (7-22). For resistance, inductance, and capacitance, respectively, we have:

$$Z = R \angle 0°$$

$$Z = \omega L \angle 90°$$

$$Z = \frac{1}{\omega C} \angle -90°$$

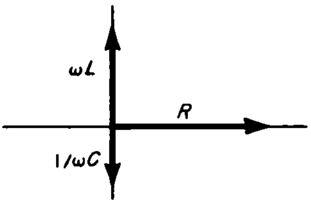


FIG. 7.5.

These impedances are shown separately in Fig. 7.5. A series combination of R and L gives the impedance

$$Z = \sqrt{R^2 + (\omega L)^2} \angle \tan^{-1} \omega L/R \qquad (7\text{-}23)$$

as shown in Fig. 7.6. The series combination of R, L, and C illustrates Eqs. (7-14) and (7-15) graphically (Fig. 7.7). It is again apparent that

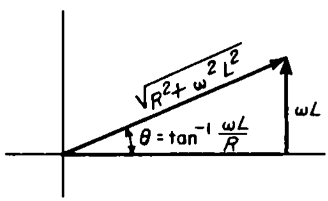


FIG. 7.6.

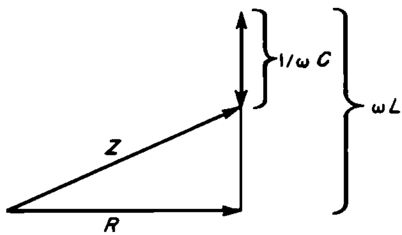


FIG. 7.7.

for $\omega L = 1/\omega C$, the impedance is a pure resistance and has its minimum magnitude as a function of ω.

Except in these simple cases, graphical combination of impedances becomes cumbersome. In the next chapter we shall develop algebraic means for dealing with impedances in series-parallel combinations.

Chapter VIII

COMPLEX NUMBERS

8-1 Review of Fundamentals. Complex numbers are numbers having both real and imaginary parts, such as $a + jb$, where a and b are real numbers, and j represents $\sqrt{-1}$, a pure imaginary. All properties of complex numbers follow from the four basic rules (these rules can be considered as mathematical postulates that define complex numbers and their algebraic properties):

If
$$a + jb = c + jd$$
then
$$a = c, \quad b = d \tag{8-1}$$

$$(a + jb) + (c + jd) = (a + c) + j(b + d) \tag{8-2}$$

$$c(a + jb) = ca + jcb \tag{8-3}$$

Since
$$jj = j^2 = -1, \quad j(a + jb) = ja - b = -b + ja \tag{8-4}$$

From these four rules, we find that

$$(a + jb)(c + jd) = (ac - bd) + j(bc + ad) \tag{8-5}$$

by carrying out the multiplication just as if j were an ordinary algebraic variable x, and using the relation $j^2 = -1$. Note that

$$(a + jb)(a - jb) = a^2 + b^2 \tag{8-6}$$

The number $(a - jb)$ is called the *complex conjugate* of $(a + jb)$, and vice versa.

We can use relation (8-6) to reduce a problem in division to one in multiplication:

$$\begin{aligned} \frac{a + jb}{c + jd} &= \frac{a + jb}{c + jd}\frac{c - jd}{c - jd} = \frac{(a + jb)(c - jd)}{c^2 + d^2} \\ &= \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + j\,\frac{bc - ad}{c^2 + d^2} \end{aligned} \tag{8-7}$$

8-2 Complex Exponentials. Since $\frac{de^{at}}{dt} = ae^{at}$ for *any* constant a, we have

$$\frac{de^{j\omega t}}{dt} = j\omega e^{j\omega t} \tag{8-8}$$

for constant ω. Differentiating again yields

$$\frac{d^2e^{j\omega t}}{dt^2} = (j\omega)^2 e^{j\omega t} = -\omega^2 e^{j\omega t}$$

so that

$$\frac{d^2e^{j\omega t}}{dt^2} + \omega^2 e^{j\omega t} = 0 \tag{8-9}$$

Now we know that the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{8-10}$$

has the general solution

$$x = A\cos\omega t + B\sin\omega t \tag{8-11}$$

so $e^{j\omega t}$ must be of this form. We find A and B to satisfy

$$\begin{aligned}(e^{j\omega t})_{t=0} &= 1\\ \left(\frac{de^{j\omega t}}{dt}\right)_{t=0} &= j\omega\end{aligned} \tag{8-12}$$

From Eq. (8-11) we find

$$(x)_{t=0} = A, \quad \left(\frac{dx}{dt}\right)_{t=0} = \omega B \tag{8-13}$$

Comparing Eq. (8-13) with Eq. (8-12) yields $A = 1$, $B = j$, so that

$$e^{j\omega t} = \cos\omega t + j\sin\omega t \tag{8-14}$$

or in general

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{8-15}$$

where θ is any quantity, either constant or variable.

8-3 Graphical Representation. It is convenient to represent complex numbers as points in a plane having real and imaginary coordinates (Fig. 8.1). The addition rule (8-2) shows that the sum of two complex numbers is represented by vector addition (Fig. 8.2), making it convenient to think of $x + jy$ not as a *point* in the complex plane, but as a *vector*. In electrical engineering, these vectors are sometimes called *phasors*, to distinguish them from the familiar three-dimensional vectors in space. From Eq. (8-15) we find that the complex number

$$Re^{j\theta} = R\cos\theta + jR\sin\theta \tag{8-16}$$

can be represented by a phasor of length R making an angle θ with the

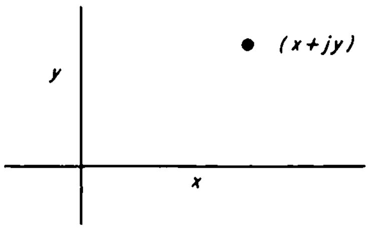


Fig. 8.1.

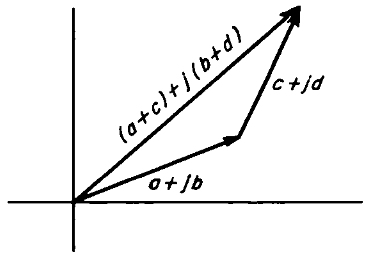


Fig. 8.2.

x-axis (Fig. 8.3). Thus any complex number can be represented in either *rectangular* or *polar* form:

$$a + jb = \sqrt{a^2 + b^2}\,(\cos\theta + j\sin\theta) = \sqrt{a^2 + b^2}\,e^{j\theta} \tag{8-17}$$

where $\theta = \tan^{-1} b/a.$

The "length" $\sqrt{a^2 + b^2}$ is called the *absolute magnitude* of $(a + jb)$.

The graphical representation of the product of two complex numbers is not apparent from the rectangular representation Eq. (8-5). If, however, we use the polar representations,

$$(a + jb) = R_1 e^{j\theta_1}$$
$$(c + jd) = R_2 e^{j\theta_2}$$

the product is (Fig. 8.4)

$$(a + jb)(c + jd) = R_1 R_2 e^{j(\theta_1 + \theta_2)} = Re^{j\theta} \tag{8-18}$$

where $R = R_1 R_2 = \sqrt{a^2 + b^2}\,\sqrt{c^2 + d^2}$

and $\theta = \theta_1 + \theta_2 = \tan^{-1} b/a + \tan^{-1} d/c$

8-4 Application to Linear Differential Equations. The current through a series combination of resistance and inductance satisfies the

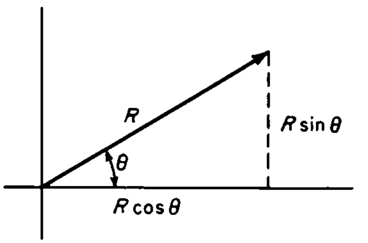


Fig. 8.3.

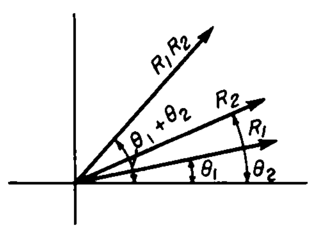


Fig. 8.4.

differential equation:

$$L\frac{di}{dt} + Ri = e \tag{8-19}$$

If $e_1 = E \cos \omega t$, we have

$$i \equiv i_1 = \frac{E(R \cos \omega t + \omega L \sin \omega t)}{R^2 + \omega^2 L^2} \tag{8-20}$$

for the solution of Eq. (8-19). For $e_2 = E \sin \omega t$, we have

$$i \equiv i_2 = \frac{E(R \sin \omega t - \omega L \cos \omega t)}{R^2 + \omega^2 L^2} \tag{8-21}$$

and for $e = jE \sin \omega t$, we would have $i = ji_2$ (by substituting jE for E in Eq. (8-21)). Because Eq. (8-19) is a *linear* differential equation, the solution for a sum of driving forces is the sum of the corresponding separate solutions (*principle of superposition*):

$$\begin{aligned} e &= e_1 + je_2 \\ &= E \cos \omega t + jE \sin \omega t = Ee^{j\omega t} \end{aligned}$$

yields

$$i = i_1 + ji_2$$

but by Eqs. (8-20) and (8-21),

$$\begin{aligned} i_1 + ji_2 &= \frac{E}{R_2 + \omega^2 L^2} \{R (\cos \omega t + j \sin \omega t) - j\omega L (\cos \omega t + j \sin \omega t)\} \\ &\qquad (\text{since } j^2 = -1) \\ &= \frac{E}{R^2 + \omega^2 L^2} (R - j\omega L)e^{j\omega t} \\ &= \frac{Ee^{j\omega t}}{R + j\omega L} \end{aligned} \tag{8-22}$$

Hence if we let $i \equiv Ie^{j\omega t}$, we have

$$I = \frac{E}{R + j\omega L} \tag{8-23}$$

(Note that although E was taken as a real number, I is a complex number.)

In practice, we use this superposition in reverse. Let $e = Ee^{j\omega t}$ and try $i = Ie^{j\omega t}$ as a solution of Eq. (8-19), with E and I both complex numbers:

$$i = Ie^{j\omega t}$$

$$\frac{di}{dt} = j\omega I e^{j\omega t}$$

and substituting into Eq. (8-19) gives

$$(j\omega L + R)Ie^{j\omega t} = Ee^{j\omega t}$$

whereupon

$$I = \frac{E}{R + j\omega L} \tag{8-23}$$

If the solution for $e = E_0 \cos \omega t$ is desired (E_0 real) we use Eq. (8-1) which says that the real and imaginary terms in an equation can be equated separately.

Real part of $i \equiv Re(i) = Re(Ie^{j\omega t})$

$$= Re\left(\frac{E_0 e^{j\omega t}}{R + j\omega L}\right) = \frac{E_0}{R^2 + \omega^2 L^2} Re((R - j\omega L)e^{j\omega t})$$

$$= \frac{E_0}{R^2 + \omega^2 L^2} \{R \cos \omega t + \omega L \sin \omega t\}$$

which is the result (8-20).

The point of all this discussion is that by letting e and i be complex, the solution (8-23) $I = E/(R + j\omega L)$ in complex numbers is obtained much more readily than the real solutions (8-20) and (8-21). The complex current (8-23) is much simpler in form than (8-20) and (8-21) but carries the information contained in *both* of these other equations. Furthermore, Eq. (8-23) is of the form $I = E/Z$, which is an extension of Ohm's law to complex impedance Z. The polar form of $Z = R + j\omega L$ is

$$Z = \sqrt{R^2 + \omega^2 L^2}\, e^{j\theta}$$

with $\theta = \tan^{-1} \omega L/R$. (Compare with Eq. (7-23) and Fig. 7.6.)

A higher order differential equation (appropriate to a more complicated circuit) benefits even more by the use of complex solutions. Let the equation be

$$a \frac{d^3x}{dt^3} + b \frac{d^2x}{dt^2} + c \frac{dx}{dt} + fx = E_0 \cos \omega t \tag{8-24}$$

The brute-force approach is to assume

$$x = A \cos \omega t + B \sin \omega t,$$

form the various derivatives, substitute into the differential equation, and to *separately* equate the coefficients of $\sin \omega t$ and $\cos \omega t$. This yields simultaneous algebraic equations for A and B.

On the other hand, $E_0 \cos \omega t = Re(E_0 e^{j\omega t})$, so that x is the real part of the solution of

$$a \frac{d^3x}{dt^3} + b \frac{d^2x}{dt^2} + c \frac{dx}{dt} + fx = E_0 e^{j\omega t} \tag{8-25}$$

Let $x = Ie^{j\omega t}$ and substitute:

$$I\{-ja\omega^3 - b\omega^2 + jc\omega + f\}e^{j\omega t} = E_0 e^{j\omega t}$$

yielding:

$$I = \frac{E_0}{-ja\omega^3 - b\omega^2 + jc\omega + f} = \frac{E_0}{(f - b\omega^2) + j(c\omega - a\omega^3)}$$
$$= E_0 \frac{(f - b\omega^2) - j(c\omega - a\omega^3)}{(f - b\omega^2)^2 + (c\omega - a\omega^3)^2}$$

The real part of x is

$$Re(Ie^{j\omega t}) = Re\{I(\cos \omega t + j \sin \omega t)\}$$
$$= \frac{E_0}{(f - b\omega^2)^2 + (c\omega - a\omega^3)^2} \{(f - b\omega^2) \cos \omega t + (c\omega - a\omega^3) \sin \omega t\} \tag{8-26}$$

8-5 Complex Current and Voltage. It is convenient to express real currents and voltages by a complex number representation. A current of magnitude $|I|$ and phase φ can be represented in several ways:

$$i = |I| \cos (\omega t + \varphi) = Re\{|I|e^{j(\omega t+\varphi)}\}$$
$$= Re\{|I|e^{j\varphi}e^{j\omega t}\} \equiv Re\{Ie^{j\omega t}\} \tag{8-27}$$

with the complex number I given by

$$I = |I|e^{j\varphi} = |I| \cos \varphi + j|I| \sin \varphi$$
$$\equiv I_r + jI_i$$

If a related voltage is given by

$$e = |E| \cos (\omega t + \varphi + \theta) = |Z| \cdot |I| \cos (\omega t + \varphi + \theta)$$
$$\equiv Re\{Ee^{j\omega t}\} \tag{8-28}$$

then

$$E = |E|e^{\theta(\varphi+\theta)} = |Z|\ |I|e^{j(\varphi+\theta)}$$
$$= |Z|e^{j\theta}|I|e^{j\varphi} \equiv ZI \tag{8-29}$$

with the complex Z defined by

$$Z = |Z|e^{j\theta} \tag{8-30}$$

Thus the *real* circuit variables relations expressed by

$$i = |I| \cos (\omega t + \varphi)$$
$$e = |E| \cos (\omega t + \varphi + \theta) \tag{8-31}$$
$$|E| = |Z| \cdot |I|$$

can be expressed as

$$e = Re(Ee^{j\omega t})$$
$$i = Re(Ie^{j\omega t}) \tag{8-32}$$

$$E = ZI \tag{8-33}$$

Note that the relations (8-32) are "standard" forms, expressing the conventional physical meaning of complex E and I, and hold for all problems and future discussions. The entire relation between the voltage and current of Eq. (8-31) is therefore expressed by the simple relation (8-33) among complex numbers *representing* voltage, current, and impedance. For simplicity, we speak of the complex numbers E, I, and Z as *being* the voltage, current, and impedance.

8-6 Impedance. The basic circuit elements R, L, C have properties governed by

$$\begin{aligned} e &= Ri \\ e &= L\,di/dt \\ i &= C\,de/dt \end{aligned} \tag{8-34}$$

For *cisoidal*[1] voltages and currents, $e = Ee^{j\omega t}$ and $i = Ie^{j\omega t}$ (E and I complex numbers), these become

$$\begin{aligned} E &= RI \\ E &= j\omega LI \\ I &= j\omega CE, \quad E = I/j\omega C \end{aligned} \tag{8-35}$$

These can all be expressed in one standard form, $E = ZI$, the generalization of Ohm's law to complex voltages and currents. The impedances of a resistor, inductor, and capacitor are respectively R, $j\omega L$, $1/j\omega C$. Because $E = ZI$ is of the same form as $E = RI$, and because the real and imaginary parts of currents and voltages add separately Eq. (8-2), the arguments of Chapter I leading to formulas for series-parallel combination of R are applicable to Z; i.e., series-parallel combinations of Z follow the same rules as for R.

In general, all of the linear E vs. I network relations derived in earlier chapters are still valid, using complex impedance instead of resistance. In particular, the various formulas of Chapter IV for alternative representations of a two-port are valid. The *only* equations which are affected by our generalization from the dc resistance-networks to ac impedance-networks are those relating to *power*, since the product EI appears. Proper formulas for power are given in this chapter.

The general form of Z is taken as

$$Z = R + jX \tag{8-36}$$

where $Re(Z) = R$ is the resistance and $Im(Z) = X$ is the *reactance*.

[1] Mathematicians use i rather than j for $\sqrt{-1}$. The oscillation $e^{i\omega t} = \cos \omega t + i \sin \omega t$ is called a *cisoid*, corresponding to *sinusoid*.

The reciprocal of an impedance is the *admittance*:

$$Y = \frac{1}{Z} = \frac{R - jX}{R^2 + X^2} \equiv G + jS \tag{8-37}$$

$$G = \frac{R}{R^2 + X^2}; \quad S = -\frac{X}{R^2 + X^2}$$

The real part of admittance (G) is *conductance* (corresponding to dc conductance $1/R$) and the imaginary part (S) is called *susceptance.* Note that *positive* reactance implies *negative* susceptance.

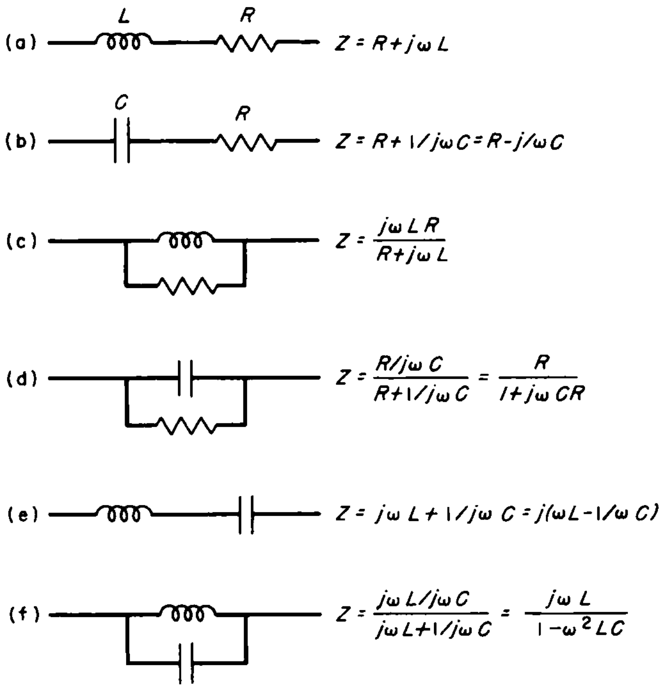


FIG. 8.5.

8-7 Series-Parallel Combinations. From Eq. (8-35) we readily compute the impedances of the simple combinations of Fig. 8.5. Note that for Fig. 8.5e, $\omega^2 LC = 1$ gives $Z = 0$, and for Fig. 8.5f, the same condition gives $Z = \infty$.

8-8 Addition of Currents or Voltages. If an alternating current of magnitude I passes through a series RL combination, the *magnitude* of the total voltage drop is $I\sqrt{R^2 + \omega^2 L^2}$. The *magnitude* of the drop across the resistance is RI, that across the inductance is ωLI. Since these voltages are not in phase, the total drop has the magnitude $\sqrt{R^2 + \omega^2 L^2}\, I$ rather than $(R + \omega L)I$. Graphically, the two voltage drops are added at right angles, or in *quadrature* (Fig. 8.6), and the net voltage amplitude is *less* than the sum of its parts.

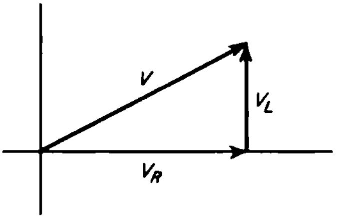


FIG. 8.6.

For the series LC combination of Fig. 8.5e, the individual voltage drops are 180° out of phase, and their phasor sum (the net voltage drop) is the *difference* of their individual magnitudes. Thus for one ampere at 150 cps through one microfarad and one henry, we have:

$$\omega = 2\pi(150) = 942$$
$$V_L = j\omega L = 942j \text{ volts}$$
$$V_C = 1/j\omega C = -1062j \text{ volts}$$
$$V_L + V_C = -120j \text{ volts}$$

Thus the total voltage drop is only 120 volts, although the inductance and capacitance have individual drops of 942 and 1062 volts!

The series RC combination has the impedance $Z = R - j/\omega C$. For the current I, the voltage across the capacitance is $-jI/\omega C$, while that across the combination is $I(R - j/\omega C)$. The ratio between these two is

$$\frac{-j/\omega C}{R - j/\omega C} = \frac{1}{1 + jR\omega C} = \frac{1 - jR\omega C}{1 + (R\omega C)^2} = \frac{e^{-j\theta}}{\sqrt{1 + (R\omega C)^2}}$$

where $\theta = \tan^{-1} R\omega C$ is the phase difference between the two voltages. This is the basic relation for RC phase-shifters

The magnitude of the impedance of the series RC combination is $\sqrt{R^2 + 1/\omega^2 C^2}$. If this is shunted across the output of an audio amplifier stage, the reduction of impedance with increase of frequency reduces the high frequency response of the amplifier. Increasing R reduces this loading at high frequencies where $R \gg 1/\omega C$, but has little effect at low frequencies where $R \ll 1/\omega C$. Variation of R in such a circuit is often used for *tone control* in radio receivers.

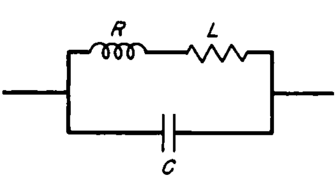


FIG. 8.7.

8-9 Parallel Resonance. Consider the tuned circuit of Fig. 8.7. The entire resistance is associated with the inductance because this is a good approximation in practice. (Practical capacitors are very "low-loss" compared with inductors.) The impedance of the RL branch is $R + j\omega L$; this in parallel with $1/j\omega C$ yields:

$$Z = \frac{(R + j\omega L)/j\omega C}{R + j\omega L + 1/j\omega C} = \frac{R + j\omega L}{1 - \omega^2 LC + j\omega CR} \tag{8-38}$$

We are interested in (1) *amplitude resonance* (minimum current or maximum magnitude of impedance) and (2) *phase resonance,* which occurs at the frequency that makes Z real, so that the net current is in phase with the applied voltage (the resonant circuit presents pure resistance to the source).

To find the magnitude of Z, we *could* multiply both numerator and denominator by the complex conjugate of the denominator, to get j out of the denominator. Thus

$$\frac{a + jb}{c + jd} = \frac{(ac - bd) + j(bc + ad)}{c^2 + d^2}$$

and then find the absolute magnitude by $|e + jf| = \sqrt{e^2 + f^2}$. This procedure, however, involves a large amount of unnecessary and confusing algebra. We look to polar representation for our clue. We have

$$\frac{a + jb}{c + jd} = \frac{\sqrt{a^2 + b^2}\, e^{j\theta_1}}{\sqrt{c^2 + d^2}\, e^{j\theta_2}}$$

$$= \sqrt{\frac{a^2 + b^2}{c^2 + d^2}}\, e^{j(\theta_1 - \theta_2)} \tag{8-39}$$

so that

$$\left|\frac{a + jb}{c + jd}\right| = \frac{|a + jb|}{|c + jd|} \tag{8-40}$$

The square of the absolute magnitude of Z is therefore

$$|Z|^2 = \frac{R^2 + \omega^2 L^2}{(1 - \omega^2 LC)^2 + (\omega CR)^2} \tag{8-41}$$

and

$$Z = \sqrt{\frac{R^2 + \omega^2 L^2}{(1 - \omega^2 LC)^2 + (\omega CR)^2}}\, e^{j\theta} \tag{8-42}$$

with $\theta = \tan^{-1} \omega L/R - \tan^{-1} \omega CR/(1 - \omega^2 LC)$. The phasor representation of Z varies with ω, both in magnitude and phase angle. Since

$$\tan(\theta_1 - \theta_2) \equiv \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

we have

$$\tan\theta = \frac{\dfrac{\omega L}{R} - \dfrac{\omega RC}{1 - \omega^2 LC}}{1 + \dfrac{\omega L}{R}\dfrac{\omega RC}{1 - \omega^2 LC}} \tag{8-43}$$

$$= \frac{\omega L}{R}(1 - \omega^2 LC) - \omega CR$$

Amplitude resonance is the condition of maximum $|Z|$ or $|Z|^2$. Thus from Eq. (8-41) the frequency of amplitude resonance is given by

$$0 = \frac{d|Z|^2}{d\omega} =$$

$$= \frac{2\omega L^2\{(1 - \omega^2 LC)^2 + (\omega CR)^2\} - (R^2 + \omega^2 L^2)\{2(1 - \omega^2 LC)(-2\omega LC) + 2\omega C^2 R^2\}}{\{(1 - \omega^2 LC)^2 + (\omega CR)^2\}^2} \tag{8-44}$$

A factor 2ω can be taken out of the numerator; the remaining factor is, after multiplying and collecting terms:

$$-\omega^4 L^4 C^2 - 2\omega^2 L^2 C^2 R^2 + L^2 + 2LCR^2 - C^2 R^4$$

This vanishes for

$$\omega^2 = -\frac{R^2}{L^2} \pm \frac{1}{LC}\sqrt{1 + 2R^2C/L} \tag{8-45}$$

Since ω^2 must be positive, only the (+) sign in Eq. (8-45) has meaning:

$$\omega^2 = \frac{1}{LC}\{\sqrt{1 + 2R^2C/L} - R^2C/L\} \tag{8-46}$$

The corresponding maximum amplitude value of Z is found by substituting ω^2 from Eq. (8-46) into Eqs. (8-41) and (8-43). This gives

$$|Z|^2 = \frac{L/C}{2\{\sqrt{1 + 2R^2C/L} - 1\} - R^2C/L} \tag{8-47}$$

$$\tan\theta = \left(\frac{R}{\omega L} + \frac{1}{\omega CR}\right)(\sqrt{1 + 2R^2C/L} - 1) - \frac{2R}{\omega L} \tag{8-48}$$

The radical can be expanded by the binomial theorem, so that for $R^2 \ll L/C$, the amplitude-resonance frequency is

$$\omega_a^2 \doteq \frac{1}{LC}\left\{1 - \tfrac{1}{2}\left(\frac{R^2C}{L}\right)^2 + \cdots\right\} \tag{8-49}$$

which is a little less than $1/LC$.

The same expansion applied to Eqs. (8-47) and (8-48) yields

$$|Z|^2 = \frac{L^2}{R^2C^2\left(1 - \frac{R^2C}{L} + \cdots\right)}$$

$$|Z| = \frac{L}{RC}\left(1 + \tfrac{1}{2}\frac{R^2C}{L} + \cdots\right) \tag{8-50}$$

$$\doteq \frac{L}{RC} + \frac{R}{2}$$

$$\tan\theta = -\frac{R}{\omega L}\left(1 - \tfrac{1}{2}\frac{R^2C}{L} + \cdots\right) \tag{8-51}$$

Thus $|Z| \doteq L/RC$, and θ is slightly negative (i.e., the impedance is capacitive at amplitude resonance).

For phase resonance, the imaginary part of Z must vanish. Again, we can operate on Eq. (8-38) to remove j from the denominator, or we can note from Eq. (8-39) that $\theta, -\theta_2 = 0$ for $b/a = d/c$. Thus Z is real for (cf. Eq. 8-43)

$$\frac{\omega L}{R} = \frac{\omega CR}{1 - \omega^2 LC} \tag{8-52}$$

requiring

$$\omega_p^2 = \frac{1}{LC}(1 - R^2C/L) \tag{8-53}$$

$$= \frac{1}{LC} - \frac{R^2}{L^2}$$

Since R^2C/L is small, its square is even smaller; the frequency for amplitude resonance is higher than for phase resonance (cf. Eq. 8-49):

$$\omega_p^2 < \omega_a^2 < 1/LC \tag{8-54}$$

The resistance presented by the tuned circuit at phase resonance is readily found to be $Z_{\text{real}} = L/CR$, which increases without bound as $R \to 0$.

The magnitude and phase of Z are plotted in (Fig. 8.8) as functions of ω^2.

8-10 Power. If

$$e = |E| \cos \omega t$$

$$i = |I| \cos (\omega t + \theta) \tag{8-55}$$

the *instantaneous* power is

$$\begin{aligned} ei &= |E|\cdot|I| \cos \omega t \cos (\omega t + \theta) \\ &= |E|\cdot|I| \{\cos^2 \omega t \cos \theta - \cos \omega t \sin \omega t \sin \theta\} \end{aligned} \tag{8-56}$$

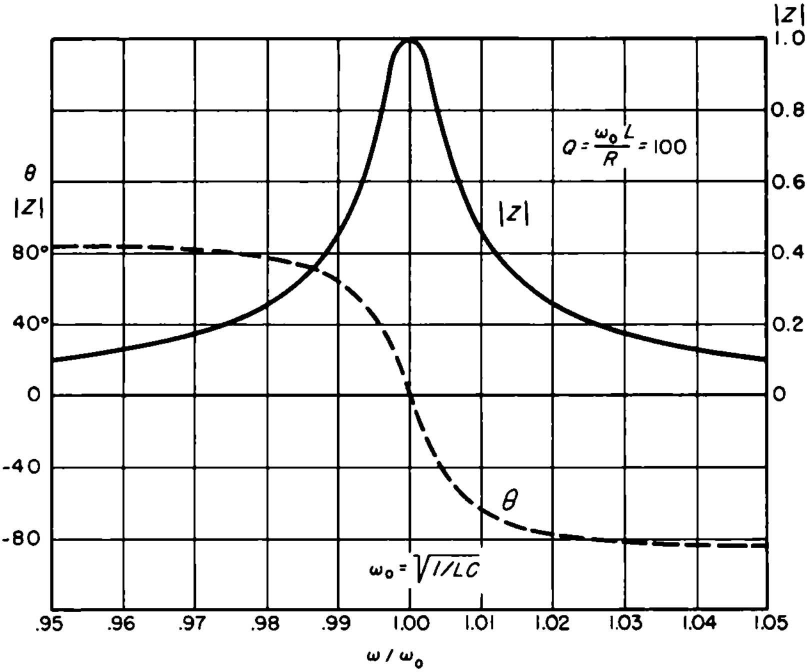


FIG. 8.8.

The *average* power is

$$P = \langle ei \rangle_{av} = \tfrac{1}{2}|E|\,|I|\cos\theta$$

The term $\cos\theta$ is called the *power factor*, since it is the factor needed to compute power when e and i are not in phase.

In an impedance $Z = R + jX$, the current and voltage drop are related by

$$V = (R + jX)I = \sqrt{R^2 + X^2}\, Ie^{j\theta} \tag{8-57}$$

with $\theta = \tan^{-1} X/R$ (see Fig. 8.9). The voltage *leads* the current by θ. The power factor of Z is therefore

$$\text{p.f.} = \cos\theta \equiv \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{1}{\sqrt{1 + X^2/R^2}} \tag{8-58}$$

$$= \frac{R}{\sqrt{R^2 + X^2}} = \frac{R}{|Z|}$$

A related quantity is Q, the quality factor of a capacitor or inductor,

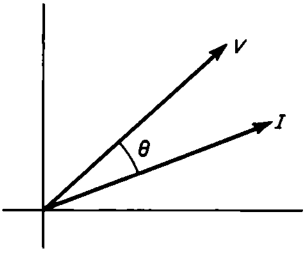


FIG. 8.9.

which is given by $X/R = \begin{cases}\omega L/R \\ 1/R\omega C\end{cases}$ For high Q, $X \gg R$, and

$$\text{p.f.} = \frac{R}{\sqrt{R^2 + X^2}} \doteq \frac{R}{X} = \frac{1}{Q}$$

Hence the power dissipated by an inductor is approximately $\frac{1}{2}VI/Q$, where V and I are the peak voltage and peak current respectively.

Care must be used in computing power when V and I are in complex form, since they then represent superposed $\sin \omega t$ and $\cos \omega t$ situations.
For

$$\begin{aligned} v &= Ve^{j\omega t} = (V_r + jV_i)(\cos \omega t + j \sin \omega t) \\ &= (V_r \cos \omega t - V_i \sin \omega t) + j(V_i \cos \omega t + V_r \sin \omega t) \qquad (8\text{-}59) \\ &= \sqrt{V_r^2 + V_i^2}\,\{\cos(\omega t + \theta) + j \sin(\omega t + \theta)\} \end{aligned}$$

where

$$\tan \theta = V_i/V_r$$

simultaneously represents the voltage

$$|V| \cos(\omega t + \theta) \text{ and the voltage } |V| \sin(\omega t + \theta).$$

The respective currents are

$$i = |I| \cos(\omega t + \varphi) \text{ and } i = |I| \sin(\omega t + \varphi)$$

where

$$\varphi = \tan^{-1} I_i/I_r.$$

Whichever component (real or imaginary) we consider to be the actual voltage and current implied by the complex expression, we have

$$\begin{aligned} P &= \tfrac{1}{2}|V|\,|I| \cos(\varphi - \theta) = \tfrac{1}{2}|V|\,|I|\,(\cos \varphi \cos \theta + \sin \varphi \sin \theta) \\ &= \tfrac{1}{2}\{|V| \cos \theta\,|I| \cos \varphi + |V| \sin \theta\,|I| \sin \varphi\} \\ &= \tfrac{1}{2}(V_r I_r + V_i I_i) \qquad (8\text{-}60) \end{aligned}$$

For the impedance $R + jX$, we have

$$\begin{aligned} V &= (R + jX)I = (R + jX)(I_r + jI_i) \\ &= (RI_r - XI_i) + j(XI_r + RI_i) \\ &\equiv V_r + jV_i \end{aligned}$$

The power is

$$\begin{aligned} P &= \tfrac{1}{2}(V_r I_r + V_i I_i) \\ &= \tfrac{1}{2}(I_r^2 R - I_r X I_i + I_i X I_r + R I_i^2) \\ &= \tfrac{1}{2}R(I_r^2 + I_i^2) \\ &= \tfrac{1}{2}R|I|^2 \end{aligned} \tag{8-61}$$

Note, however, that

$$\tfrac{1}{2}|V|^2/R = \tfrac{1}{2}(R^2 + X^2)|I|^2/R \neq P$$

but that

$$\tfrac{1}{2}G|V|^2 = \tfrac{1}{2}\frac{R}{R^2 + X^2}|V|^2 = \tfrac{1}{2}R|I|^2 = P \tag{8-62}$$

Chapter IX

ALTERNATING CURRENT NETWORKS

In Chapter II, we developed the theory of resistance networks for direct currents and voltages, and learned to write circuit equations by inspection. The use of complex E, I, and Z developed in Chapter VIII allows us to use our previous network equations without modification, as long as no mutual inductance is involved.

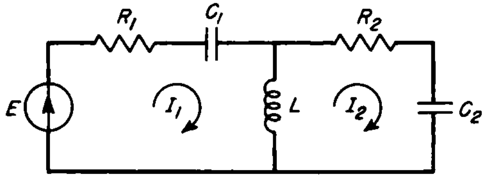


FIG. 9.1.

Consider the circuit of Fig. 9.1. The circuit equations are written by inspection, using $j\omega L$ for the impedance of the inductance L, and $1/j\omega C$ for the impedance of the capacitance C.

$$\begin{aligned} E &= R_1I_1 + I_1/j\omega C_1 + (I_1 - I_2)j\omega L \\ 0 &= -I_1j\omega L + I_2(j\omega L + R_2 + 1/j\omega C_2) \end{aligned} \tag{9-1}$$

Note that these complex circuit equations represent an algebraic procedure for solving the following simultaneous differential equations of the network (cf. Eq. (8-19) ff.):

$$\begin{aligned} e &= R_1i_1 + q_1/C_1 + L(di_1/dt - di_2/dt) \\ 0 &= L(di_2/dt - di_1/dt) + R_2i_2 + q_2/C_2 \end{aligned} \tag{9-2}$$

where $q = \int i dt$. For $i_1 = I_1e^{j\omega t}$, we have

$$\frac{di_1}{dt} = j\omega I_1e^{j\omega t}$$

$$q_1 = I_1e^{j\omega t}/j\omega, \text{ etc.}$$

Setting $e = Ee^{j\omega t}$, and cancelling the factor $e^{j\omega t}$ on both sides, Eq. (9-2) go over into Eq. (9-1).

Equations (9-1) can be solved algebraically, but carrying through the algebra for the *particular* case of Fig. 9.1 is not instructive. It is better to solve this network configuration *in general* (Fig. 9.2).

$$\begin{aligned} E &= I_1(Z_1 + Z_3) - I_2Z_3 \\ 0 &= -I_1Z_3 + I_2(Z_2 + Z_3) \end{aligned} \tag{9-3}$$

These are readily solved, yielding:

$$\begin{aligned} I_2 &= \frac{Z_3}{(Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2} E \\ I_1 &= \frac{Z_2 + Z_3}{(Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2} E \end{aligned} \tag{9-4}$$

Note that Z_3^2 is complex; it is $Z_3 \cdot Z_3$ and not $|Z_3|^2$. Note also that the de-

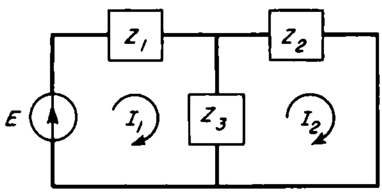


FIG. 9.2.

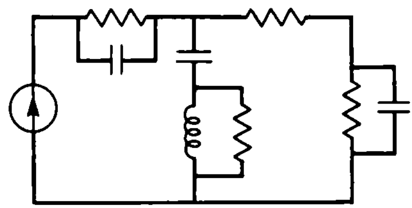


FIG. 9.3.

nominator can be written as

$$Z_1Z_2 + Z_1Z_3 + Z_2Z_3$$

but we have left it in the form of Eq. (9-4) intentionally.

The impedance $Z_1 + Z_3$ is the total self-impedance of the first loop (i.e., with the second loop opened).

Let the self-impedance of the first loop be z_1; that of the second loop, z_2; and the impedance that is common to both loops (the coupling impedance), z_c. The coupling impedance is sometimes called the *mutual* impedance; this can cause confusion when it is an inductance, but there is no "mutual inductance" in the network, as in Fig. 9.1. In terms of these new variables, we readily see that Eq. (9-4) becomes

$$\begin{aligned} I_2 &= \frac{z_c}{z_1z_2 - z_c^2} E \\ I_1 &= \frac{z_2}{z_1z_2 - z_c^2} E \end{aligned} \tag{9-5}$$

Problem.

Find I_2 for Fig. 9.1, using Eq. (9-5) or (9-4) and show that I_2 is in phase with E for

$$\frac{1}{\omega^2} = R_1R_2C_1C_2 + L(C_1 + C_2)$$

The use of Eq. (9-5) allows the straightforward analysis of Fig. 9.3 on a *two-loop basis* by utilizing the formula for impedances in parallel (the analysis is straightforward, but the algebra is complicated).

9-1 Mutual Inductance. We saw in Chapter VI that two coils having self-inductances L_1, L_2, and mutual inductance M have voltage drops and currents related by:

$$\begin{aligned} v_1 &= L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \\ v_2 &= M\frac{di_1}{dt} + L_2\frac{di_2}{dt} \end{aligned} \qquad (9\text{-}6)$$

In terms of complex voltages and currents, these relations are:

$$\begin{aligned} V_1 &= j\omega L_1 I_1 + j\omega M I_2 \\ V_2 &= j\omega M I_1 + j\omega L_2 I_2 \end{aligned} \qquad (9\text{-}7)$$

Interchanging the connections to *one* coil changes the sign of M. We are immediately faced with the problem of determining the sign of M in any specific case.

In Chapter II, we established our conventions for the signs of voltage and current by assigning an arbitrary forward-direction arrow to each branch of a network. We now assign such arrows to all coils in a network. Figure 9.4 shows such a set of coils and assigned positive directions, without regard to the remainder of the network. Consider a current in coil 1, with a steady rate of increase in the direction of the arrow, $di_1/dt = +a$. The voltage *induced* in coil 1 by its self-inductance is a back-voltage, by Lenz's law. The self-induced voltage is therefore negative (using the arrow for positive direction). Recall that in our circuit equations v refers to a voltage *drop*, the opposite of a voltage *rise*. Hence the voltage *drop* in coil 1 due to its self-inductance is

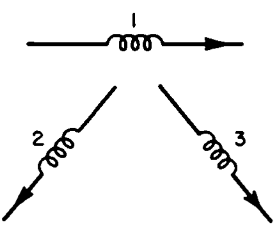


Fig. 9.4.

$$v_1 = L_1\, di_1/dt = aL_1$$

The sign of the voltage (rise) induced in either of the other coils, say coil 3, can be readily determined by experiment. Let us assume that the induced voltage is positive, i.e., the arrowhead end of coil 3 becomes positive, the other end negative. Since the voltage induced in coil 3, as measured by a voltmeter, is a *rise*, the resulting voltage *drop* in coil 3 has the opposite sign, or is negative. Then

$$v_3 = -aM$$

or

$$v_3 = -M\, di_1/dt \quad \text{in general,}$$

and the mutual inductance M_{13} between coils 1 and 3 is negative $(-M)$.

If *either* (but not both) of the arbitrary reference arrows on coils 1 and 3 had been chosen in the opposite sense, the mutual inductance M_{13} would have been $+M$. *The sign of the mutual inductance between coils depends upon the arbitrarily assigned reference directions.* This is not a paradox, for reversing the arrow on coil 1 would change the sign of a fixed current and also the sign of M, leaving the sign of v_3 unchanged. The actual voltage across coil 3 therefore depends on the actual current in coil 1, and is independent of any choice of reference arrows.

Simultaneous consideration of all the currents and voltage drops in the coils of Fig. 9.4 yields the set of equations:

$$\begin{aligned} v_1 &= L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} + M_{13} \frac{di_3}{dt} \\ v_2 &= M_{21} \frac{di_1}{dt} + L_2 \frac{di_2}{dt} + M_{23} \frac{di_3}{dt} \\ v_3 &= M_{31} \frac{di_1}{dt} + M_{32} \frac{di_2}{dt} + L_3 \frac{di_3}{dt} \end{aligned} \tag{9-8}$$

Note that we have not assumed $M_{12} = M_{21}$, i.e., that the mutual inductance between any two coils is symmetrical.

The circuit equilibrium equations follow as usual by equating the voltage drops (v) to the applied voltage rises (e) of any sources connected across the coils.

The total power being supplied to the coils by the sources is

$$\begin{aligned} \frac{dT}{dt} = P &= e_1 i_1 + e_2 i_2 + e_3 i_3 \\ &= v_1 i_1 + v_2 i_2 + v_3 i_3 \end{aligned} \tag{9-9}$$

where T is the energy stored in the magnetic field of the coils. We have from Eq. (9-8):

$$\begin{aligned}\frac{dT}{dt} &= L_1 i_1 \frac{di_1}{dt} + \left(M_{12} i_1 \frac{di_2}{dt} + M_{21} i_2 \frac{di_1}{dt}\right) \\ &+ \left(M_{13} i_1 \frac{di_3}{dt} + M_{31} i_3 \frac{di_1}{dt}\right) \\ &+ L_2 i_2 \frac{di_2}{dt} + \left(M_{23} i_2 \frac{di_3}{dt} + M_{32} i_3 \frac{di_2}{dt}\right) \\ &+ L_3 i_3 \frac{di_3}{dt}\end{aligned} \tag{9-10}$$

If we start with a state of zero magnetic energy ($i_1 = i_2 = i_3 = 0$) and increase the currents in any arbitrary fashion, finally establishing a steady condition at values I_1, I_2, I_3, the energy stored in the magnetic field depends only upon I_1, I_2, I_3 and not upon the manner in which these final values are reached. This is because the steady-state magnetic field depends only upon the various steady currents. This allows us to compute T by choosing a *convenient* manner of increasing the currents, namely, by increasing all three proportionately. We take $i_1 = xI_1$, $i_2 = xI_2$, $i_3 = xI_3$, where x is a fraction that increases from 0 to 1 in an arbitrary fashion. Equation (9-10) becomes

$$\begin{aligned}\frac{dT}{dt} &= \{ L_1 I_1^2 + (M_{12} + M_{21}) I_1 I_2 \\ &+ (M_{13} + M_{31}) I_1 I_3 + L_2 I_2^2 \\ &+ (M_{23} + M_{32}) I_2 I_3 + L_3 I_3^2 \} \, x \frac{dx}{dt}\end{aligned} \tag{9-11}$$

with x the only varying quantity on the right. The equation is now readily integrated, and since $\int_0^1 x dx = \frac{1}{2}$, we have

$$\begin{aligned}T &= \tfrac{1}{2} \{ L_1 I_1^2 + L_2 I_2^2 + L_3 I_3^2 \\ &+ (M_{12} + M_{21}) I_1 I_2 \\ &+ (M_{13} + M_{31}) I_1 I_3 \\ &+ (M_{23} + M_{32}) I_2 I_3 \}\end{aligned} \tag{9-12}$$

as the energy stored in the magnetic field. If we now change I_1, keeping I_2 and I_3 fixed, the change of T is given by

$$\begin{aligned}dT = \tfrac{1}{2} \{ 2L_1 I_1 &+ (M_{12} + M_{21}) I_2 \\ &+ (M_{13} + M_{31}) I_3 \} \, dI_1\end{aligned}$$

or the power supplied at any instant is

$$\frac{dT}{dt} = \tfrac{1}{2}\{2L_1I_1 + (M_{12} + M_{21})I_2 + (M_{13} + M_{31})I_3\}\frac{dI_1}{dt} \tag{9-13}$$

But this must agree with Eq. (9.10) with I_1 the only varying current:

$$\frac{dT}{dt} = (L_1I_1 + M_{21}I_2 + M_{31}I_3)\frac{dI_1}{dt} \tag{9-14}$$

For Eqs. (9-13) and (9-14) to agree, we must have $M_{12} = M_{21}$, $M_{13} = M_{31}$. It is readily seen that the same argument works for any pair of coils in any set of coils, so that in general we have $M_{ij} = M_{ji}$.

9-2 Reciprocal Inductances. Equations (9-8) determine the voltages in terms of the currents. If these are integrated with respect to time, we have

$$\begin{aligned} \int v_1dt &= L_1i_1 + M_{12}i_2 + M_{13}i_3 = \Phi_1 \\ \int v_2dt &= M_{21}i_1 + L_2i_2 + M_{23}i_3 = \Phi_2 \\ \int v_3dt &= M_{31}i_1 + M_{32}i_2 + L_3i_3 = \Phi_3 \end{aligned} \tag{9-15}$$

recalling that $\Phi_1 = \int v_1dt$ is the flux linkage of the first coil. These simultaneous equations can be solved for the i's in terms of the Φ's if the determinant of the coefficients does not vanish. This condition is always satisfied, for if the determinant should vanish, we could find a set of currents for which Eq. (9-12) would yield a *negative* value of T, the energy stored in the magnetic field. This is physically impossible. (Proof of this result is beyond the scope of this book, but the theorem is well-known to mathematicians who study *quadratic forms*, such as Eq. (9-12).)

Solving Eq. (9-15)—by any of the methods of Chapter III—would yield a result of the form

$$\begin{aligned} i_1 &= c_{11}\Phi_1 + c_{12}\Phi_2 + c_{13}\Phi_3 \\ i_2 &= c_{21}\Phi_1 + c_{22}\Phi_2 + c_{23}\Phi_3 \\ i_3 &= c_{31}\Phi_1 + c_{32}\Phi_2 + c_{33}\Phi_3 \end{aligned} \tag{9-16}$$

where $c_{12} = c_{21}$ etc. The coefficients c_{ij} are sometimes called *reciprocal inductances*. They are discussed here simply to show that equations of the form (9-16) exist, and can be found if desired (for nodal analysis of networks containing mutual inductance).

9-3 Network Equations. Using impedance notation, we can rewrite Eq. (9-8) as

$$\begin{aligned} V_1 &= j\omega L_1 I_1 + j\omega M_{12} I_2 + j\omega M_{13} I_3 \\ V_2 &= j\omega M_{21} I_1 + j\omega L_2 I_2 + j\omega M_{23} I_3 \\ V_3 &= j\omega M_{31} I_1 + j\omega M_{32} I_2 + j\omega L_3 I_3 \end{aligned} \tag{9-17}$$

If the voltage drops in the loops of a network are written as complex numbers, using the impedance concepts of Chapter VIII, and the additional voltage drops contributed by Eq. (9-17) included, the steady-state behavior of any linear network can be handled just as it was in the dc case of a resistance network. Consider the network of Fig. 9.5.

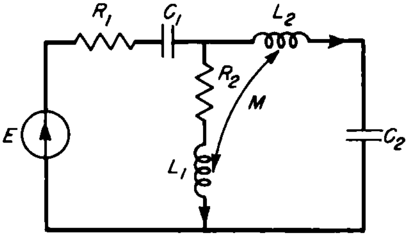


FIG. 9.5.

If there is *no* mutual inductance between the two coils, we have

$$\begin{aligned} E &= I_1(R_1 + 1/j\omega C_1 + R_2 + j\omega L_1) - I_2(R_2 + j\omega L_1) \\ 0 &= I_1(-R_2 - j\omega L_1) + I_2(j\omega L_2 + 1/j\omega C_2 + j\omega L_1 + R_2) \end{aligned} \tag{9-18}$$

Note that the reference arrows on the coils are irrelevant. They play a role only for the mutual inductance.

With mutual inductance, we have additional voltage drops. In the first mesh, the additional drop is $+j\omega M_{12}I_2$, due to I_2 in coil 2. The reference directions of I_2 and coil 2 agree, making the induced drop in coil 1 have the direction of the coil 1 arrow (for M_{12} positive). Since the coil 1 arrow "agrees" with the I_1 direction, the added voltage drop is positive. For the second mesh, we have two coupling contributions: (1) the current I_2 in coil 2 induces the drop in coil 1 just discussed (*opposing* the reference arrow I_2 of the second mesh), and (2) a branch current $(I_2 - I_1)$ in coil 1 inducing a drop in coil 2. Since $(I_2 - I_1)$ opposes the arrow on coil 1, the drop induced in coil 2 in the direction of the coil 2 arrow is $-j\omega M_{21}(I_2 - I_1)$.

Adding the above terms to Eq. (9-18) yields

$$\begin{aligned} E &= I_1(R_1 + 1/j\omega C_1 + R_2 + j\omega L_1) - I_2(R_2 + j\omega L_1) + I_2(j\omega M_{12}) \\ 0 &= I_1(-R_2 - j\omega L_1) + I_2(j\omega L_2 + 1/j\omega C_2 + j\omega L_1 + R_2) - I_2(j\omega M_{12}) \\ &\qquad - (I_2 - I_1)j\omega M_{21} \end{aligned} \tag{9-19}$$

Collecting terms, these equations can be written

$$E = I_1(R_1 + R_2 + j\omega L_1 + 1/j\omega C_1) + I_2(-R_2 - j\omega L_1 + j\omega M_{12})$$

$$0 = I_1(-R_2 - j\omega L_1 + j\omega M_{21}) + I_2(j\omega L_2 + 1/j\omega C_2 + j\omega L_1 + R_2 - j\omega M_{12} - j\omega M_{21}) \quad (9\text{-}20)$$

Although $M_{12} = M_{21}$, the notation has been kept distinct in the above to show the physical origin (direction of coupling) of the mutual inductance terms, since M_{12} refers to a voltage in coil 1 produced by a current in coil 2, and vice versa. With practice, the equations in the form (9-20) can be written directly from inspection of the network by keeping tabs on the induced voltages in all coils produced separately by each mesh current through each coil.

Example.

Show that the equation relevant to Fig. 9.6 is

$$E = I(j\omega L_1 + j\omega L_2 - 2j\omega M)$$

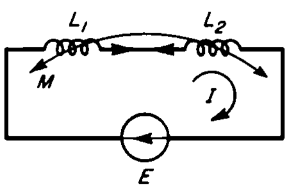


FIG. 9.6.

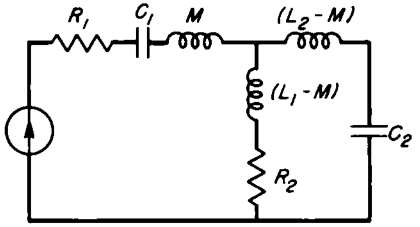


FIG. 9.7.

and that if the connections to *either* coil are reversed,

$$E = I(j\omega L_1 + j\omega L_2 + 2j\omega M)$$

Recalling from Chapter VI that $M = k\sqrt{L_1L_2}$, note that

$$E = j\omega I(L_1 \pm 2k\sqrt{L_1L_2} + L_2)$$

in these two cases. The two coils together present an inductance

$$L_1 \pm 2k\sqrt{L_1L_2} + L_2$$

which can be either greater or less than $(L_1 + L_2)$ depending upon the sign of the coupling. The least value of the total inductance is zero; resulting from $L_1 = L_2$ and $k = 1$ (unity coupling).

Referring back to Fig. 9.2 and Eq. (9-3), we see that Eq. (9-3) can be identified with Eq. (9-20) by setting

$$Z_3 = R_2 + j\omega L_1 - j\omega M$$

$$Z_1 + Z_3 = R_1 + 1/j\omega C_1 + R_2 + j\omega L_1$$

so that

$$Z_1 = R_1 + 1/j\omega C_1 + j\omega M$$

and similarly

$$Z_2 = 1/j\omega C_2 + j\omega L_2 - j\omega M$$

Hence the circuit of Fig. 9.5 is equivalent to that of Fig. 9.7, where there are *no* mutual inductances. All networks containing mutual inductance can be reduced in this fashion to circuits not containing mutual inductance. The self-inductances of the new circuit are not necessarily positive, hence the equivalent circuit may not be physically realizable. In the case of Fig. 9.7, with $M > 0$, one of the other coils is a negative inductance if $M > L_1$ or $M > L_2$. Since $M = k\sqrt{L_1L_2}$, $0 \leq k \leq 1$, a negative coil will appear if $k > \sqrt{L_1/L_2}$ or $k > \sqrt{L_2/L_1}$. Since $k \leq 1$, only one of these conditions can hold, and if k^2 is greater than the smaller of the ratios L_1/L_2 and L_2/L_1, there will be one negative coil. The equivalent circuit is still valid as a *representation*, however, since the currents I_1 and I_2 deduced therefrom are correct.

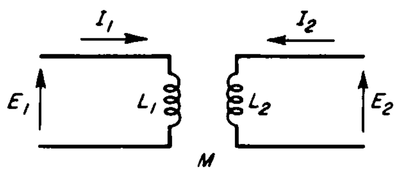


FIG. 9.8.

9-4 Transformer Representation. Consider the coupled coils of Fig. 9.8. We shall treat them as a two-port network as in Chapter IV (the E's are voltage *rises*):

$$E_1 = AE_2 - BI_2$$
$$I_1 = CE_2 - DI_2 \qquad (9\text{-}21)$$
$$AD - BC = 1$$

For $I_2 = 0$, we have

$$E_1 = j\omega L_1 I_1$$

$$E_2 = j\omega M I_1 = \frac{M}{L_1} E_1$$

giving

$$A = L_1/M, \quad C = 1/j\omega M$$

For $I_1 = 0$ we find

$$I_2 = E_2/j\omega L_2 = \frac{C}{D} E_2$$

so that

$$D = j\omega L_2 C = L_2/M$$

Finally,

$$B = \frac{AD - 1}{C} = \frac{\frac{L_1L_2}{M^2} - 1}{1/j\omega M} = \frac{-\omega^2(L_1L_2 - M^2)}{j\omega M}$$

Substituting these results into Eq. (9-21) yields

$$\begin{aligned} E &= \frac{L_1}{M} E_2 + \frac{\omega^2(L_1L_2 - M^2)}{j\omega M} I_2 \\ I_1 &= \frac{1}{j\omega M} E_2 - \frac{L_2}{M} I_2 \end{aligned} \qquad (9\text{-}22)$$

For *unity coupling,* $M^2 = L_1L_2$, making $B = 0$ and

$$E_1 = \frac{L_1}{M} E_2 = \sqrt{\frac{L_1}{L_2}} E_2 \equiv E_2/n$$

independent of any load conditions. From Chapter VI, we recognize $\sqrt{L_2/L_1}$ as N_2/N_1, the turns ratio of the transformer. Furthermore if L_1 and L_2 are very large, the second equation of (9-22) reduces to

$$I_1 = -\frac{L_2}{M} I_2 = -\sqrt{\frac{L_2}{L_1}} I_2 = -\frac{N_2}{N_1} I_2 \equiv -nI_2$$

Both these conditions are fairly well approximated in well-designed iron-core transformers.

9-5 Ideal Transformers. A transformer which satisfies the relations

$$\begin{aligned} E_2 &= nE_1 \\ I_1 &= -nI_2 \end{aligned} \qquad (9\text{-}23)$$

is called an *ideal transformer.*

For a unity-coupled transformer in which the inductances are not large enough to make the above approximation of an ideal transformer, we have the equations, from Eq. (9-22),

$$E_1 = \frac{1}{n} E_2$$

$$I_1 = \frac{1}{j\omega M} E_2 - nI_2$$

where we have put

$$n = \sqrt{L_2/L_1} = L_2/M = M/L_1$$

The value of E_2 from the first equation can be substituted into the second, yielding

$$I_1 = \frac{n}{j\omega M} E_1 - nI_2 = \frac{E_1}{j\omega L_1} - nI_2$$

FIG. 9.9.

The equations can now be interpreted as describing the combination of inductance and an ideal transformer shown in Fig. (9.9).

The impedance transforming properties of an ideal transformer are simple and interesting. Let us consider the circuit of Fig. 9.10, and de-

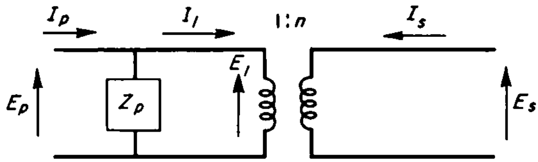


FIG. 9.10.

scribe it by equations of the form

$$\begin{aligned} E_p &= AE_s - BI_s \\ I_p &= CE_s - DI_s \end{aligned} \qquad (9\text{-}24)$$

We reserve E_1, E_2, I_1, and I_2 for use at the terminals of the ideal transformer.

Since

$$E_1 = E_p \quad \text{and} \quad I_2 = I_s,$$

we have

$$\begin{aligned} E_p &= E_1 = \frac{1}{n} E_2 = \frac{1}{n} E_s \\ I_p &= \frac{E_p}{Z_p} + I_1 = \frac{E_1}{Z_p} - nI_2 = \frac{E_2}{nZ_p} - nI_2 = \frac{E_s}{nZ_p} - nI_s \end{aligned} \qquad (9\text{-}25)$$

We shall compare these relations with the corresponding ones for Fig. 9.11:

$$E_p = \frac{1}{n} E_2 = \frac{1}{n} E_s$$

$$I_p = -nI_2 = -n\left(I_s - \frac{E_s}{Z_s}\right) = \frac{nE_s}{Z_s} - nI_s$$

These relations among E_p, I_p, E_s, and I_s are identical with those of Eq. (9-25) provided that $Z_s = n^2 Z_p$.

Similarly, the series arrangement of Fig. 9.12a yields

$$E_p = Z_pI_p + E_1 = \frac{1}{n}E_s - nZ_pI_s$$

$$I_p = I_1 = -nI_2 = -nI_s$$

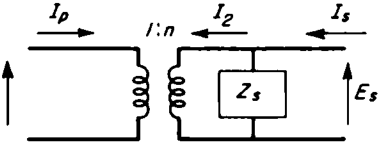


FIG. 9.11.

while that of Fig. 9.12b yields

$$E_p = E_1 = \frac{1}{n}E_2 = \frac{1}{n}(E_s - Z_sI_s) = \frac{E_s}{n} - \frac{Z_s}{n}I_s$$

$$I_p = -nI_2 = -nI_s$$

Again, equivalence is given by $Z_s = n^2Z_p$. Note that if $Z_p = j\omega L$, $Z_s = n^2j\omega L = j\omega(n^2L)$, but if $Z_p = 1/j\omega C$, $Z_s = n^2/j\omega C = \dfrac{1}{j\omega(C/n^2)}$.

By using these results stepwise, we find the equivalents of Fig. 9.13. Note the relative locations of the series impedances Z_1 and n^2Z_1.

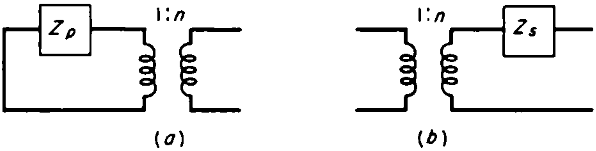


FIG. 9.12.

9-6 Leakage Inductance. The transformer of Fig. 9.8 can be represented as a combination of a unity-coupled transformer with uncoupled self-inductance. The inductances x_1 and x_2 represent the *leakage inductance* of the transformer. The term "leakage" comes from iron-core transformer thinking, where flux that does not stay in the core is responsible for the uncoupled inductance, i.e., for lack of unity coupling. The A, B, C, D coefficients for Fig. 9.14 are readily found: $I_2 = 0$ gives

$$I_1 = E_1/j\omega(x_1 + l_1)$$

$$E_2 = j\omega\sqrt{l_1l_2}\,I_1 = \sqrt{l_1l_2}\,E_1/(x_1 + l_1)$$

making

$$A = \frac{x_1 + l_1}{\sqrt{l_1l_2}}, \quad C = \frac{1}{j\omega\sqrt{l_1l_2}}$$

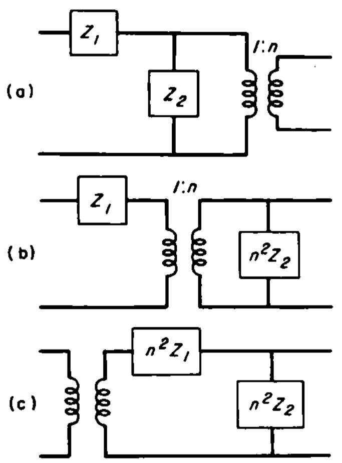
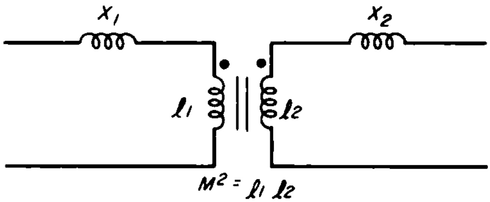


FIG. 9.13.



The dots indicate the positions of the coil arrowheads for M positive

FIG. 9.14.

while $I_1 = 0$ gives

$$I_2 = \frac{E_2}{j\omega(x_2 + l_2)} = \frac{C}{D} E_2$$

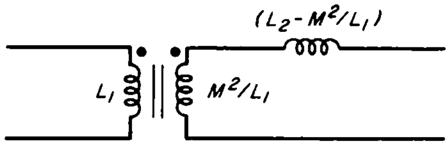
so that

$$D = \frac{x_2 + l_2}{\sqrt{l_1 l_2}}$$

Equating the A, C, D just found to those relevant to Fig. 9.8 yields:

$$\begin{aligned} \sqrt{l_1 l_2} &= M \\ x_1 + l_1 &= L_1 \\ x_2 + l_2 &= L_2 \end{aligned} \tag{9-26}$$

Finding the new B and equating to the old B yields nothing additional, since only three of the coefficients are independent.



Unity coupling

FIG. 9.15.

In Eq. (9-26) we have *four* unknowns (l_1, l_2, x_1, x_2) and only three equations. We can therefore add an arbitrary condition. One such is $x_1 = 0$, giving

$$\begin{aligned} l_1 &= L_1 \\ l_2 &= M^2/L_1 \\ x_2 &= L_2 - M^2/L_1 \end{aligned}$$

and the transformer of Fig. 9.8 is equivalent to the combination of Fig. 9.15.

In this case, all the leakage inductance is referred to the second side. Conversely, we could choose $x_2 = 0$, giving Fig. 9.16.

Still another alternative is to give the unity-coupled transformer a one-to-one voltage ratio: $l_1 = l_2$. From Eq. (9-26) we find

$$l_1 = M$$

$$x_1 = L_1 - M$$

$$x_2 = L_2 - M$$

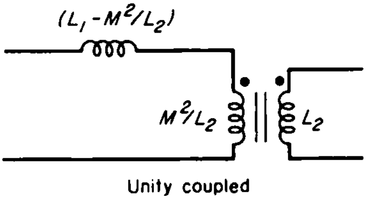


FIG. 9.16.

which is shown in Fig. 9.17. This equivalent can be nonphysical, in the sense that one of the self-inductances can be negative ($k^2 > L_1/L_2$ or $k^2 > L_2/L_1$).

In Figs. 9.15 and 9.16, however, this effect cannot arise, since $M^2 \leq L_1L_2$.

In Fig. 9.17, the voltages across the two windings are necessarily identical. Changing the connections to either of those shown in Fig. 9.18 can

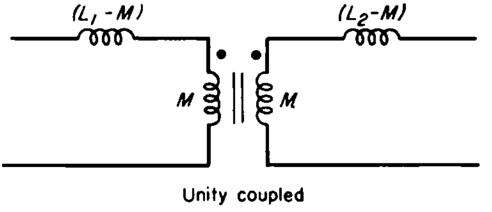


FIG. 9.17.

therefore have no effect. Since the unity-coupled unit ratio transformer can be considered as made by winding a *pair* of wires onto a core, the parallel connected arrangement of Fig. 9.18a is the same as a single wire (of twice the cross section) wound on the same core, with the same number of turns. Hence the parallel-connected transformer is the same as a simple inductance M, and Fig. 9.19 is another equivalent of Fig. 9.8.

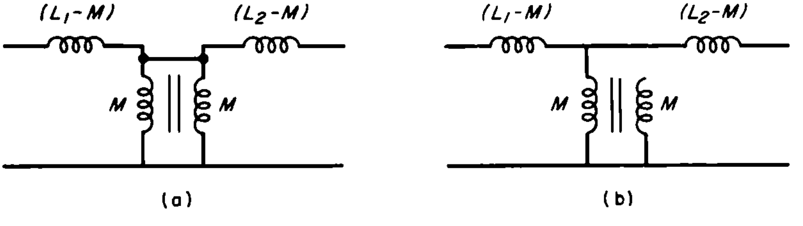


FIG. 9.18.

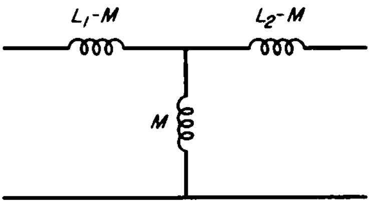


FIG. 9.19.

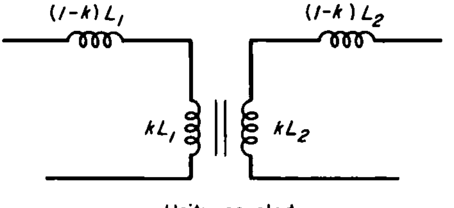


FIG. 9.20.

Problem.

Check the preceding statement by finding the A, C, D coefficients for Fig. 9.19.

We noticed in connection with Figs. 9.15, 9.16, and 9.17 that there is no unique assignment of leakage inductance to the two sides of the transformer. There is one assignment, however, that has intuitive appeal. If the coefficient of coupling in Fig. 9.8 is $k = M/\sqrt{L_1L_2}$, we let $l_1 = kL_1$. Equations (9-26) yield

$$l_2 = kL_2$$
$$x_1 = (1 - k)L_1$$
$$x_2 = (1 - k)L_2$$

The equivalent is illustrated in Fig. 9.20.

9-7 Practical Transformers. The "transformers" discussed in the preceding section were pure inductive devices. The physical realization of these devices inevitably involves distributed capacitance between the coils, and between turns of an individual coil. There is, of course, also the resistance of the windings; in iron-core transformers there are additional power losses that act like resistance.

An air-core transformer, such as an inductive coupling at radio frequencies, or an intermediate-frequency (IF) transformer in a superheterodyne receiver, suffers mainly from distributed capacitance (Fig. 9.21). In the equivalent network, these "parasitic" capacitances are indicated by dotted-line connections; they are there electrically, but would not appear in a

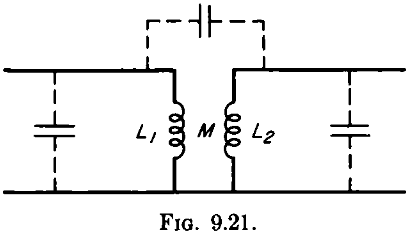


FIG. 9.21.

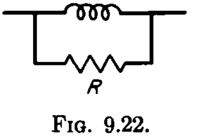


FIG. 9.22.

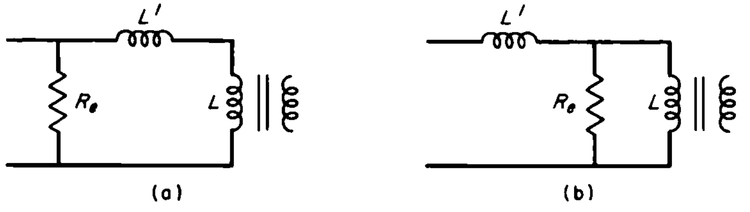


FIG. 9.23.

wiring diagram. The stray capacitance across each coil does not complicate the circuit, for it is in parallel with whatever tuning capacitance is used. The *bridging* capacitance modifies the transmission properties of the transformer and must be allowed for in computing the bandwidth and other properties of an amplifier.

In iron-core transformers, such as used in audio amplifiers, we again find distributed capacitances. Since audio circuits are usually intended to have flat frequency characteristics, there is no tuning capacitance, and all three parasitic capacitances modify the response. These effects are important at the high frequencies of the audio band.

Iron-core transformers exhibit power loss due to (1) winding resistance, (2) eddy current loss, and (3) hysteresis loss. The loss due to the resistance of the windings is called "copper loss." The eddy current loss arises in the circulating currents in the core. Consider what would happen if a piece of iron pipe were used for a core. The cross section of the pipe would be a shorted turn, and would carry an induced current. A solid core would be equivalent to a concentric "nest" of pipes of various diameters. To avoid this excessive loss, practical cores are laminated, i.e., built up of thin layers insulated from one another. Eddy current losses are thereby greatly

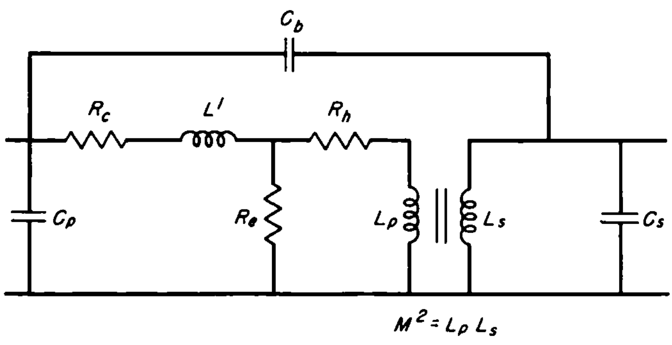


FIG. 9.24.

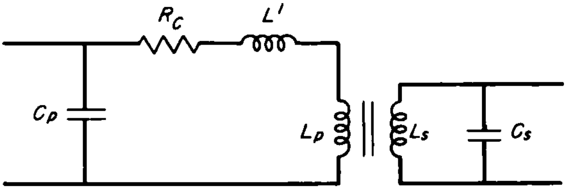


FIG. 9.25.

reduced, but are still present, because each lamination is a thin solid core. The laminations are therefore made as thin as practicable.

The eddy-current power loss is found experimentally (and theoretically) to be proportional to the square of the current, and the *square of the frequency*. The square of the current suggests an equivalent series resistance, but the power loss in resistance does not vary with frequency. Consider however the combination of Fig. 9.22; the impedance of the L, R parallel combination is

$$Z = \frac{j\omega LR}{j\omega L + R} = \frac{R\omega^2L^2 + j\omega LR^2}{R^2 + \omega^2L^2}$$

For $R \gg \omega L$, which is valid for good coils except at very high frequency,

$$Z \doteq \frac{\omega^2L^2}{R} + j\omega L \equiv r + j\omega L$$

with

$$r = \omega^2(L^2/R)$$

The equivalent *series* resistance is therefore proportional to ω^2, and its power loss,

$$I^2r = I^2\omega^2(L^2/R)$$

is proportional to both I^2 and ω^2 as desired. Hence eddy-current loss is representable by resistance in parallel with the inductance. The question arises as to a choice between Figs. 9.23a and 9.23b for the equivalent circuit. Here L' represents the leakage inductance, and R_e the equivalent resistance due to eddy-current loss. The leakage inductance is associated with leakage flux, i.e., flux that does not link the other coil. This leakage flux fol-

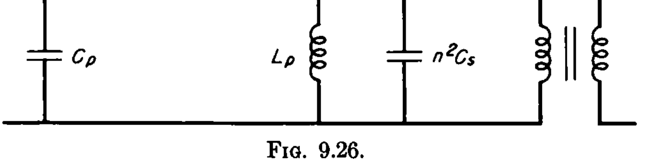


FIG. 9.26.

lows mainly an air path, hence is essentially not subject to eddy-current effects. For this reason, Fig. 9.23b is the better representation.

The hysteresis loss is associated with nonlinear behavior of the core, hence cannot have a precise representation in terms of constant resistance. Luckily this loss is relatively small and can be sufficiently well represented by additional resistance associated with the unity-coupled inductance.

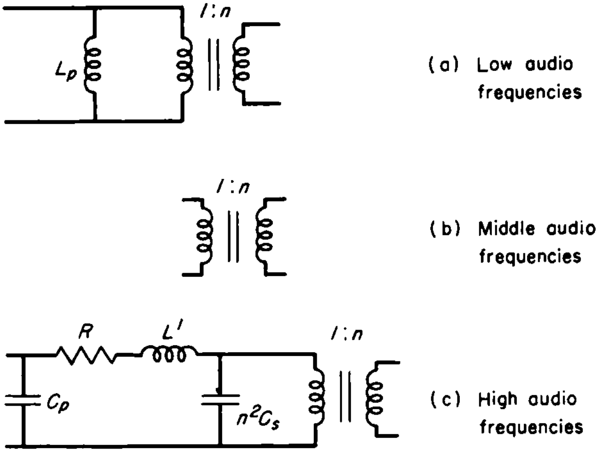


FIG. 9.27.

Combining all these parasitic capacitances and resistances, we obtain an equivalent circuit for an iron-core transformer as shown in Fig. 9.24.

For a well-designed transformer, we can neglect R_h and R_e in comparison with R_c, and C_b can also be ignored for audio frequencies. The resulting simplified equivalent is shown in Fig. 9.25. Replacing the unity-coupled transformer by its ideal-transformer equivalent, and referring C_s to the primary side yields Fig. 9.26.

Since $L' \ll L_p$, we have simplified approximations for various frequency ranges (Fig. 9.27a, b, c).

Chapter X

SPECIFIC AC NETWORKS

10-1 Simple Radio-Frequency Transformers. The tuned circuit of Fig. 10.1 is excited by the direct insertion of a voltage generator. The put voltage (across the capacitance) is

$$V_o = I/j\omega C = \frac{1}{j\omega C}\frac{E}{R + j\omega L + \dfrac{1}{j\omega C}} = \frac{E}{1 - \omega^2 LC + jR\omega C} \quad (10\text{-}1)$$

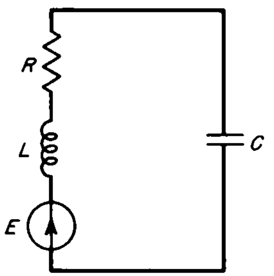


Fig. 10.1.

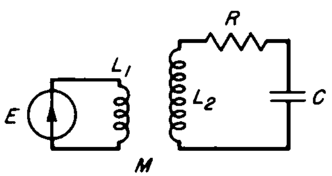


Fig. 10.2.

At resonance, $\omega^2 LC = 1$, making

$$|V_o| = \left|\frac{E}{jR\omega C}\right| = |E|\,\frac{\omega L}{R} = Q|E| \quad (10\text{-}2)$$

so that the voltage is stepped-up by the factor Q. In practice, however, we do not find sources inserted directly into tuned circuits; the source is usually the output of an amplifying tube or transistor which is *coupled* to the tuned circuit. A low-impedance source can be coupled as in Fig. 10.2. The circuit equations are

$$E = j\omega L_1 I_1 + j\omega M I_2$$
$$0 = j\omega M I_1 + I_2(j\omega L_2 + R + 1/j\omega C)$$

Eliminating I_1 yields

$$E = I_2\left\{j\omega M + \frac{j\omega L_1}{j\omega M}\left(R + j\omega L_2 + \frac{1}{j\omega C}\right)\right\}$$

and

$$V_o = \frac{I_2}{j\omega C} = \frac{E}{-\omega^2 MC + \frac{L_1}{M}(1 - \omega^2 L_2 C + j\omega CR)}$$

$$= \frac{EM/L_1}{1 - \omega^2 C(L_2 + M^2/L_1) + j\omega CR}$$

$$= \frac{Ek\sqrt{L_2/L_1}}{[1 - \omega^2 CL_2(1 + k^2)] + j\omega CR}$$

which is similar in form to Eq. (10-1). Note that the effective inductance in the tuned circuit is $(1 + k^2)L_2$. The voltage step-up at resonance is

$$\left|\frac{V_o}{E}\right| = \frac{\omega L_2}{R}(1 + k^2)k\sqrt{\frac{L_2}{L_1}}$$

The circuit is equivalent to that of Fig. 10.1 with $L = (1 + k)L_2$, $e = k\sqrt{L_2/L_1}\,E$, so that $k\sqrt{L_2/L_1}$ is the additional voltage step-up due to the transformer action.

The coils L_1 and L_2 need not be distinct, but may comprise an *auto-transformer*, a single tapped coil, as in Fig. 10.3. The total inductance of the coil is $L = L_1 + L_2 = 2M$, for a current I through the coil would give the voltage drop (cf. Fig. 9.6):

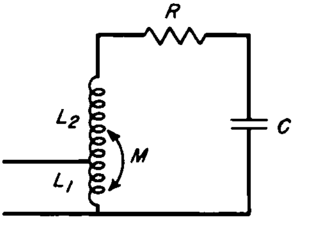


FIG. 10.3.

$$E = (j\omega L_1 I + j\omega M_{12} I) + (j\omega L_2 I + j\omega M_{21} I)$$

$$= j\omega(L_1 + L_2 + 2M)I$$

Applying a voltage E across the tapped portion L_1 yields

$$E = I_1 j\omega L_1 - I_2 j\omega L_1 - I_2 j\omega M$$

$$0 = -I_1 j\omega L_1 - I_1 j\omega M + I_2(j\omega L_1 + j\omega L_2 + 2j\omega M) + I_2 R + I_2/j\omega C$$

making

$$I_1 = I_2\frac{R + j\omega L + 1/j\omega C}{j\omega(L_1 + M)}$$

$$E = I_2\left\{\frac{L_1}{L_1 + M}(R + j\omega L + 1/j\omega C) - j\omega L_1 - j\omega M\right\}$$

$$= I_2\frac{RL_1 + L_1/j\omega C + j\omega(L_1 L_2 - M^2)}{L_1 + M}$$

and finally

$$\frac{V_o}{E} = \frac{L_1 + M}{L_1 - \omega^2 C(L_1 L_2 - M^2) + j\omega C L_1 R}$$

$$= \frac{1 + M/L_1}{1 - \omega^2 C(L_2 - M^2/L_1) + j\omega CR} = \frac{1 + k\sqrt{L_2/L_1}}{1 - \omega^2 C L_2(1 - k^2) + j\omega CR}$$

These simple resonant circuits have single peaks in their frequency response; in the next section we shall study circuits that can be adjusted for double peaks, or for a flat top broad peak.

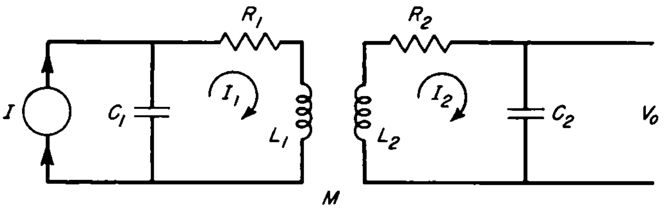


Fig. 10.4.

10-2 Double-Tuned Transformers. An intermediate-frequency (IF) amplifier commonly employs a chain of pentode amplifying tubes, successive tubes coupled via *double-tuned* transformers. As far as its plate circuit is concerned, a pentode is practically a constant-current source. (We do not mean "steady" current or dc, but a current source of complex I which is unaffected by load.) The interstage circuit is therefore represented by Fig. 10.4.

Problem.

Using Figs. 9.16, 9.9, and 9.13 in this order, transform Fig. 10.4 into Fig. 10.5, where $k^2 = M^2/L_1L_2$, $n = L_2/M$.

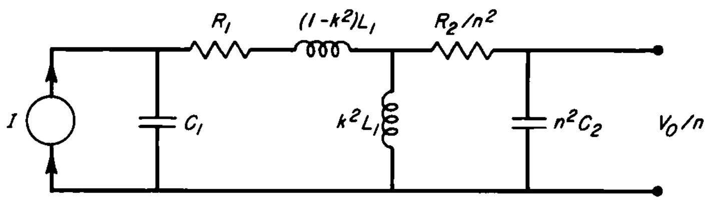


Fig. 10.5.

The circuit equations for Fig. 10.4 are

$$\begin{aligned} 0 &= (I_1 - I)/j\omega C_1 + I_1(R_1 + j\omega L_1) - I_2 j\omega M \\ 0 &= -I_1 j\omega M + I_2(R_2 + j\omega L_2 + 1/j\omega C_2) \qquad (10\text{-}3) \\ V_o &= I_2/j\omega C_2 \end{aligned}$$

The second equation can be solved for I_1; this is substituted in the first, yielding I_2 in terms of I. The third equation then gives

$$V_o = I \frac{M}{j\omega C_1 C_2} \{(R_1 + j\omega L_1 + 1/j\omega C_1)(R_2 + j\omega L_2 + 1/j\omega C_2) + \omega^2 M^2\}^{-1} \tag{10-4}$$

Problem.

Solve Eq. (10-3) and check Eq. (10-4).

Since $(j\omega L + 1/j\omega C) = j\omega L(1 - 1/\omega^2 LC)$, which vanishes at resonance, $\omega^2 LC = 1$, it is convenient to introduce the resonance frequencies

$$\omega_1^2 = 1/L_1C_1$$

$$\omega_2^2 = 1/L_2C_2$$

This allows Eq. (10-4) to be written

$$V_o = \frac{IM}{j\omega C_1 C_2 \omega^2 L_1 L_2 \left\{\left[\frac{R_1}{\omega L_1} + j\left(1 - \frac{\omega_1^2}{\omega^2}\right)\right]\cdot\left[\frac{R_2}{\omega L_2} + j\left(1 - \frac{\omega_2^2}{\omega^2}\right)\right] + k^2\right\}}$$

$$= \frac{-j\omega k\sqrt{L_1L_2}\, I}{\frac{\omega^2}{\omega_1^2}\frac{\omega^2}{\omega_2^2}\left\{k^2 + \frac{R_1}{\omega L_1}\frac{R_2}{\omega L_2} - \left(1 - \frac{\omega_1^2}{\omega^2}\right)\left(1 - \frac{\omega_2^2}{\omega^2}\right) + j\left[\frac{R_1}{\omega L_1}\left(1 - \frac{\omega_2^2}{\omega^2}\right) + \frac{R_2}{\omega L_2}\left(1 - \frac{\omega_1^2}{\omega^2}\right)\right]\right\}} \tag{10-5}$$

Equation (10-5) is exact and general. To simplify the analysis, we shall now assume that the primary and secondary are tuned to resonate at the same frequency, $\omega_1 = \omega_2$. We also express $\omega L_1/R_1$ as Q_1, the Q of the coil. With these simplifications Eq. (10-5) becomes

$$V_o = \frac{-j\omega k\sqrt{L_1L_2}\, I}{\left(\frac{\omega^2}{\omega_1^2}\right)^2\left\{k^2 + \frac{1}{Q_1Q_2} - \left(1 - \frac{\omega_1^2}{\omega^2}\right)^2 + j\left(\frac{1}{Q_1} + \frac{1}{Q_2}\right)\left(1 - \frac{\omega_1^2}{\omega^2}\right)\right\}} \tag{10-6}$$

Note that Q varies with frequency, but for small frequency ranges near resonance may be assumed constant, since $(1 - \omega_1^2/\omega^2)$ contributes the major effect.

10-3 Critical Coupling. At resonance ($\omega = \omega_1$), Eq. (10-6) reduces to

$$V_o = -j\omega_1 I\sqrt{L_1L_2}\frac{k}{k^2 + \frac{1}{Q_1Q_2}}$$

As a function of k, this is maximized (in amplitude) for

$$0 = \frac{d}{dk}\left(\frac{k}{k^2 + \frac{1}{Q_1Q_2}}\right) = \frac{\left(k^2 + \frac{1}{Q_1Q_2}\right) - 2k^2}{\left(k^2 + \frac{1}{Q_1Q_2}\right)^2}$$

that is, for $k = 1/\sqrt{Q_1Q_2} \equiv k_c$, the *critical coupling*. The corresponding maximum output voltage is given by

$$|V_o| = \omega_1 I\sqrt{L_1L_2}\sqrt{Q_1Q_2}/2 \tag{10-7}$$

Increasing k to more than the critical value, k_c, *reduces* the output voltage. This implies that the circuit is no longer properly tuned; the two tuned circuits interfere with each other. We return to Eq. (10-5), which we rewrite using $x_1 \equiv \omega_1^2/\omega^2$, $x_2 \equiv \omega_2^2/\omega^2$:

$$V_o = \frac{-j\omega\sqrt{L_1L_2}\,Ikx_1x_2}{k^2 + \frac{1}{Q_1Q_2} - (1 - x_1)(1 - x_2) + j\left[\frac{1 - x_2}{Q_1} + \frac{1 - x_1}{Q_2}\right]} \tag{10-8}$$

We are interested in the absolute magnitude of the voltage, whose square is

$$|V_o|^2 = \frac{\omega^2L_1L_2I^2k^2x_1^2x_2^2}{\left[k^2 + \frac{1}{Q_1Q_2} - (1 - x_1)(1 - x_2)\right]^2 + \left[\frac{1 - x_2}{Q_1} + \frac{1 - x_1}{Q_2}\right]^2} \tag{10-9}$$

For simplicity, we ignore the slow variation of the numerator, and of the Q terms in the denominator. Our tuning problem is to adjust x_1 and x_2 (i.e., ω_1 and ω_2) so as to minimize the denominator. We can simplify the minimization problem by temporarily writing y_1 for $(1 - x_1)$ and y_2 for $(1 - x_2)$. The denominator is

$$D = \left(k^2 + \frac{1}{Q_1Q_2} - y_1y_2\right)^2 + \left(\frac{y_2}{Q_1} + \frac{y_1}{Q_2}\right)^2 \tag{10-10}$$

For small variations of y_1 and y_2, the corresponding variation of D is:

$$\begin{aligned} dD &= 2\left(k^2 + \frac{1}{Q_1Q_2} - y_1y_2\right)(-y_1dy_2 - y_2dy_1) + 2\left(\frac{y_2}{Q_1} + \frac{y_1}{Q_2}\right)\left(\frac{dy_2}{Q_1} + \frac{dy_1}{Q_2}\right) \\ &= \left[-2y_2\left(k^2 + \frac{1}{Q_1Q_2} - y_1y_2\right) + \frac{2}{Q_2}\left(\frac{y_2}{Q_1} + \frac{y_1}{Q_2}\right)\right]dy_1 \\ &\quad + \left[-2y_1\left(k^2 + \frac{1}{Q_1Q_2} - y_1y_2\right) + \frac{2}{Q_1}\left(\frac{y_2}{Q_1} + \frac{y_1}{Q_2}\right)\right]dy_2 \end{aligned} \tag{10-11}$$

For D to be simultaneously minimized with respect to *both* y_1 and y_2, the coefficients of dy_1 and dy_2 must both vanish, giving the simultaneous equations

$$y_2\left(k^2 + \frac{1}{Q_1Q_2} - y_1y_2\right) = \frac{1}{Q_2}\left(\frac{y_2}{Q_1} + \frac{y_1}{Q_2}\right)$$
$$y_1\left(k^2 + \frac{1}{Q_1Q_2} - y_1y_2\right) = \frac{1}{Q_1}\left(\frac{y_2}{Q_1} + \frac{y_1}{Q_2}\right) \tag{10-12}$$

Dividing one by the other gives

$$\frac{y_2}{y_1} = \frac{Q_1}{Q_2} \tag{10-13}$$

Using this relation to eliminate y_2 in the second equation gives

$$y_1\left(k^2 + \frac{1}{Q_1Q_2} - y_1{}^2\frac{Q_1}{Q_2}\right) = \frac{2y_1}{Q_1Q_2} \tag{10-14}$$

This equation is satisfied by $y_1 = 0$, or by

$$k^2 + \frac{1}{Q_1Q_2} - y_1{}^2\frac{Q_1}{Q_2} = \frac{2}{Q_1Q_2}$$

which yields

$$y_1{}^2 = \frac{k^2Q_1Q_2 - 1}{Q_1{}^2}$$

Equation (10-13) gives the corresponding solutions for y_2, and we have the two sets

$$y_1 = 0, y_2 = 0;$$
$$y_1{}^2 = \frac{k^2Q_1Q_2 - 1}{Q_1{}^2}, \quad y_2{}^2 = \frac{k^2Q_1Q_2 - 1}{Q_2{}^2}$$

or

$$y_1 = \pm\frac{\sqrt{k^2Q_1Q_2 - 1}}{Q_1}, \quad y_2 = \pm\frac{\sqrt{k^2Q_1Q_2 - 1}}{Q_2} \tag{10-15}$$

but the *same* choice of sign must be made because of Eq. (10-13). Returning to the frequency ratios x_1 and x_2, we have the *three* solutions

$$x_1 = 1, \quad x_2 = 1$$
$$x_1 = 1 + \frac{\sqrt{k^2Q_1Q_2 - 1}}{Q_2}, \quad x_2 = 1 + \frac{\sqrt{k^2Q_1Q_2 - 1}}{Q_1}$$
$$x_1 = 1 - \frac{\sqrt{k^2Q_1Q_2 - 1}}{Q_2}, \quad x_2 = 1 - \frac{\sqrt{k^2Q_1Q_2 - 1}}{Q_1} \tag{10-16}$$

If $k^2Q_1Q_2 < 1$, i.e., if $k < k_c$, the only real solution is $x_1 = x_2 = 1$. This is our original case of both coils turned to resonance. For $k > k_c$, the coils can also be tuned both above resonance, or both below resonance.

The corresponding stationary values of k^2/D occurring in $|V_o|^2$ (Eq. (10-9)) are,

for resonance:
$$\frac{k^2}{\left(k^2 + \frac{1}{Q_1Q_2}\right)^2}$$

with a maximum of $\frac{Q_1Q_2}{4}$ for $k = k_c$,

and in both the other cases:
$$\frac{k^2}{\left(k^2 + \frac{1}{Q_1Q_2} - \frac{k^2Q_1Q_2 - 1}{Q_1Q_2}\right)^2 + \left(\frac{\sqrt{k^2Q_1Q_2 - 1}}{Q_1Q_2} + \frac{\sqrt{k^2Q_1Q_2 - 1}}{Q_1Q_2}\right)^2}$$
$$= \frac{k^2}{\left(\frac{2}{Q_1Q_2}\right)^2 + (k^2Q_1Q_2 - 1)\left(\frac{2}{Q_1Q_2}\right)^2} = \frac{Q_1Q_2}{4}$$

Hence for $k > k_c$, the maximum value of V_o is the same as the maximum attained at resonance, with $k = k_c$. But for $k > k_c$, the circuits must be *detuned, either above or below resonance.* This suggests that for *fixed* tuning, there are maximum responses at frequencies both above and below resonance. The stationary value *at* resonance is now a minimum, or a valley between the peaks.

From Eq. (10-16), we see that the response is symmetrical about resonance if $Q_1 = Q_2$, putting the stationary points at

$$\begin{aligned} x_1 &= x_2 = 1 \\ x_1 &= x_2 = 1 - \frac{\sqrt{k^2Q^2 - 1}}{Q} = 1 - k\sqrt{1 - \frac{1}{Q^2}} \\ x_1 &= x_2 = 1 + \frac{\sqrt{k^2Q^2 - 1}}{Q} = 1 + k\sqrt{1 - \frac{1}{Q^2}} \end{aligned} \qquad (10\text{-}17)$$

These equations correspond to

$$\omega_1 = \omega_2$$
$$\frac{\omega_1}{\omega} = 1 \pm k\sqrt{1 - \frac{1}{Q^2}}$$

The peaks occur at frequencies

$$\begin{aligned} \omega_a &= \frac{\omega_1}{1 - k\sqrt{1 - \frac{1}{Q^2}}} \doteq \frac{\omega_1}{1 - k} \\ \omega_b &= \frac{\omega_1}{1 + k\sqrt{1 - \frac{1}{Q^2}}} \doteq \frac{\omega_1}{1 + k} \end{aligned} \qquad (10\text{-}18)$$

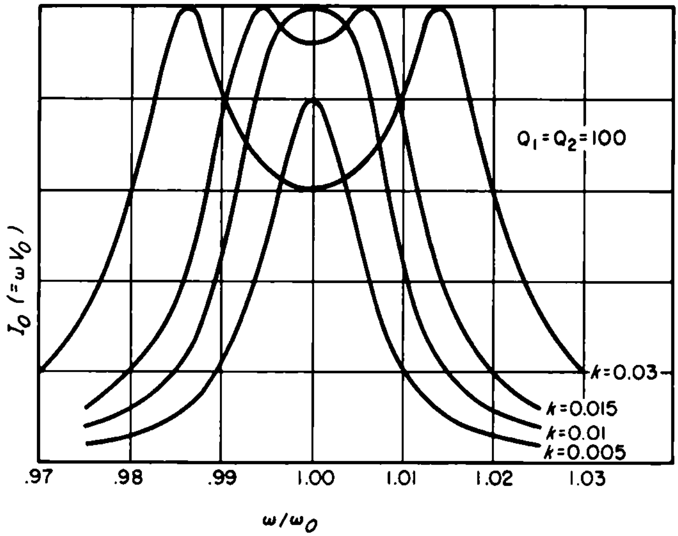


FIG. 10.6.

for large Q. The response curve is shown in Fig. 10.6 for several values of k.

10-4 Frequency-Modulation Discriminator. The coupled tuned circuits of Fig. 10.4 had the solution Eq. (10-6). For large equal Q's and for $\omega \doteq \omega_1$, we have

$$V_o \doteq \frac{-j\omega_1 k\sqrt{L_1L_2}\, I}{k^2 + \dfrac{2j}{Q}\left(1 - \dfrac{\omega_1^2}{\omega^2}\right)}$$

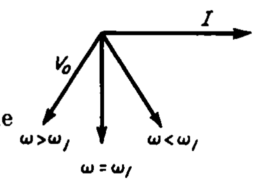


FIG. 10.7.

For small variations of ω, $|V_o|$ = constant, but the phase of V_o varies as

$$-\frac{\pi}{2} - \tan^{-1}\frac{2(1 - \omega_1^2/\omega^2)}{k^2Q},$$

shown in Fig. 10.7. Consider the center of the secondary coil grounded as in Fig. 10.8. The voltages V_1 and V_2, with respect to ground, are

$$V_1 = V_o/2, \quad V_2 = -V_o/2,$$

as shown in Fig. 10.9. We next return the coil center, not to ground, but to a voltage in phase with I, such as may be obtained by either arrangement of Fig. 10.10. The voltages V_a and V_b are the vector sums shown in Fig. 10.11; the magnitudes of V_a and V_b vary with frequency.

In fact, the difference between the *magnitudes* of V_a and V_b is approximately linear with the deviation of ω from resonance. Now the *magnitude* of a radio-frequency voltage can be obtained as the output of a linear rectifier; a pair of rectifiers back-to-back can yield $|V_a| - |V_b|$. The combination of discriminator and rectifiers makes a frequency-modulation (FM) detector (Fig. 10.12).

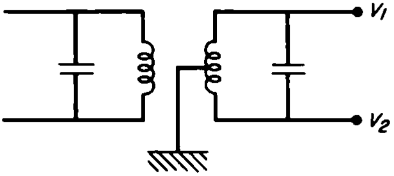


FIG. 10.8.

10-5 Bridge Circuits. The simple bridge of Fig. 10.13 is formally the same as the Wheatstone bridge of Chapter II; here we have general impedances in the arms instead of pure resistance. The condition for balance ($V_o = 0$) is

$$\frac{Z_1}{Z_3} = \frac{Z_2}{Z_4} \quad \text{or} \quad Z_1Z_4 = Z_2Z_3$$

Since the Z's vary with frequency, the balance may depend upon frequency as well as the fixed parameters (R, L, C) of the branches. By proper choice of the branch elements, the bridge can be made suitable for measuring resistance, inductance, capacitance, or frequency. In fact it can be arranged to measure both the inductance and resistance of a coil. We shall examine only a few of the well-known bridge arrangements.

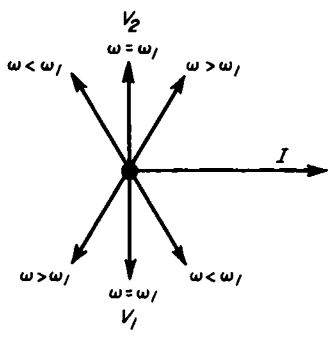


FIG. 10.9.

The Maxwell bridge (Fig. 10.14) balances a series RL against a parallel RC. The balance condition is

$$R_3R_2 = (R_4 + j\omega L_4)\frac{R_1/j\omega C_1}{R_1 + 1/j\omega C_1}$$

which becomes

$$R_3R_2 + R_3R_2R_1j\omega C_1 = R_4R_1 + j\omega L_4R_1$$

The real and imaginary terms must equate separately, so there are two simultaneous conditions to be satisfied for balance:

$$R_3R_2 = R_1R_4$$
$$R_3R_2C_1 = L_4$$

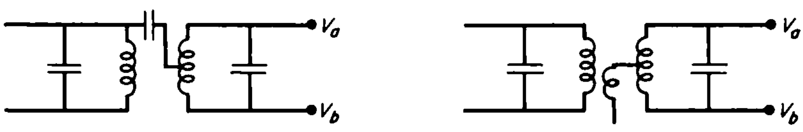
With R_2 and R_3 fixed, adjusting R_1 and C_1 for balance yields the unknown L_4 and R_4 of a coil, in terms of the (calibrated) variable resistor and capacitor. For this bridge, the balance conditions are independent of frequency.

The Wien bridge (Fig. 10.15), however, is frequency-dependent and in fact a useful device for measuring an audio frequency. The balance condition is

$$\frac{R_4R_1}{1 + j\omega C_1R_1} = R_2(R_3 + 1/j\omega C_3)$$

Equating real and imaginary terms separately yields:

$$\omega^2C_1C_3R_1R_3 = 1$$
$$\frac{C_1}{C_3} = \frac{R_4}{R_2} - \frac{R_3}{R_1} \qquad (10\text{-}19)$$



(a) (b)

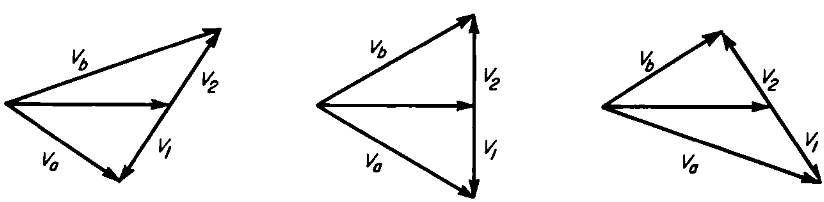
Fig. 10.10.

A very useful frequency-independent bridge for capacitance measurement is the Schering bridge (Fig. 10.16). The balance condition is

$$\frac{R_2}{j\omega C_3} = \frac{R_1}{1 + j\omega R_1C_1}(R_4 + 1/j\omega C_4)$$

which yields

$$C_3R_4 = C_1R_2$$
$$C_3R_1 = C_4R_2$$



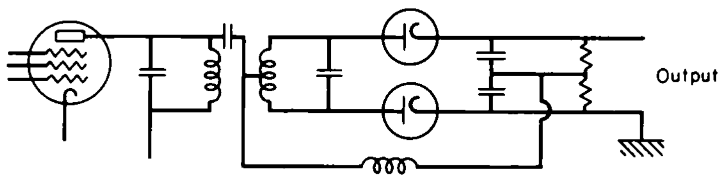
$\omega > \omega_1$ $\omega = \omega_1$ $\omega < \omega_1$

Fig. 10.11.

Note that we can measure C_3 in the presence of an *unknown* C_1. For example, let $R_1 = R_2$ be fixed, with C_4 a calibrated capacitor and R_4 a variable resistor, which need not be calibrated. The adjustments for balance are

$$R_4 = R_2C_1/C_3$$
$$C_4 = C_3$$

This bridge can be used to measure interelement capacitances in vacuum tubes. Consider the triangle of capacitances shown in Fig. 10.17. These can be measured *separately* without disconnecting them from one another.



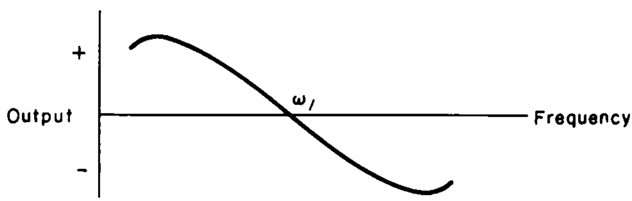


Fig. 10.12.

The arrangement of Fig. 10.18 makes C_a the previous C_3 of a Schering bridge, and C_b the C_1 of the bridge. The third capacitance C_c appears across a bridge *diagonal* and has no effect on the balance condition.

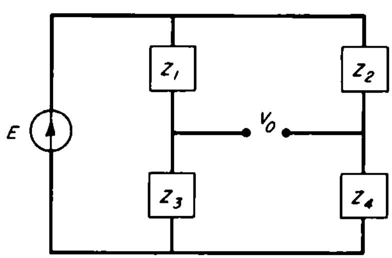


Fig. 10.13.

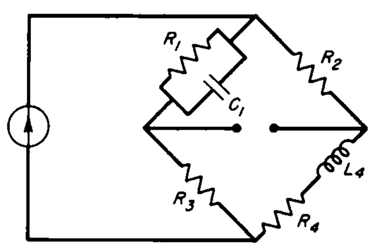


Fig. 10.14.

10-6 Bridged-T's. All the above bridge circuits have the practical disadvantage that the input and output cannot have a common ground terminal. Either the generator or the detector must "float." The bridged-T of Fig. 10.19 has properties similar to the preceding "square"

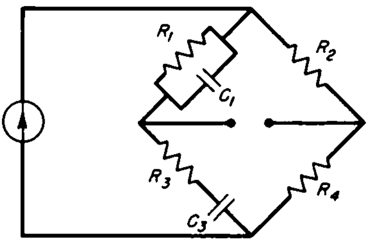


FIG. 10.15.

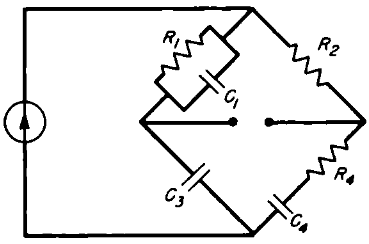


FIG. 10.16.

bridges. The circuit equations are

$$E = I_1(Z_1 + Z_3) - I_2Z_1 \tag{10-20}$$

$$0 = -I_1Z_1 + I_2(Z_1 + Z_2 + Z_4) \tag{10-21}$$

while

$$V_o = I_2Z_2 + I_1Z_3 \tag{10-22}$$

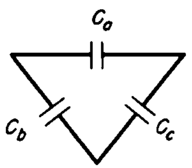


FIG. 10.17.

For $V_o = 0$, Eqs. (10-21) and (10-22) require

$$-I_1Z_1 + I_2(Z_1 + Z_2 + Z_4) = 0$$

$$I_1Z_3 + I_2Z_2 = 0$$

and Eq. (10-20) is of no interest. For these homogeneous simultaneous equations for I_1 and I_2 to have a nontrivial solution, i.e., for them to *not* require $I_1 = 0$, $I_2 = 0$, the determinant of the coefficients must vanish:

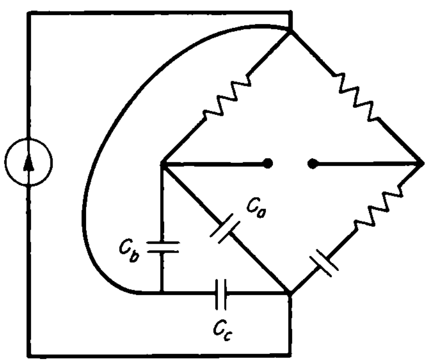


FIG. 10.18.

$$Z_1Z_2 + Z_3(Z_1 + Z_2 + Z_4) = 0 \tag{10-23}$$

This condition cannot be satisfied with all branches pure positive resistance. Unless all the branches are pure reactance, which cannot be achieved in practice, there will be some positive real terms in the above

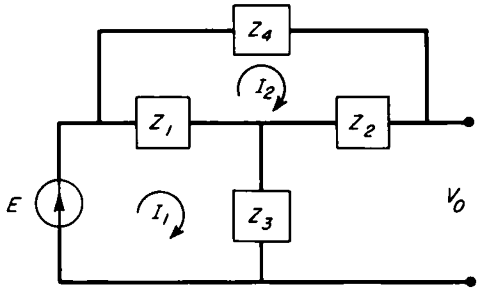


Fig. 10.19.

expression. To cancel these, we must have at least two complex impedances to produce a negative real term in their product. A particular case of interest is given by

$$Z_1 = Z_2 = 1/j\omega C$$

making the balance condition

$$Z_3(Z_4 + 2/j\omega C) = 1/\omega^2C^2 \tag{10-24}$$

We are at liberty to make either Z_3 or Z_4 pure resistance. Let us examine both cases.

First, let $Z_3 = R$. Equation (10-24) becomes

$$R(Z_4 + 2/j\omega C) = 1/\omega^2C^2$$

requiring

$$Z_4 = -2/j\omega C + 1/R\omega^2C^2 = r + jx$$

so that Z_4 can be a series combination of resistance and *inductance*:

$$r = 1/R\omega^2C^2$$

$$\omega L = 2/\omega C$$

These equations are therefore the balance relations for the bridged-T of Fig. 10.20.

If, on the other hand, we had chosen $Z_4 = R$:

$$Z_3(R + 2/j\omega C) = 1/\omega^2C^2$$

$$Z_3 = \frac{1}{\omega^2C^2R - 2j\omega C}$$

making the *reciprocal* of Z_3 a simple sum.

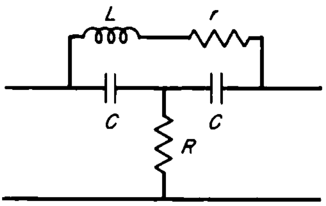


FIG. 10.20.

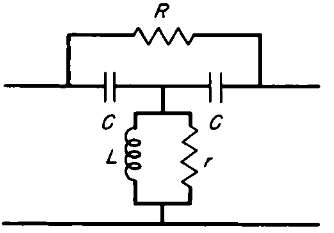


FIG. 10.21.

This suggests interpreting Z_3 as a *parallel* combination. (A *series* combination could be used, but yields more complicated balance conditions.) Since

$$\frac{1}{Z_3} = \omega^2C^2R - 2j\omega C \equiv \frac{1}{r} + \frac{1}{j\omega L}$$

we have

$$r = 1/\omega^2RC^2$$
$$\omega L = 1/2\omega C$$

as the balance conditions for Fig. 10.21.

10-7 Parallel-T. A modified-T that is even more useful than the bridged-T is the so-called twin-T, comprising two simple T-sections in parallel (Fig. 10.22). We shall examine only the very special case of the

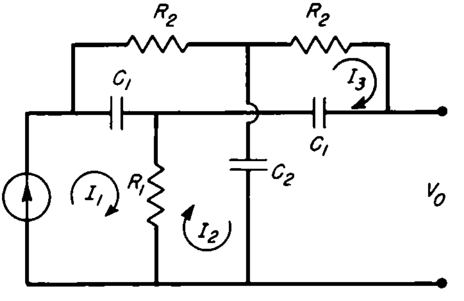


FIG. 10.22.

symmetrical *R-C* parallel-T of the figure. Writing the circuit equations for the twin-T as shown is "tricky," because I_2 is a loop current, but not a *mesh* current.

$$E = (I_1 - I_2 - I_3)/j\omega C_1 + (I_1 - I_2)R_1$$
$$0 = (I_2 - I_1)R_1 + (I_2 + I_3 - I_1)/j\omega C_1 + (I_2 + I_3)R_2 + I_2/j\omega C_2$$
$$0 = I_3R_2 + I_3/j\omega C_1 + (I_3 + I_2 - I_1)/j\omega C_1 + (I_3 + I_2)R_2 \qquad (10\text{-}25)$$

The output voltage is

$$V_o = I_3/j\omega C_1 + (I_1 - I_2)R_1$$

For $V_o = 0$ (balance), we again have a set of homogeneous equations (three this time), and again the (first) equation containing E is not of interest. For the set to have a nontrivial solution (for I_1, I_2, and I_3) the determinant of coefficients must vanish:

$$\begin{vmatrix} R_1 & -R_1 & -jX_1 \\ -R_1 + jX_1 & R_1 + R_2 - jX_1 - jX_2 & R_2 - jX_1 \\ jX_1 & R_2 - jX_1 & 2(R_2 - jX_1) \end{vmatrix} = 0 \quad (10\text{-}26)$$

where we have written X_1 for $1/\omega C_1$ for convenience. The determinant can be expanded by brute force. We shall, however, simplify the determinant by using some of the manipulations of Chapter III. Adding the first column to the second yields

$$\begin{vmatrix} R_1 & 0 & -jX_1 \\ -R_1 + jX_1 & R_2 - jX_2 & R_2 - jX_1 \\ jX_1 & R_2 & 2R_2 - 2jX_1 \end{vmatrix}$$

This can be split into the sum:

$$\begin{vmatrix} R_1 & 0 & 0 \\ -R_1 + jX_1 & R_2 - jX_2 & R_2 \\ jX_1 & R_2 & 2R_2 - jX_1 \end{vmatrix} + \begin{vmatrix} R_1 & 0 & -jX_1 \\ -R_1 + jX_1 & R_2 - jX_2 & -jX_1 \\ jX_1 & R_2 & -jX_1 \end{vmatrix}$$

The second of these contains $-jX_1$ as a factor, and is equal to:

$$-jX_1 \begin{vmatrix} R_1 & 0 & 1 \\ -R_1 + jX_1 & R_2 - jX_2 & 1 \\ jX_1 & R_2 & 1 \end{vmatrix}$$

The determinants are now in good form for expanding. Equation (10-26) becomes

$$\begin{aligned} &R_1\{(R_2 - jX_2)(2R_2 - jX_1) - R_2^2\} \\ &\qquad -jX_1\{R_1[(R_2 - jX_2) - R_2] + [(-R_1 + jX_1)R_2 - jX_1(R_2 - jX_2)]\} \\ &\qquad \equiv R_1R_2^2 - 2R_1X_1X_2 + j[X_2X_1^2 - 2X_2R_1R_2] \qquad (10\text{-}27) \end{aligned}$$

For this to vanish, we have the two relations

$$R_2^2 = 2X_1X_2 = 2/\omega^2C_1C_2$$

$$2R_1R_2 = X_1^2 = 1/\omega^2C_1^2$$

These can be re-expressed as

$$\omega^2 = \frac{1}{2R_1R_2C_1^2}$$

$$\frac{C_2}{C_1} = 4\frac{R_1}{R_2}$$

Comparison with Eq. (10-19) shows that this parallel-T is essentially a Wien bridge with a common input-output terminal.

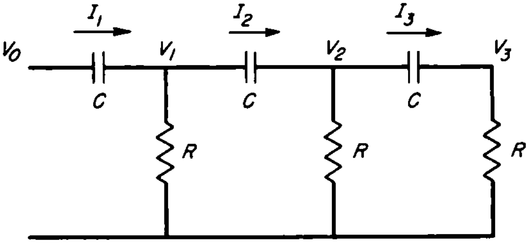


FIG. 10.23.

We conclude this chapter with a particular *RC* ladder (Fig. 10.23) network often used in phase-shift audio oscillators. It is not a bridge circuit, but is frequency-selective for providing a phase reversal (180° phase shift). Instead of writing the usual mesh equations, we take advantage of the simple structure of the ladder by starting at the right-hand end (output). The relations for stepping back section-by-section are apparent in Eq. (10-28).

$$I_3 = V_3/R$$

$$V_2 = V_3 + I_3/j\omega C = V_3\left(1 + \frac{1}{j\omega CR}\right)$$

$$I_2 = \frac{V_2}{R} + I_3 = V_3\left(\frac{2}{R} + \frac{1}{j\omega CR^2}\right)$$

$$V_1 = V_2 + \frac{I_2}{j\omega C} = V_3\left(1 + \frac{1}{j\omega CR} + \frac{2}{j\omega CR} - \frac{1}{\omega^2C^2R^2}\right) \qquad (10\text{-}28)$$

$$I_1 = \frac{V_1}{R} + I_2 = V_3\left(\frac{3}{R} + \frac{4}{j\omega CR^2} - \frac{1}{\omega^2C^2R^3}\right)$$

$$V_0 = V_1 + \frac{I_1}{j\omega C} = V_3\left(1 + \frac{6}{j\omega CR} - \frac{5}{\omega^2C^2R^2} - \frac{1}{j\omega^3C^3R^3}\right)$$

V_0 is the first term we have encountered in which the coefficient of V_3 can be made real: let

$$\frac{6}{\omega CR} = \frac{1}{\omega^3 C^3 R^3}$$

$$(\omega CR)^2 = 1/6 \tag{10-29}$$

Then

$$V_0 = V_3(1 - 30) = -29V_3$$

so that V_3 is 180° out of phase with V_0, and has 1/29 the magnitude. As will be shown in a later chapter, this implies that this RC ladder can be used as a plate-to-grid feedback circuit for an oscillator of frequency ω given by Eq. (10-29), if the tube develops a voltage amplification of at least 29.

Problem.

Show that adding a fourth section to the ladder of Fig. 10.23 will provide a 180° phase shift for $(\omega CR)^2 = 10/7$, and that the corresponding voltage ratio is approximately 18:1.

Chapter XI

IMPEDANCE MATCHING

11-1 Thevenin's Theorem. We are interested in the various effects that occur when an arbitrary load impedance is placed across the output terminals of an arbitrary network. To avoid discussing the (irrelevant) details of the network that drives the load, we have recourse to Thevenin's theorem. This theorem was encountered in our earlier discussions of dc circuits. We now give a simple proof of the theorem for any network of linear complex impedances.

Let the source network be represented by a "black box" having hidden internal voltage or current sources, and characterized by the open circuit voltage E_o and internal impedance Z_s as seen from the terminals.

FIG. 11.1.

Impedance Z_s is the impedance presented by the box to an external circuit *with all internal generators inactivated.* If we now connect an external voltage source E_o (Fig. 11.2a) so that the new terminals exhibit no voltage difference, and then connect a load impedance Z_L across these new terminals (Fig. 11.2b), there will be no current in Z_L. Now a linear system exhibits superposition: the current through Z_L is the sum of the currents produced by the various sources acting separately. Hence the current due to the internal sources alone equals the negative of the current due to the external source alone. Algebraically:

$$0 = I - \frac{E_o}{Z_L + Z_s} \qquad (11\text{-}1)$$

where I is the current due to internal sources alone ($e = 0$ in Fig. 11.2).

But Eq. (11-1) is identically the relation governing the behavior of the simple circuit of Fig. 11.3; since the equivalence holds for *all* Z_L, the source of Fig. 11.3 is equivalent to that of Fig. 11.1. The single-port "black box" of Fig. 11.1 is completely described, for *external* behavior, by the simplified source of Fig. 11.3.

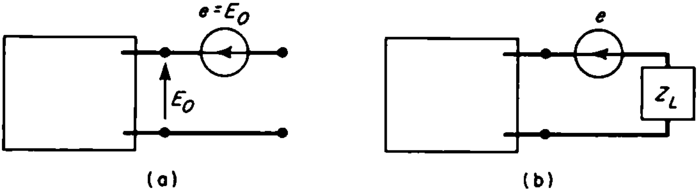


FIG. 11.2.

This result can be described verbally: The current in any impedance Z_L connected to any network is the same as if Z_L were connected to a generator whose voltage is the open circuit voltage of the network, and whose internal impedance Z_s is the impedance looking in from the terminals of Z_L, with all generators replaced by impedances equal to the internal impedances of these generators.

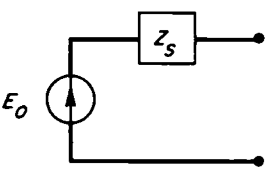


FIG. 11.3.

11-2 Output Power. Consider first the case where both Z_s and Z_L are pure resistance, R_s and R_L. The load current is $I_L = E_o/(R_s + R_L)$, the load voltage is $V_L = R_L I_L$, hence the load power is

$$V_L I_L = R_L I_L^2 = \frac{E_o^2 R_L}{(R_s + R_L)^2} = \frac{E_o^2}{R_s} \frac{R_L/R_s}{(1 + R_L/R_s)^2} \tag{11-2}$$

The load power is plotted as a function of the ratio R_L/R_s in Fig. 11.4; for $R_L = R_s$ the power has its maximum value,

$$P_{max} = E_o^2/4R_s \tag{11-3}$$

This maximum power, $E_o^2/4R_s$, is called the "available power" of the source.

For complex impedances, we have

$$\begin{aligned} Z_s &\equiv R_s + jX_s \\ Z_L &\equiv R_L + jX_L \end{aligned} \tag{11-4}$$

$$I_L = \frac{E_o}{Z_s + Z_L} = \frac{E_o}{R_s + R_L + j(X_s + X_L)}$$

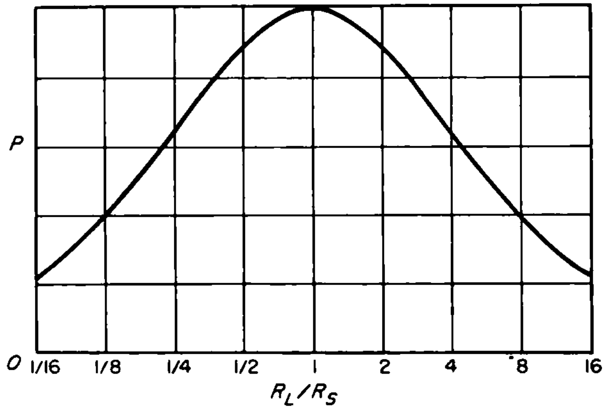


FIG. 11.4.

$$P_L = R_L|I_L|^2 = \frac{|E_o|^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

and the load power is maximized by $X_s = -X_L$, and again the maximum is the available power

$$P_{\max} = P_a = |E_o|^2/4R_s$$

This condition of $X_s = -X_L$, $R_s = R_L$ can be expressed succinctly as $Z_L = \bar{Z}_s$. The condition for maximum load power is that the load impedance be the *conjugate* of the source impedance. In physical terms, we have tuned the overall system to resonance at the frequency for which

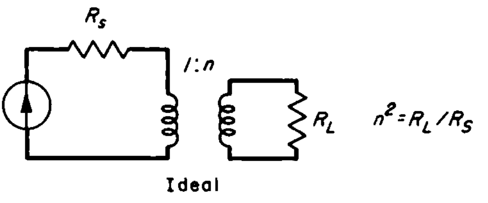


FIG. 11.5.

we desire the power maximization. For narrow-band radio-frequency communication, this is a desirable condition; for audio-frequency circuits a resonance would be distasteful and we cannot generally match the load for maximum power (unless the source is purely resistive).

When the source and load impedances are both resistive, but unequal, an ideal transformer can be inserted for matching the load to the source (Fig. 11.5).

When tuned circuits are acceptable, networks such as those in Fig. 11.6 can be used in place of a transformer to achieve the required step-up or step-down of voltage. The various configurations of Fig. 11.6 can be designed to simultaneously supply the required voltage transformation and any desired phase shift. The different configurations have different ranges of easily obtained phase shift, and are very useful for coupling radio transmitters to antenna arrays. The familiar broadcast station array of three towers, for example, obtains its directional coverage pattern from the proper combinations of amplitude and phase of the several tower currents.

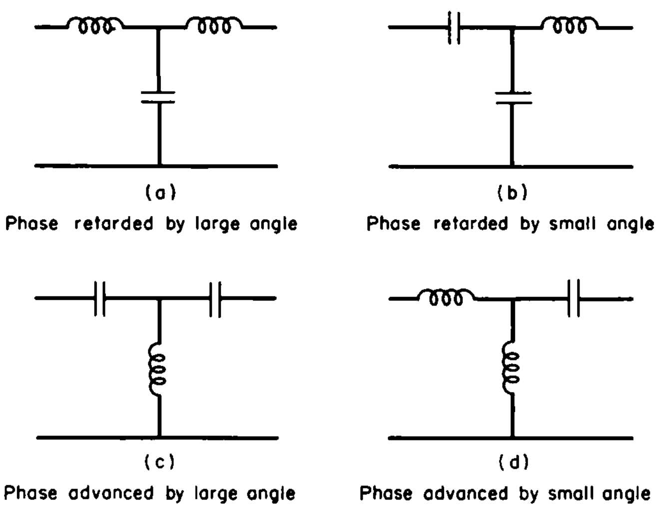


FIG. 11.6.

11-3 Matching Sections. We have seen previously that a two-port network can be described (for external behavior) in terms of the three independent impedances of its equivalent T-network or Π-network. We have also described it in terms of the general circuit parameters (A,B,C,D), of which only three are independent:

$$
\begin{aligned}
E_1 &= AE_2 - BI_2 \\
I_1 &= CE_2 - DI_2 \\
AD - BC &= 1
\end{aligned}
\qquad (11\text{-}5)
$$

There are still other sets of three parameters describing a two-port network, one of which bears directly on its impedance transforming properties.

From Eq. (11-5) we observe that if the output (port 2) is open-circuited ($I_2 = 0$), the input impedance is A/C; for port 2 shorted ($E_2 = 0$), the input impedance is B/D, etc., yielding the four relations:

$$\begin{aligned} Z_{oc1} &= A/C \qquad Z_{oc2} = D/C \\ Z_{sc1} &= B/D \qquad Z_{sc2} = B/A \end{aligned} \tag{11-6}$$

The dependence relationship is

$$\frac{Z_{oc1}}{Z_{sc1}} = \frac{AD}{BC} = \frac{Z_{oc2}}{Z_{sc2}}$$

For a given output load Z_2, ($E_2 = -Z_2I_2$), the input impedance is

$$Z_i = \frac{AZ_2 + B}{CZ_2 + D} = Z_{oc1}\frac{Z_2 + Z_{sc2}}{Z_2 + Z_{oc2}} \tag{11-7}$$

and conversely, the output impedance (or impedance looking back into port 2) is:

$$Z_o = Z_{oc2}\frac{Z_1 + Z_{sc1}}{Z_1 + Z_{oc1}} \tag{11-8}$$

where Z_1 is the impedance of the source that drives the network.

We shall see later that the most common way of using matching sections is on a matched-impedance basis; i.e., such that $Z_o = Z_2$, $Z_i = Z_1$. Note that this is not a *conjugate* match, but an equality match: the load impedance is equal to the source impedance. Substituting these relations into Eq. (11-7) and (11-8) gives

$$\begin{aligned} Z_1 &= Z_{oc1}\frac{Z_2 + Z_{sc2}}{Z_2 + Z_{oc2}} \\ Z_2 &= Z_{oc2}\frac{Z_1 + Z_{sc1}}{Z_1 + Z_{oc1}} \end{aligned} \tag{11-9}$$

The simultaneous Eqs. (11-9) have the solutions

$$Z_1^2 = Z_{oc1}Z_{sc1}$$
$$Z_2^2 = Z_{oc2}Z_{sc2}$$

These particular impedances that produce simultaneous impedance matching at both ends of the two-port are called the *image impedances* of the section:

$$\begin{aligned} Z_{I_1} &= \sqrt{Z_{oc1}Z_{sc1}} \\ Z_{I_2} &= \sqrt{Z_{oc2}Z_{sc2}} \end{aligned} \tag{11-10}$$

and can be taken as two of the three independent parameters of the network. The third parameter is the common ratio

$$r = \frac{Z_{oc1}}{Z_{sc1}} = \frac{Z_{oc2}}{Z_{sc2}} \tag{11-11}$$

In terms of these parameters, Eq. (11-5) becomes

$$E_1 = \sqrt{\frac{Z_{I_1}}{Z_{I_2}}}\sqrt{\frac{1}{r-1}}\,\{\sqrt{r}\,E_2 - Z_{I_2}I_2\}$$
$$I_1 = \sqrt{\frac{Z_{I_2}}{Z_{I_1}}}\sqrt{\frac{1}{r-1}}\left\{\frac{E_2}{Z_{I_2}} - \sqrt{r}\,I_2\right\} \tag{11-12}$$

11-4 Matched Impedance Operation. If we operate a network under image impedance conditions, as in Fig. 11.7, the input-output

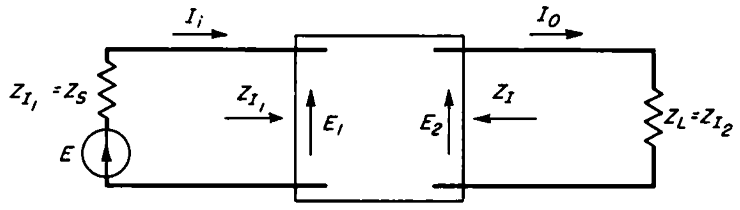


Fig. 11.7.

voltage and current ratios assume an interesting form. Since $I_2 = -I_o$ and $E_2 = E_o = I_o Z_{I_2}$, Eqs. (11-12) yield:

$$\frac{E_o}{E_i} = \sqrt{\frac{Z_{I_2}}{Z_{I_1}}}\sqrt{\frac{\sqrt{r}-1}{\sqrt{r}+1}}$$
$$\frac{I_o}{I_i} = \sqrt{\frac{Z_{I_1}}{Z_{I_2}}}\sqrt{\frac{\sqrt{r}-1}{\sqrt{r}+1}} \tag{11-13}$$

These equations show that the matched network produces an impedance transformation like that of a transformer of turns ratio $n = Z_{I_2}/Z_{I_1}$, but has an additional factor $\sqrt{(\sqrt{r}-1)/(\sqrt{r}+1)}$. If the impedance ratio gives a step-up of voltage, it gives a step-down of current, and vice versa, but the factor $\sqrt{(\sqrt{r}-1)/(\sqrt{r}+1)}$ is *not* inverted in the two equations of (11-13). One might suspect that this factor inverts if the direction of transmission is reversed; i.e., from port 2 to port 1. Let us examine this. Equations (11-12) are readily solved for

$$E_2 = \sqrt{\frac{Z_{I_2}}{Z_{I_1}}}\sqrt{\frac{1}{r-1}}\,\{\sqrt{r}\,E_1 - Z_{I_1}I_1\}$$
$$I_2 = \sqrt{\frac{Z_{I_1}}{Z_{I_2}}}\sqrt{\frac{1}{r-1}}\left\{\frac{E_1}{Z_{I_1}} - \sqrt{r}\,I_1\right\} \tag{11-14}$$

Comparing with Eq. (11-12), we see that all the 1, 2 subscripts have been interchanged, but that r appears in precisely the same way as before. This is because r is invariant to the interchange of subscripts by Eq. (11-11).

Transmission from port 2 to port 1 now gives

$$\frac{E_o}{E_i} = \sqrt{\frac{Z_{I_1}}{Z_{I_2}}}\sqrt{\frac{\sqrt{r}-1}{\sqrt{r}+1}}$$
$$\frac{I_o}{I_i} = \sqrt{\frac{Z_{I_2}}{Z_{I_1}}}\sqrt{\frac{\sqrt{r}-1}{\sqrt{r}+1}} \qquad (11\text{-}15)$$

so that the impedance transforming ratio is inverted relative to Eq. (11-13), as it should be, but the factor $\sqrt{(\sqrt{r}-1)/(\sqrt{r}+1)}$ occurs in the same way for transmission in either direction. This factor therefore represents an effect of propagation *through* the section, *regardless of direction.*

It is convenient to define the *image transfer factor* θ by the relation

$$e^{-\theta} = \sqrt{\frac{\sqrt{r}-1}{\sqrt{r}+1}} \qquad (11\text{-}16)$$

Problem.

Show that

$$\frac{\sqrt{r}-1}{\sqrt{r}+1} = \frac{Z_{I_1} - Z_{sc1}}{Z_{I_1} + Z_{sc1}} = \frac{Z_{I_2} - Z_{sc2}}{Z_{I_2} + Z_{sc2}}$$

removing the ambiguity in the sign of $\sqrt{r}$.

11-5 Cascaded Sections. If several sections are cascaded (Fig. 11.8), the output current of each section is the input current of the next section. If the resulting network is matched on an image impedance basis through-

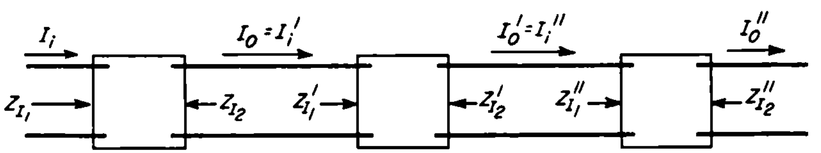


FIG. 11.8.

out, i.e., if $Z_{I_1}' = Z_{I_2}$, $Z_{I_1}'' = Z_{I_2}'$, etc., repeated applications of Eq. (11-13) (using Eq. 11-16) yield:

$$I_o = I_i\sqrt{\frac{Z_{I_1}}{Z_{I_2}}}\,e^{-\theta_1}$$

$$I_o' = I_i'\sqrt{\frac{Z_{I_1}'}{Z_{I_2}'}}\,e^{-\theta_2}$$

$$= I_i \sqrt{\frac{Z_{I_1}}{Z_{I_2}'}}\, e^{-(\theta_1+\theta_2)}$$

etc.

making the overall behavior

$$I_o'' = I_i \sqrt{\frac{Z_{I_1}}{Z_{I_2}''}}\, e^{-(\theta_1+\theta_2+\theta_3)}$$

$$E_o'' = E_i \sqrt{\frac{Z_{I_2}''}{Z_{I_1}}}\, e^{-(\theta_1+\theta_2+\theta_3)}$$

The set of cascaded sections therefore has the image impedances Z_{I_1} and Z_{I_2}'' of the end sections, and an image transfer factor given by the sum of the separate transfer factors: $\theta = \theta_1 + \theta_2 + \theta_3$.

When the individual sections have $Z_{I_1} = Z_{I_2}$ (as can be achieved by making the sections symmetrical), there is no impedance transformation. The voltages and currents at the successive junctions simplify to:

$$\begin{aligned} E_1 &= E_o e^{-\theta_1} & I_1 &= I_o e^{-\theta_1} \\ E_2 &= E_o e^{-(\theta_1+\theta_2)} & I_2 &= I_o e^{-(\theta_1+\theta_2)} \\ E_3 &= E_o e^{-(\theta_1+\theta_2+\theta_3)} & I_3 &= I_o e^{-(\theta_1+\theta_2+\theta_3)} \end{aligned} \tag{11-17}$$

etc.

In this case, the net effect of each section is to multiply both input voltage and current by $e^{-\theta}$; θ is then called the *propagation factor*.

In general, θ is a complex number, $\alpha + j\beta$. Squaring Eq. (11-16) gives

$$e^{-2\theta} \equiv e^{-2\alpha}e^{-2j\beta} = \frac{\sqrt{r} - 1}{\sqrt{r} + 1}$$

so that

$$e^{-2\alpha} = \left| \frac{\sqrt{r} - 1}{\sqrt{r} + 1} \right| \tag{11-18}$$

and 2β is the phase angle of the complex number $(\sqrt{r} - 1)/(\sqrt{r} + 1)$. For a lossless network (one made of reactance only), the impedances Z_{oc} and Z_{sc} are pure imaginary, hence from Eq. (11-11), r is real. Now Z_{oc} and Z_{sc} vary with frequency, so r may be positive for some ranges of frequency, and negative for the remaining ranges. When $r > 0$, $\sqrt{r}$ is real, and $e^{-2\alpha} < 1$, so that each section weakens or *attenuates* the signal. When $r < 0$, $\sqrt{r}$ is pure imaginary, and Eq. (11-18) states that $e^{-2\alpha} = 1$. In this case the sections are pure phase shifters. (The cascade combination is a *delay line*.) If we temporarily let $Z_{oc1} = jx$, $Z_{sc1} = jy$, we have

$$r = jx/jy = x/y$$

$$Z_{I_1} = \sqrt{(jx)(jy)} = \sqrt{-xy}$$

so that positive r is associated with *imaginary* image impedance, and negative r (giving no attenuation) is associated with *real* image impedance. Thus a lossless network can be used to match a resistance load to a resistive source, but for frequency ranges making $r > 0$, the network becomes an attenuator. This is the basic phenomenon in the design of band-pass filters.

11-6 Mismatch Effects. In Fig. 11.9a we show a section mismatched at one end, and in Fig. 11.9b a circuit that is equivalent, since there is no

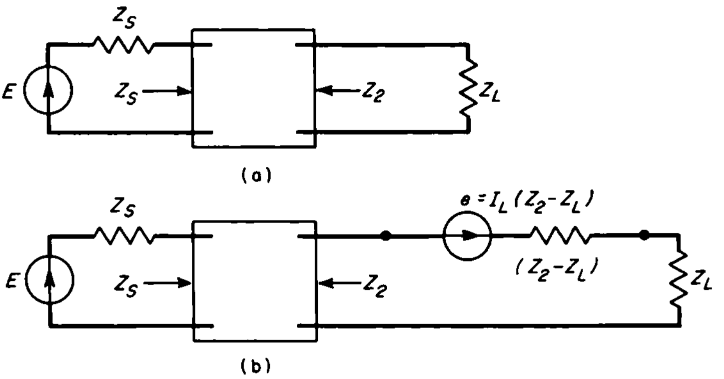


FIG. 11.9.

voltage drop across the combination that has been added. (Although drawn with resistance symbols, Z_s and Z_L are arbitrary impedances.) The arrangement of Fig. 11.9b is, however, matched at *both* ends, but has sources at both ends. By superposition, the load current is the sum of the load currents yielded by the two sources separately; these are readily computed since the new circuit is image matched. The load current has two contributions: the first is the output current I produced by the actual source (E) alone, the second is

$$\frac{e}{2Z_2} = \frac{I_L(Z_2 - Z_L)}{2Z_2}$$

Therefore,

$$I_L = I_o + I_L(Z_2 - Z_L)/2Z_2 \tag{11-19}$$

This yields

$$I_L = I_o \frac{2Z_2}{Z_L + Z_2} = I_o\left[1 - \frac{Z_L - Z_2}{Z_L + Z_2}\right] \tag{11-20}$$

which we write as

$$I_L = I_o(1 - R) \tag{11-21}$$

with

$$R \equiv \frac{Z_L - Z_2}{Z_L + Z_2}$$

(Here, R is a *number*, not a resistance.) Note that I_o is the load current that would be obtained if the load were chosen to match the network ($Z_L = Z_2$). The load voltage is

$$E_L = I_L Z_L = I_o \frac{2Z_2 Z_L}{Z_L + Z_2} = E_o \frac{2Z_L}{Z_L + Z_2} \tag{11-22}$$

where $E_o = I_o Z_2$ is the output voltage that would be obtained if the load matched the network. Equation (11-22) can also be expressed in terms of R:

$$E_L = E_o\left[1 + \frac{Z_L - Z_2}{Z_L + Z_2}\right] = E_o(1 + R) \tag{11-23}$$

If the load impedance is real, the load power is given by

$$P_L = E_L I_L = E_o I_o (1 - R^2)$$

The *input* current is similarly the sum of two contributions:

$$I_i = \frac{E}{2Z_s} + \frac{e}{2Z_2}\sqrt{\frac{Z_s}{Z_2}}\, e^{-\theta} \tag{11-24}$$

By substituting

$$e = I_L(Z_2 - Z_L)$$

$$I_L = I_o(1 - R)$$

and

$$I_o = \frac{E}{2Z_s}\sqrt{\frac{Z_s}{Z_2}}\, e^{-\theta}$$

we find that Eq. (11-24) becomes

$$I_i = \frac{E}{2Z_s}(1 - Re^{-2\theta})$$

$$= I_{im}(1 - Re^{-2\theta}) \tag{11-25}$$

where I_{im} is the input current for a matched load. The additional input current can be interpreted in terms of the matched input current I_{im} propagated through the section ($I_{im}e^{-\theta}$), multiplied by a *reflection coefficient* ($-RI_{im}e^{-\theta}$), and propagated back to the input ($-RI_{im}e^{-\theta}e^{-\theta}$).

Similarly, the voltage across the input is

$$E_i = \frac{E}{2} - \frac{e}{2}\sqrt{\frac{Z_s}{Z_2}}\, e^{-\theta} \tag{11-26}$$

which, with the same substitutions, becomes

$$E_i = E_{im}(1 + Re^{-2\theta}) \tag{11-27}$$

Hence R (defined in Eq. 11-21) can be considered as a voltage reflection coefficient, and $-R$ as a current reflection coefficient.

This reflection interpretation is real. For systems involving appreciable time delay in transmission, such as coaxial cables or other transmission lines, the reflected signals become "echoes" and are very annoying. It is primarily for this reason that impedance matching to eliminate reflections is more important than conjugate matching for maximum power. When the image impedances are real, the two conditions are equivalent.

The preceding discussion of mismatch was from the viewpoint of a mismatched *load*, i.e., the load current was compared with that obtainable if the *load* were changed to accomplish matching. If we are interested in specified load and source impedances, we should compare the load current with that obtainable by changing the *network* to accomplish matching. We consider changes of the image impedance Z_2, but not of the transfer factor θ. (If θ is imaginary, it introduces only a phase shift, which does not affect the *magnitude* of the load current. If θ is real, it is not "fair" to reduce the attenuation in the matching procedure.)

Since

$$I_o = \frac{E}{2Z_s}\sqrt{\frac{Z_s}{Z_2}}\,e^{-\theta}$$

we substitute this into Eq. (11-20), obtaining

$$I_L = \frac{E}{Z_L + Z_2}\sqrt{\frac{Z_2}{Z_s}}\,e^{-\theta} \tag{11-28}$$

But if the network were matched ($Z_2 = Z_L$), the load current would be

$$I_m = \frac{E}{2Z_s}\sqrt{\frac{Z_s}{Z_L}}\,e^{-\theta} = \frac{Ee^{-\theta}}{2\sqrt{Z_s Z_L}} \tag{11-29}$$

Substituting this into Eq. (11-28) yields

$$I_L = \frac{2\sqrt{Z_2 Z_L}}{Z_2 + Z_L}\,I_m \equiv k_2 I_m \tag{11-30}$$

so that the effect of mismatch of the network output image impedance is given by the *mismatch factor*:

$$k_2 = \frac{2\sqrt{Z_2 Z_L}}{Z_2 + Z_L} \tag{11-31}$$

This can be expressed as

$$k_2 = \sqrt{1 - R_2^2}$$

where R_2 is the output reflection coefficient given by Eq. (11-21).

If the load is connected directly to the source (Fig. 11.10), the load current is

$$I_d = \frac{E}{Z_s + Z_L} \tag{11-32}$$

Comparing this with the load current if the source were matched to the load through a *nonattenuating* section, Eq. (11-29) with $\theta = 0$,

$$I_m' = E/2\sqrt{Z_s Z_L}$$

we see that

$$I_d = \frac{2\sqrt{Z_s Z_L}}{Z_s + Z_L} I_m' \equiv k I_m' \tag{11-33}$$

with k the mismatch factor of the source and load. Cf. Eq. (11-30). If we break the circuit of Fig. 11.10 and insert our section having $Z_{I_1} = Z_s$, $Z_{I_2} = Z_2 \neq Z_L$, the load current given by Eqs. (11-30) and (11-29) can be expressed as

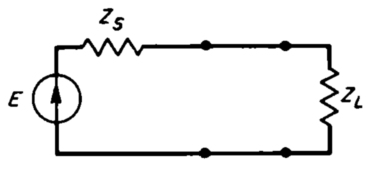


FIG. 11.10.

$$I_L = \frac{k_2 E e^{-\theta}}{2\sqrt{Z_s Z_L}} = \frac{k_2}{k} e^{-\theta} I_d \tag{11-34}$$

so that inserting the network has changed the load current by the factor $k_2 e^{-\theta}/k$. Before interpreting this insertion factor, we must consider the general case allowing for a network mismatch at *both* ends.

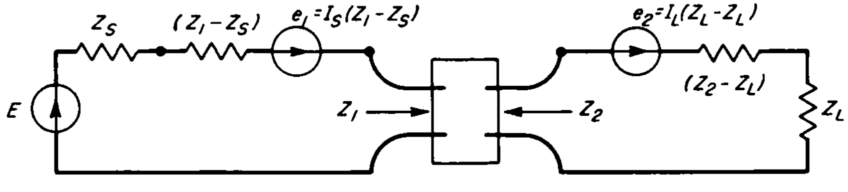


FIG. 11.11.

We shall again employ the technique used in Fig. 11.9, but this time at both ends of the network (Fig. 11.11). The sources on the left give current contributions:

$$I_{i1} = \frac{E + I_s(Z_1 - Z_s)}{2Z_1}$$

$$I_{o1} = I_{i1}\sqrt{\frac{Z_1}{Z_2}}\, e^{-\theta} \tag{11-35}$$

while the source on the right gives

$$I_{o2} = \frac{I_L(Z_2 - Z_L)}{2Z_2}$$

$$I_{i2} = I_{o2}\sqrt{\frac{Z_2}{Z_1}}\, e^{-\theta} \tag{11-36}$$

The total input and output currents are

$$I_s = I_{i1} + I_{i2} = \frac{E + I_s(Z_1 - Z_s)}{2Z_1} + I_L\frac{Z_2 - Z_L}{2Z_2}\sqrt{\frac{Z_2}{Z_1}}\, e^{-\theta}$$

$$I_L = I_{o1} + I_{o2} = \frac{E + I_s(Z_1 - Z_s)}{2Z_1}\sqrt{\frac{Z_1}{Z_2}}\, e^{-\theta} + I_L\frac{(Z_2 - Z_L)}{2Z_2} \tag{11-37}$$

The I_L and I_s terms can be collected on the left:

$$I_s\frac{Z_1 + Z_s}{Z_1} - I_L\frac{Z_2 - Z_L}{\sqrt{Z_1Z_2}}\, e^{-\theta} = \frac{E}{Z_1}$$

$$-I_s\frac{Z_1 - Z_s}{\sqrt{Z_1Z_2}}\, e^{-\theta} + I_L\frac{Z_2 + Z_L}{Z_2} = \frac{E}{\sqrt{Z_1Z_2}}\, e^{-\theta} \tag{11-38}$$

Solving for I_s and I_L yields:

$$I_s = \frac{E}{Z_1 + Z_s}(1 - R_2e^{-\theta})\sigma$$

$$I_L = \frac{E}{Z_1 + Z_s}\frac{2\sqrt{Z_1Z_2}}{Z_2 + Z_L}\,\sigma e^{-\theta} \tag{11-39}$$

where

$$\sigma \equiv \frac{1}{1 - R_1R_2e^{-2\theta}}$$

$$R_1 \equiv \frac{Z_s - Z_1}{Z_s + Z_1}$$

$$R_2 \equiv \frac{Z_L - Z_2}{Z_L + Z_2}$$

The load current can be written in terms of the direct-connected current Eq. (11-32):

$$\frac{I_L}{I_d} = \frac{k_1k_2}{k}\,\sigma e^{-\theta} \tag{11-40}$$

where k and k_2 have their previous significance, and $k_1 \equiv 2\sqrt{Z_1Z_s}/(Z_1 + Z_s)$. The quantity on the right of Eq. (11-40) is called the *insertion loss factor* of the network. If the quantity is greater than unity, there is an insertion *gain*, for the load current will have been increased by inserting the network. Since we are considering a fixed load impedance, the load power is proportional to $|I_L|^2$; the power gain (in decibels) due to insertion is

$$\begin{aligned} \text{Gain} &= 10 \log \left| \frac{k_1 k_2 \sigma e^{-\theta}}{k} \right|^2 \\ &= 20 \log \left| \frac{k_1 k_2 \sigma e^{-\theta}}{k} \right| \end{aligned} \tag{11-41}$$

or the loss is

$$\text{Loss} = -20 \log \left| \frac{k_1 k_2 \sigma e^{-\theta}}{k} \right|$$

The *interaction factor* σ has an interesting physical interpretation. If we apply the (binomial) expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1$$

to the formula for σ, we find

$$\sigma = 1 + R_1 R_2 e^{-2\theta} + (R_1 R_2)^2 e^{-4\theta} + \cdots \tag{11-42}$$

In terms of current waves, an input of 1 becomes $e^{-\theta}$ upon passage through the network, is reflected ($R_2 e^{-\theta}$) and propagates back ($R_2 e^{-\theta} e^{-\theta}$), is reflected at the input mismatch ($R_1 R_2 e^{-\theta} e^{-\theta}$), is propagated again, etc. The expansion (11-42) is the sum of the input, the additional input due to one round trip with reflection at both ends, the input due to two round trips, etc. If there is appreciable attenuation, the multiple reflections weaken rapidly. Note that if the section has its image impedance matched at *either* end, there is no reflection at that end, and $\sigma = 1$.

11-7 Hyperbolic Functions. The trigonometric functions can be expressed in exponential form as

$$\begin{aligned} \cos \theta &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin \theta &= \frac{e^{j\theta} - e^{-j\theta}}{2j} \end{aligned} \tag{11-43}$$

Similar *real* expressions define the *hyperbolic cosine* and *hyperbolic sine*:

$$\begin{aligned} \cosh \theta &= \frac{e^{\theta} + e^{-\theta}}{2} \\ \sinh \theta &= \frac{e^{\theta} - e^{-\theta}}{2} \end{aligned} \tag{11-44}$$

Squaring and adding yields the identity

$$\cosh^2 \theta - \sinh^2 \theta = 1 \tag{11-45}$$

(cf., $\cos^2 \theta + \sin^2 \theta = 1$).

The hyperbolic tangent and cotangent are defined by:

$$\tanh \theta \equiv \frac{\sinh \theta}{\cosh \theta} = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}$$

and

$$\coth \theta \equiv \frac{\cosh \theta}{\sinh \theta}$$

with the identities:

$$\begin{aligned} 1 - \tanh^2 \theta &\equiv \frac{1}{\cosh^2 \theta} \\ \coth^2 \theta - 1 &\equiv \frac{1}{\sinh^2 \theta} \end{aligned} \qquad (11\text{-}46)$$

Comparing Eq. (11-44) with Eq. (11-43) shows that the hyperbolic functions are trigonometric functions of an imaginary angle:

$$\begin{aligned} \cos j\theta &= \frac{e^{-\theta} + e^\theta}{2} = \cosh \theta \\ \sin j\theta &= \frac{e^{-\theta} - e^\theta}{2j} = j \sinh \theta \end{aligned} \qquad (11\text{-}47)$$

Using hyperbolic functions, we can solve

$$e^{-\theta} = \sqrt{\frac{\sqrt{r} - 1}{\sqrt{r} + 1}} \qquad (11\text{-}16)$$

for r:

$$e^{-2\theta} = \frac{\sqrt{r} - 1}{\sqrt{r} + 1}$$

$$\sqrt{r} = \frac{1 + e^{-2\theta}}{1 - e^{-2\theta}} = \frac{e^\theta + e^{-\theta}}{e^\theta - e^{-\theta}} = \coth \theta$$

and Eq. (11-46) yields

$$r - 1 = 1/\sinh^2 \theta$$

so that

$$\begin{aligned} \sinh \theta &= \frac{1}{\sqrt{r - 1}} \\ \cosh \theta &= \sqrt{\frac{r}{r - 1}} \end{aligned} \qquad (11\text{-}48)$$

Substituting this result into Eq. (11-14) gives the relations for an image-matched section:

$$\begin{aligned} E_2 &= \sqrt{\frac{Z_{I_2}}{Z_{I_1}}} \{E_1 \cosh \theta - Z_{I_1} I_1 \sinh \theta\} \\ I_2 &= \sqrt{\frac{Z_{I_1}}{Z_{I_2}}} \left\{E_1 \frac{\sinh \theta}{Z_{I_1}} - I_1 \cosh \theta\right\} \end{aligned} \qquad (11\text{-}49)$$

11-8 Logarithmic Expressions. Logarithms and exponentials bear an inverse relationship toward each other. By definition

$$\ln e^{\theta} = \theta \qquad (11\text{-}50)$$

where ln is read "natural logarithm of." For a complex transfer factor, we have

$$\ln e^{-\alpha - j\beta} = -\alpha - j\beta$$

The imaginary part, $-\beta$, is the phase shift (in radians) of the transfer. The real part expresses the change of magnitude; the *attenuation* (in nepers) is α. Note that the attenuation is the negative of the logarithm of the absolute value of the transfer ratio:

$$\alpha = -\ln |e^{-\theta}|$$

In practical engineering, logarithms to the base 10 are more common. By definition:

$$\log 10^x = x$$

so that

$$e = 10^{\log e}$$

$$e^{\theta} = 10^{\theta \log e}$$

and

$$\log e^{\theta} = \theta \log e = 0.4343\,\theta$$

The ratio of two powers, say input power and output power, is commonly expressed logarithmically. The power *gain* of a system is $\log P_o/P_i$; the *loss*, $\log P_i/P_o = -\log P_o/P_i$. Thus if one quarter of the input power is dissipated in a matched network,

$$P_o = 3/4\, P_i$$

and the *loss* is $P_i/P_o = \log 4/3 = 0.124$ *bels.* A smaller measure, the *decibel*, is most commonly used. In decibels (db), the loss is

$$\text{db loss} = 10 \log P_i/P_o \qquad (11\text{-}51)$$

so that in the above example the loss is 1.24 db.

If the input resistance and load resistance of a network (which may be an amplifier) are *equal,*

$$P_i/P_o = E_i^2/E_o^2 = I_i^2/I_o^2$$

and the loss is

$$10 \log P_i/P_o = 10 \log (E_i/E_o)^2 = 20 \log E_i/E_o$$

If the input and output resistances are *not* equal, the loss is *not* 20 log E_i/E_o. Unfortunately, this 20 log (voltage ratio) is sometimes used in this meaningless case. Confusion would not arise if the attenuation of a voltage and current were always expressed in nepers, but this involves natural logarithms, and engineers prefer logarithms to base 10. *By definition* and long-established usage, *decibels* refer to *power ratios*. This dilemma has been resolved by the Institute of Radio Engineers and American Standards Association, who recommend the base 10 measure of voltage or current ratios in *decilogs* (dg):

$$\text{dg attenuation} = 10 \log E_1/E_2$$

When dealing with insertion loss, we are discussing the change of current in a *fixed* load caused by insertion of a network. In this case,

$$P_1/P_2 = |I_1|^2R/|I_2|^2R = |I_1|^2/|I_2|^2$$

so that *insertion loss* can be legitimately expressed in decibels.

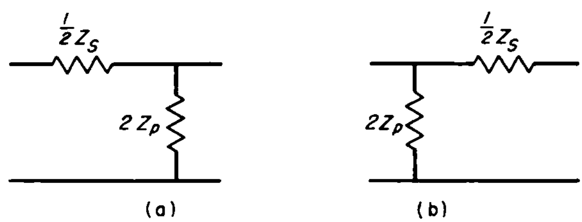


FIG. 11.12.

11-9 Symmetrical Sections. The L-sections of Fig. 11.12 can be combined to make either a symmetrical T-section or a symmetrical Π-section (Fig. 11.13). For the T-section, we have

$$\begin{aligned} Z_{oc1} &= Z_{oc2} = \tfrac{1}{2}Z_s + Z_p \\ Z_{sc1} &= Z_{sc2} = \tfrac{1}{2}Z_s + \frac{\tfrac{1}{2}Z_sZ_p}{\tfrac{1}{2}Z_s + Z_p} \\ &= \frac{\tfrac{1}{4}Z_s^2 + Z_sZ_p}{\tfrac{1}{2}Z_s + Z_p} \end{aligned} \tag{11-52}$$

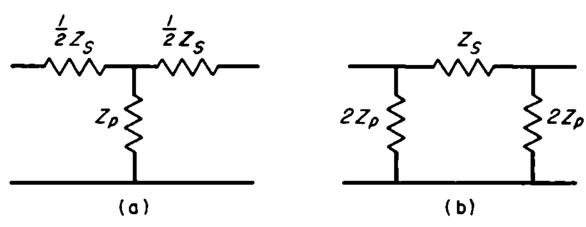


FIG. 11.13.

From Eq. (11-10):

$$Z_{I_1} = Z_{I_2} = \sqrt{Z_{oc}Z_{sc}}$$
$$= \tfrac{1}{2}\sqrt{Z_s^2 + 4Z_sZ_p} = \tfrac{1}{2}\sqrt{Z_s(Z_s + 4Z_p)} \tag{11-53}$$

From Eqs. (11-47) and (11-11):

$$\cosh\theta = \sqrt{\frac{r}{r-1}} = \sqrt{\frac{Z_{oc}}{Z_{oc} - Z_{sc}}} = \frac{Z_s + 2Z_p}{2Z_p} \tag{11-54}$$

The condition for no attenuation is that θ be pure imaginary, hence (Eq. (11-47)) that $-1 \leq \cosh\theta \leq 1$. If, for example, $Z_s = j\omega L$, $Z_p = 1/j\omega C$, we have

$$\cosh\theta = \frac{j\omega L + 2/j\omega C}{2/j\omega C} = 1 - \frac{\omega^2 LC}{2}$$

and the condition for zero attenuation becomes

$$0 \leq \omega^2 LC \leq 4$$

A chain of such sections is therefore a *low-pass filter*, since all frequencies below $\omega_c = 2\sqrt{1/LC}$ are transmitted without attenuation. $\omega_c/2\pi$ is the *cutoff* frequency.

Problem.

Let $Z_s = 1/j\omega C$, $Z_p = j\omega L$ and find the pass band.

The image impedance is

$$Z_{I_1} = Z_{I_2} = \sqrt{\frac{L}{C}}\sqrt{1 - \frac{\omega^2 LC}{4}}$$

which, in the pass band, is a pure resistance varying from $\sqrt{L/C}$ at $\omega = 0$ to 0 at the cutoff frequency. Such a filter would transmit into a load $R = \sqrt{L/C}$ nicely at very low frequencies; as ω_c is approached, reflection loss is encountered because of the impedance mismatch. Practical filters are commonly made up of a chain of symmetrical sections, plus special matching sections at the ends to make the image impedance approximately constant over the pass band. These special sections are added to reduce the reflection loss due to mismatch. (Detailed discussion of filter design is outside the scope of this book, but is readily available in advanced texts.[1])

For the Π-section

$$Z_{oc} = \frac{2Z_p(Z_s + 2Z_p)}{2Z_p + (Z_s + 2Z_p)}$$
$$Z_{sc} = \frac{2Z_pZ_s}{2Z_p + Z_s} \tag{11-55}$$

[1] *Transmission Networks and Wave Filters*, T. E. Shea, D. Van Nostrand Co., Princeton, N. J., 1929.

yielding

$$Z_I = \frac{2Z_pZ_s}{\sqrt{Z_s(Z_s + 4Z_p)}}$$

$$\cosh\theta = \frac{Z_s + 2Z_p}{2Z_p} \tag{11-56}$$

The transfer factor is the same as for the T-section, but the image impedance has the radical in the denominator. For our previous low-pass filter section, we find

$$Z_I = \sqrt{\frac{L}{C}} \bigg/ \sqrt{1 - \frac{\omega^2LC}{4}}$$

which varies from ∞ to $\sqrt{L/C}$.

11-10 Long Chains. Returning to the L-section of Fig. 11.12a, and applying Eq. (11-10) as before, we find:

$$Z_{oc1} = \tfrac{1}{2}Z_s + 2Z_p$$

$$Z_{sc1} = \tfrac{1}{2}Z_s$$

$$Z_{oc2} = 2Z_p$$

$$Z_{sc2} = \frac{2Z_pZ_s}{4Z_p + Z_s}$$

yielding

$$Z_{I_1} = \tfrac{1}{2}\sqrt{Z_s(Z_s + 4Z_p)}$$

$$Z_{I_2} = 2Z_pZ_s/\sqrt{Z_s(Z_s + 4Z_p)} \tag{11-57}$$

One side of the half-section provides an image-impedance match for a T-section; the other, for a Π-section. Hence the half-section can be inserted between a chain of T's and a chain of Π's, as Fig. 11.14. Note that the periodic structure of a T-chain, a Π-chain, or the joined chain of Fig. 11.14, is the same as that of an L-chain of the sections of Fig. 11.15. The

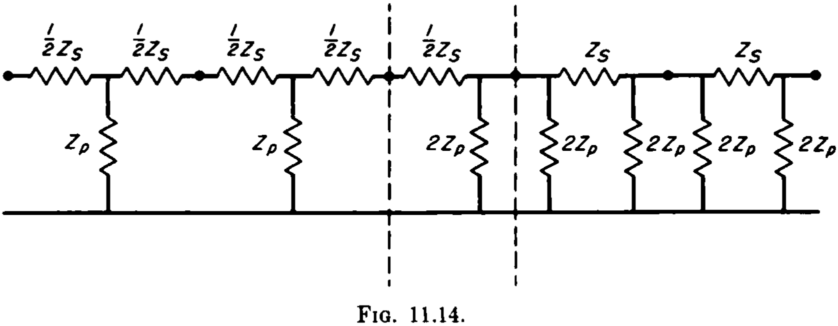


FIG. 11.14.

only distinctions among such chains are in the configurations at the ends.

For symmetrical sections, operated under image-impedance match conditions, the input impedance equals the load impedance. The section is "transparent," the source "sees" the load right through the section. This "transparency" condition can be formulated for any two-port network; the load impedance Z_k that makes the input impedance of a particular network equal to Z_k, is called the *iterative impedance* of that network.

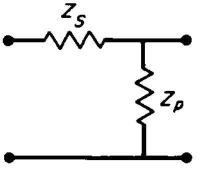


FIG. 11.15.

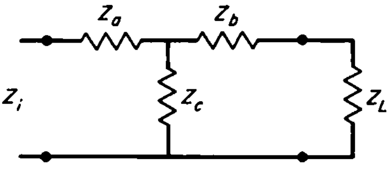


FIG. 11.16.

Consider the T-representation of an arbitrary section (Fig. 11.16). We have

$$Z_i = Z_a + \frac{(Z_b + Z_L)Z_c}{(Z_b + Z_L) + Z_c}$$

For $Z_i = Z_L = Z_k$, we find

$$Z_k^2 + Z_k(Z_b - Z_a) + Z_aZ_b + Z_aZ_c + Z_bZ_c = 0$$

If the network is lossless ($Z_a = jX_a$, $Z_b = jX_b$, $Z_c = jX_c$), this relation becomes

$$Z_k^2 + jZ_k(X_b - X_a) - (X_aX_b + X_aX_c + X_bX_c) = 0$$

making Z_k real if and only if $X_b = X_a$, i.e., if the section is symmetrical. In this case, the iterative impedance is the image impedance. The iterative impedance is of practical importance only in the symmetrical case.

A chain of sections terminated in its iterative impedance Z_k presents an input impedance Z_k for *any* number of sections. Hence any number of sections can be added to the chain without affecting its input impedance. In particular, an *unlimited* number can be added, or the *terminated* chain presents the same impedance as an infinitely long chain.

Chapter XII

DIODES

12-1 Nonlinear Circuit Elements. All the preceding chapters on circuit theory have dealt with *linear* network elements, those in which the current is proportional to the voltage; the proportionality factor was sometimes complex, but it was a *constant* for a fixed frequency. This linearity was responsible for the superposition properties of our networks: the solution for a number of source generators was the sum of the solutions for each source acting separately.

The practical applications of networks involve the addition of electronic devices, such as vacuum tubes with thermionic electron emitters (hot cathodes), semiconductor diodes (including the old-fashioned galena crystal detector), and the modern transistors. These devices are *nonlinear*. This simple statement needs elaboration.

The linear description of any natural phenomenon is only an approximation, but it may be an excellent approximation. For example, "the voltage drop in a wire is proportional to the current" is a valid statement for "reasonable" currents. It is not *strictly* true, for too large a current will overheat the wire and cause an appreciable change in its resistance, destroying the simple proportionality. For even larger currents, such as encountered in lightning strokes, other phenomena may occur. Hence by *nonlinear*, we really mean elements whose nonlinearities are evident even for "reasonable" currents, i.e., in the range of currents we expect the element to handle.

The simplest nonlinearity is that exhibited by an idealized diode:

$$I = aV \quad \text{for} \quad V > 0, \quad I = 0 \quad \text{for} \quad V < 0.$$

As long as the net voltage across such a diode has the proper polarity, the diode behaves as a simple resistor. But if the voltage ever reverses, the current cuts off. This yields phenomena entirely unobtainable with pure resistance.

Nonlinearities are both desirable and undesirable. In an amplifier, for example, nonlinearity is undesirable, since it introduces distortion. If the input is a sine wave, and the output is not proportional to the input, it will not be a sine wave. *Harmonics* are produced. On the other hand, without nonlinear elements there would be no such phenomena as the *modulation* and *demodulation* (detection) of a radio-frequency carrier wave. Without nonlinear elements, we could still have telegraphs and telephones, but no radio or television.

12-2 Thermionic Diodes. A thermionic diode comprises a heated cathode as a source of electrons, and an anode as a collector of electrons. A positive voltage applied to the anode accelerates the negatively charged electrons across the space between the cathode and anode. Increasing the anode potential increases this current. Inside the "tube," this current consists of individual electrons traversing the interelectrode space, but while they are in flight, they comprise a "cloud" of electrons between the cathode and anode. This "cloud" sets up a retarding electric field; since "like charges repel," the presence of the cloud repels newcomers from the emitting cathode. Under these conditions, the diode current is said to be *space-charge limited.* Mathematical analysis of this phenomenon, ignoring the initial emission velocity of the electrons, yields the result that the current is proportional to the three-halves power of the voltage: $I \propto V^{3/2}$. This is known as Child's law, and is the basic nonlinearity of the thermionic diode. For large voltage, the rate of emission from the cathode is a limiting factor—*saturation* is reached. The net result is a characteristic curve such as shown in Fig. 12.1.

When oxide-coated cathodes are used, the field produced by the anode potential helps the emission, and the saturation is not as pronounced (Fig. 12.2).

For semiconductor diodes (*p-n* junction), theory indicates a characteristic of the form

$$I \propto (\epsilon^{eV/kT} - 1) \tag{12-1}$$

where

ϵ is the base of natural logarithms,
e is the charge on the electron,
V is the voltage across the junction,
T is the absolute temperature,
k is Boltzmann's constant.

For normal room temperature, this becomes

$$I \propto (\epsilon^{39V} - 1)$$

which is shown in Fig. 12.3. In real diodes (Fig. 12.4) there is a very interesting and useful phenomenon indicated by the broken line. There is

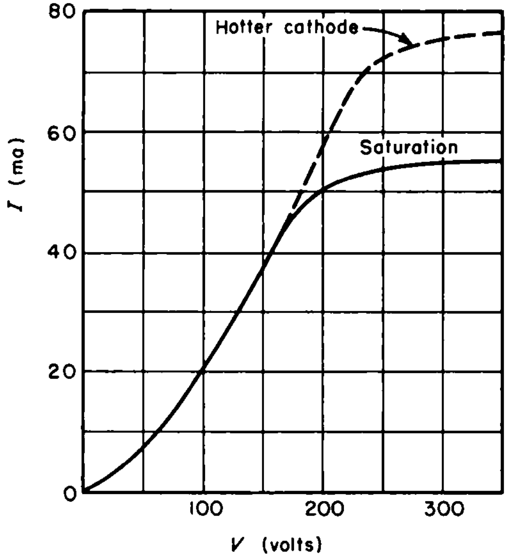


FIG. 12.1.

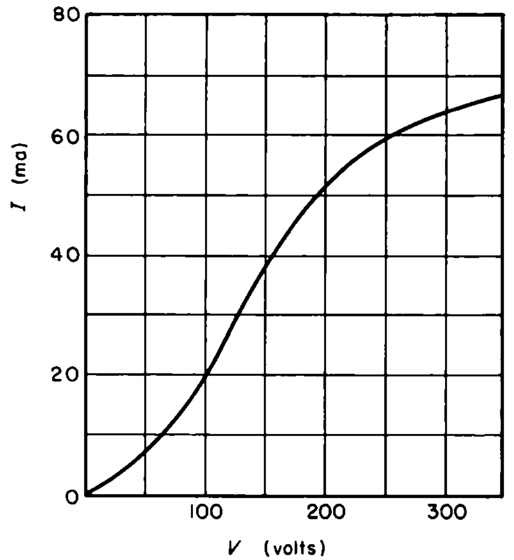


FIG. 12.2.

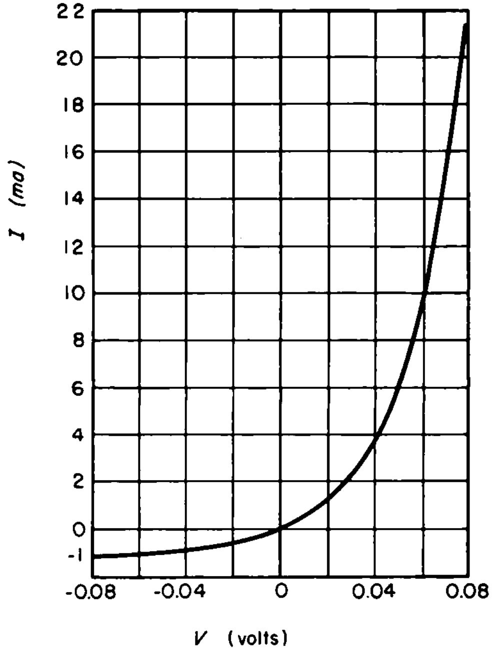


FIG. 12.3.

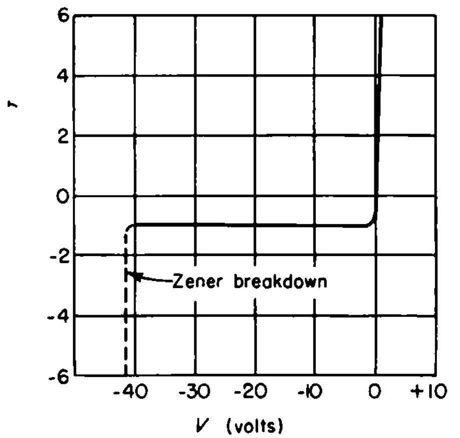


FIG. 12.4A.

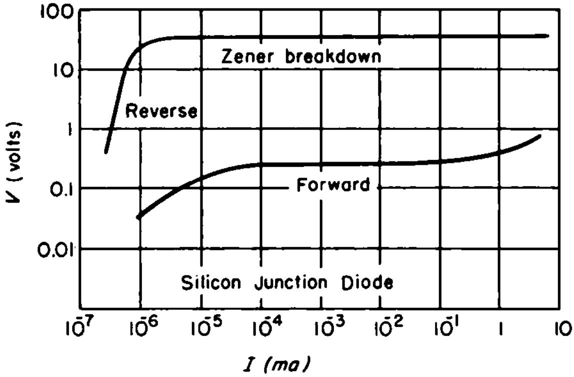


FIG. 12.4B.

a critical reverse voltage (Zener voltage) at which a complete breakdown takes place; this can be used in voltage regulating devices.

12-3 Graphical Solutions. Consider a diode connected in series with a resistor and a battery (Fig. 12.5). By Ohm's law, the voltage across the diode is $V = E - RI$, while the diode characteristic is a relation between V and I. Hence V and I are determined by two simultaneous equations, of which that required by the diode is shown graphically in Fig. 12.2 or Fig. 12.4. The Ohm's law equation can also be expressed graphically (Fig. 12.6) by the straight line $V = E - RI$. Exhibiting these "equations" on a single plot (Fig. 12.7) shows that the intersection represents the only V and I that simultaneously satisfy both requirements. We also see that for a *small* change in E, the solution changes as though the diode characteristic were a straight line tangent to the actual characteristic curve (Fig. 12.8). In fact, since the resistance requires $E = V + RI$, for changes it requires

$$dE = dV + RdI$$

or

$$\frac{dE}{dI} = \frac{dV}{dI} + R \tag{12-2}$$

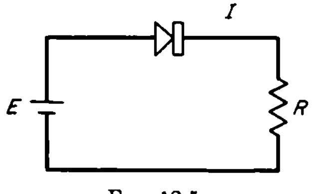


FIG. 12.5.

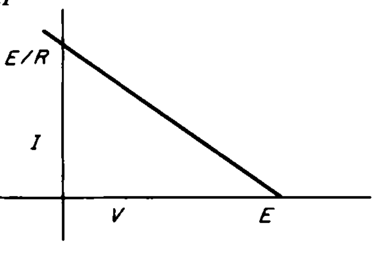


FIG. 12.6.

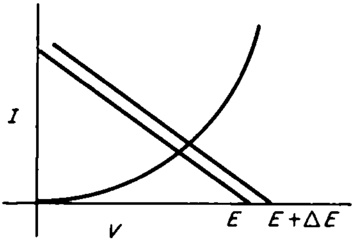


FIG. 12.7.

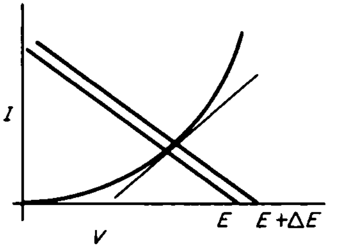


FIG. 12.8.

so that for very small changes in E, the resulting current change is the same as if the diode were a resistance dV/dI, the reciprocal of the slope (dI/dV) of the tangent of Fig. 12.8. The derivative dV/dI at any operating point is called the incremental resistance or *dynamic resistance* of the diode at that operating point. Thus for a small ac signal superposed on dc, the diode behaves approximately as a resistance dV/dI, insofar as the ac signal component is concerned. For larger ac signals, or for a better approximation, the curvature of the diode characteristic must be considered. In fact, for small excursions ΔI of I about an average value I_0, we have the Taylor series expansion of the diode characteristic:

$$V = V_0 + \left(\frac{dV}{dI}\right)_0 \Delta I + \frac{1}{2!}\left(\frac{d^2V}{dI^2}\right)_0 (\Delta I)^2 + \cdots \tag{12-3}$$

where V_0, $(dV/dI)_0$, etc. represent V, dV/dI etc. evaluated at $I = I_0$. The corresponding excursions of V are

$$\Delta V = V - V_0 = \left(\frac{dV}{dI}\right)_0 \Delta I + \frac{1}{2}\left(\frac{d^2V}{dI^2}\right)_0 (\Delta I)^2 + \cdots$$

so that for $\Delta I = a \sin \omega t$, we have

$$\begin{aligned}\Delta V &= \left(\frac{dV}{dI}\right)_0 a \sin \omega t + \frac{1}{2}\left(\frac{d^2V}{dI^2}\right)_0 a^2 \sin^2 \omega t + \cdots \\ &= \frac{a^2}{4}\left(\frac{d^2V}{dI^2}\right)_0 + a\left(\frac{dV}{dI}\right)_0 \sin \omega t - \frac{a^2}{4}\left(\frac{d^2V}{dI^2}\right)_0 \cos 2\omega t + \cdots \end{aligned} \tag{12-4}$$

since $\sin^2 \omega t = (1 - \cos 2\omega t)/2$. The curvature produces a dc term (rectification) and a second harmonic (distortion). The implications for modulation and demodulation will be discussed in a later chapter.

12-4 Rectifiers. Diodes are used as rectifiers of low-frequency ac to (1) allow the use of dc meter movements for ac measurements, and (2) to convert 60-cps commercial voltage supply to dc for use in electronic equipment. (There are other important commercial uses of diode rectifiers for electroplating, railway locomotives, etc., but these do not concern us here.)

The simple arrangement of an ideal diode (no back-current) and capacitor shown in Fig. 12.9 constitutes a *peak* rectifier. For whenever the source voltage is more positive than the capacitor voltage, the diode will pass current and charge the capacitor to the source voltage. When the source voltage is less positive than the capacitor voltage, there is no current. Hence in the ideal case the capacitor will charge up to the highest (positive) voltage ever reached by the source. If a *high-resistance* voltmeter is connected across the capacitor, there will be a slow loss of charge, which will be replenished periodically at the positive peaks of E. The dc meter will indicate the average voltage across itself (since the movement will not follow the rapid fluctuations at source frequency), hence slightly less than the peak voltage of the source. A medium-resistance voltmeter will give too low a reading, due to the loss of charge between peaks.

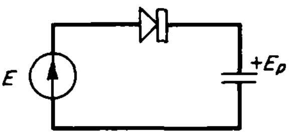


FIG. 12.9.

Rectifiers can also be used to make a so-called *averaging* voltmeter. In Fig. 12.10 we have a *full-wave bridge rectifier* supplying current to the resistance of the meter. The meter current (assuming a sinusoidal source

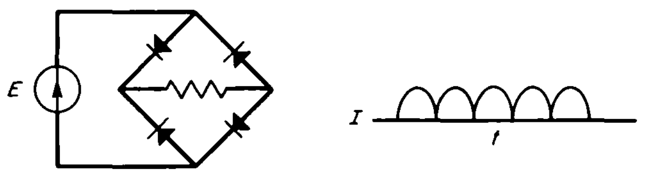


FIG. 12.10.

voltage) consists of successive half sine waves of the same polarity. Again the meter responds to the average current, which is given by

$$\langle I \rangle_{av} = \frac{1}{\pi}\int_0^{\pi} I_p \sin\theta d\theta = \frac{2}{\pi} I_p \doteq 0.637 I_p \qquad (12\text{-}5)$$

The root-mean-square (rms), or "effective" current, indicated by ac meters using thermocouples, is

$$I_{rms} = \sqrt{\langle I^2 \rangle_{av}} = \left\{\frac{1}{\pi}\int_0^{\pi} I_p{}^2 \sin^2\theta d\theta\right\}^{1/2} = \frac{I_p}{\sqrt{2}} \doteq 0.707 I_p \qquad (12\text{-}6)$$

For sine waves, the averaging meter can be calibrated to read either I_{av} or I_{rms}, but for other wave shapes the indicated rms will be incorrect, since the ratio of I_{av} to I_{rms} depends on wave form. It should be noted that I_{av}

is a customary notation, but that strictly it represents the *average absolute value* of the current. The *average* value of a sine wave is zero.

12-5 Power Supplies. The dc power supply for the various anodes in electronic equipment is commonly a rectifier-filter combination. The principles involved are adequately illustrated by the full-wave rectifier with center-tapped transformer, shown in Fig. 12.11. The voltage at the

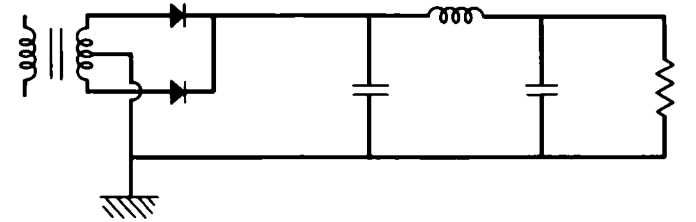


FIG. 12.11.

rectifier output is a rectified sine wave, as in Fig. 12.10. A Fourier series analysis (Chapter XIV) shows the wave to be composed of dc and even harmonics of the supply frequency:

$$E = E_p \frac{2}{\pi} \left\{ 1 + \frac{2}{3} \cos 2\omega t - \frac{2}{15} \cos 4\omega t + \frac{2}{35} \cos 6\omega t + \cdots \right\} \quad (12\text{-}7)$$

The filter could be designed as a low-pass filter with a cutoff frequency $2f$ (ordinarily 120 cps), but is ordinarily a brute-force filter; the inductance and capacitances are as large as is economical. The inductance presents a

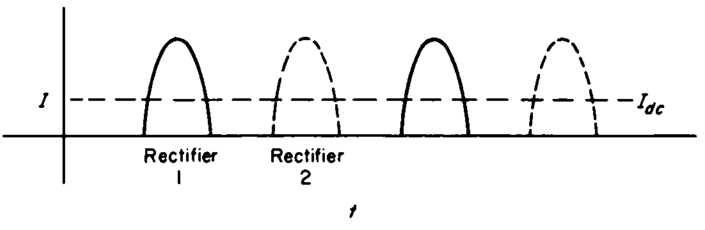


FIG. 12.12.

large series impedance to the ripple, the capacitance a small impedance. For very light loads (high load resistance, small current), the input capacitance tends to charge up to the voltage peak, and the dc output voltage approximates the peak voltage. For heavier loads, the output tends to be the average (absolute) voltage. Hence for changing loads, the output voltage changes considerably; the *regulation* is poor. In radio and television receivers, the load variation is relatively small and can be swamped out by adding a fixed *bleeder resistance* load.

The input capacitor increases the output voltage (i.e., tends to make the system a peak rectifier rather than an averaging rectifier), but has the disadvantage that the current through the rectifiers consist of short-duration large-current pulses. The rectifiers conduct only when the supply voltage

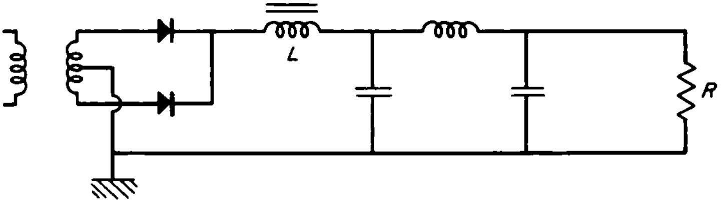


FIG. 12.13.

is above the output voltage, but the average current through the rectifiers must equal the average (dc) output current (Fig. 12.12). For anything except low power, this is "unkind" to the rectifiers and transformer.

For medium- and high-power applications, such as in transmitters, the inductance input filter of Fig. 12.13 is preferable. Since $V = L\,di/dt$, a very large inductance would force the current to be nearly constant, hence always equal to the output current, and there would be no large surges. In fact, for L infinite, the currents would be as in Fig. 12.14, with the rectifiers alternately supplying the load current. The current through either rectifier is a square-wave, 50 percent duty cycle.

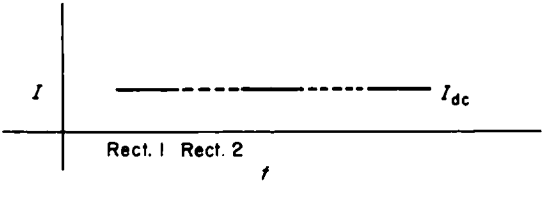


FIG. 12.14.

For L large, but finite, the current through L is primarily dc plus lowest ripple frequency (Fig. 12.15): $I \doteq I_0 + I_r \sin 2\omega t$. There is no idle-time for the rectifiers, and the regulation is good, *provided* L is large enough to maintain current at all times, i.e., provided $|I_r| \leq I_0$. For an approximate analysis, the filter capacitance can be taken as a short-circuit for the ripple. The rectifier output voltage is of the form

$$E = E_0(1 + \tfrac{2}{3} \cos 2\omega t + \cdots)$$

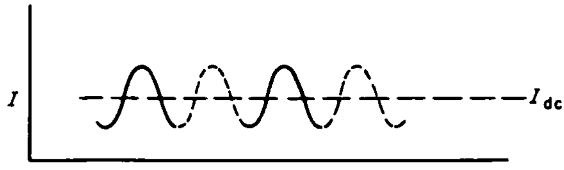


FIG. 12.15.

so the current through the inductance has the components:

$$I_0 = E_0/R$$

$$I_r = \tfrac{2}{3}E_0/2\omega L = E_0/3\omega L$$

since

$$E_L = L\,dI/dt = 2\omega L \cos 2\omega t$$

so for $|I_r| \leq I_0$, we must have

$$3\omega L \geq R$$

$$\frac{L}{R} \geq \frac{1}{3\omega} = \frac{1}{3 \cdot 2\pi \cdot 60} \doteq \frac{1}{1130} \tag{12-8}$$

where the numerical value of ω relates to the usual case of a 60-cps line supply. For all loads (R) maintaining this inequality, the filter input current is continuous and the regulation is good. It is often economical to add a bleeder resistance (in parallel with the load) to reduce the value of L needed.

Even with a reasonable bleeder, the above value of L is still apt to be uneconomically high, at least if this L is to be maintained in the presence of the dc that tends to saturate the core. Closer analysis of the problem shows, however, that since less L is needed for heavy loads than for light loads, saturation is not necessarily an evil. It is economical to design an inductor that has sufficiently large L for light loads, with a light bleeder. Heavy loads then tend to saturate the core and reduce the inductance, but the critical ratio of L to R is maintained. This type of input inductance is known as a *swinging choke.*

Another economic factor in power supply design relates to the *utilization* of the transformer. The size and cost of a transformer depend upon its power output into a resistive load. For other loads, such as reactive loads and rectifiers, the current is no longer a sine wave in phase with the voltage. The heating of a transformer, due mainly to the winding resistance, is proportional to the mean square current, irrespective of its phase relation with the voltage. Hence the rating of a transformer is given in *volt-amperes*

($E_{rms} \times I_{rms}$) rather than in *watts*. The ratio of power output to volt-amperes in the transformer is called the *utilization factor* of the transformer, in the application at hand. For linear loads, either resistive or reactive (such as lamps or induction motors, respectively), the utilization factor equals the power factor of the load. The computation of the utilization factor of a transformer in a rectifier power supply is complicated in the general case.

In the simplified case of Fig. 12.13 with the input inductance (L) *infinite*, the current through each secondary is a square wave given by I_0 half the time, and zero the remaining time. I_0 is the dc load current. The mean square current is $I_0^2/2$, hence $I_{rms} = I_0/\sqrt{2}$. The output voltage of the rectifier is

$$E = \frac{2}{\pi} E_s \left(1 + \frac{2}{3} \cos 2\omega t + \cdots \right)$$

making

$$E_0 = 2E_s/\pi, \quad \text{or} \quad E_s = \pi E_0/2$$

since for a sine wave,

$$E_{rms} = E_s/\sqrt{2}$$

we have

$$E_{rms} = E_0\pi/2\sqrt{2}$$

and for *each half* of the secondary:

$$E_{rms}I_{rms} = E_0I_0\pi/4$$

The *total* secondary rating must be twice this, or

$$\frac{\pi}{2} E_0I_0 \doteq 1.57P_0$$

The primary current is a square wave alternating between $+I_0$ and $-I_0$ (assuming unity turns ratio of the transformer). The square of the primary current is thus I_0^2 at all times; the mean square is I_0^2, and $I_{p,rms} = I_0$, while

$$E_p = E_s, \quad \text{so} \quad E_{p,rms} = E_0\pi/2\sqrt{2}.$$

The volt-ampere rating of the primary is therefore

$$E_0I_0\pi/2\sqrt{2} \doteq 1.11P_0$$

The *average* (between primary and total secondary) rating is $P_0(1.11 + 1.57)/2 = 1.34P_0$. Hence for 100 watts dc output, we would need a transformer large enough to handle 134 volt-amperes, with a primary winding capable of 111 volt-amperes, and a secondary capable of 157 volt-amperes.

This example shows some of the practical economic problems arising from nonlinearities.

The capacitor-input filter with its short-duration high-current pulses gives a much lower utilization factor, hence requires a larger transformer for the same power output.

Problem.

For an average output current of 1 ampere, compute the rms current for

(a) 2 ampere pulses, 1 second on, 1 second off.

(b) 4 ampere pulses, $\frac{1}{2}$ second on, $1\frac{1}{2}$ seconds off.

(c) 10 ampere pulses, $\frac{1}{5}$ second on, $1\frac{4}{5}$ seconds off.

Chapter XIII

AMPLIFIERS

The heart of any amplifier is an *active element*, one in which the current in one circuit is controlled by the signal (current or voltage) in another circuit. It is essential that the power *controlled by* the element is less than that needed *for control*. In the simplest case, such an active element has

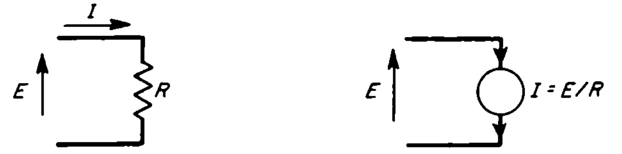


FIG. 13.1.

three terminals. Ordinary resistance may be interpreted as a two-terminal controlled generator, as in Fig. 13.1. If, however, this controlled generator has independent terminals, as in Fig. 13.2, amplification is possible. The voltage across R_2 will be $IR_2 = aR_2E_1$, which will be greater than E_1 if $aR_2 > 1$. We shall now examine some devices that have approximately these idealized properties.

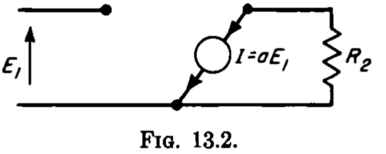


FIG. 13.2.

13-1 Thermionic Triodes. A triode is a diode with a control electrode (grid) inserted between the cathode and anode (plate). For any *fixed* grid voltage, the triode behaves like a diode and the anode current increases nonlinearly with the anode voltage. If we plot the anode current vs. anode voltage curves for various fixed grid voltages, we obtain the triode characteristics of Fig. 13.3. Since there are three variables, a *family* of curves is needed to represent the behavior of the triode. For some purposes it is more convenient to plot I_p vs. E_g for various fixed E_p; sometimes E_p vs. E_g is plotted for various fixed I_p.

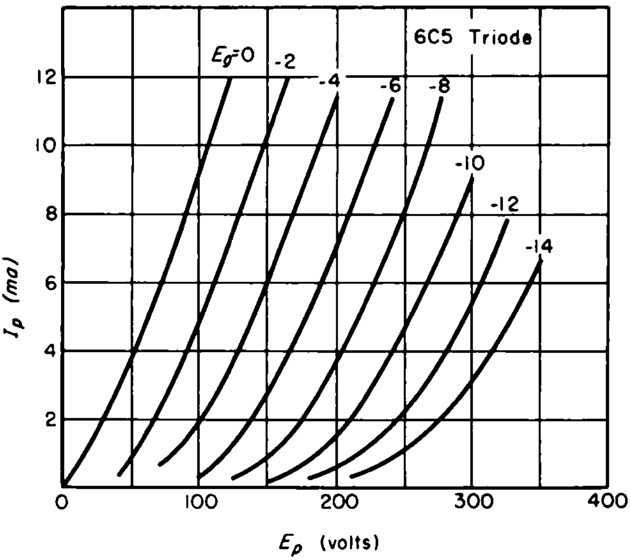


FIG. 13.3.

The curves of Fig. 13.3 can be treated graphically to find the behavior of the simple amplifier of Fig. 13.4. The anode voltage E_p must equal $E_B - RI_p$, so a load line is drawn as in Chapter XII (Fig. 13.5). The set of intersections gives the plate current and voltage, and also the voltage across the load resistance ($E_B - E_p$), for each of the various grid voltages shown. Replotting these points shows the output as a function of grid voltage. This graphical method of solution is important in large-signal problems, where distortion due to the nonlinear behavior of the tube is important. In Fig. 13.6, for example, the relative spacings of the intersections show that a 50000-ohm load would give less distortion than a 10000-ohm load, used with a 6C5 triode.

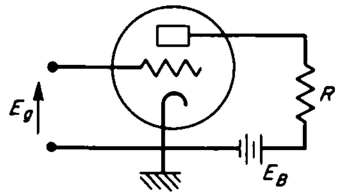


FIG. 13.4.

Analysis of the plate current characteristics of a triode shows that I_p can be expressed algebraically as a simple function of E_g and E_p:

$$I_p = k(E_g + E_p/\mu)^{3/2} \tag{13-1}$$

The triode behaves like a diode operating with an equivalent voltage $E_g + E_p/\mu$. The proportionality constant k is called the *perveance* of the tube. From Eq. (13-1), it is apparent that

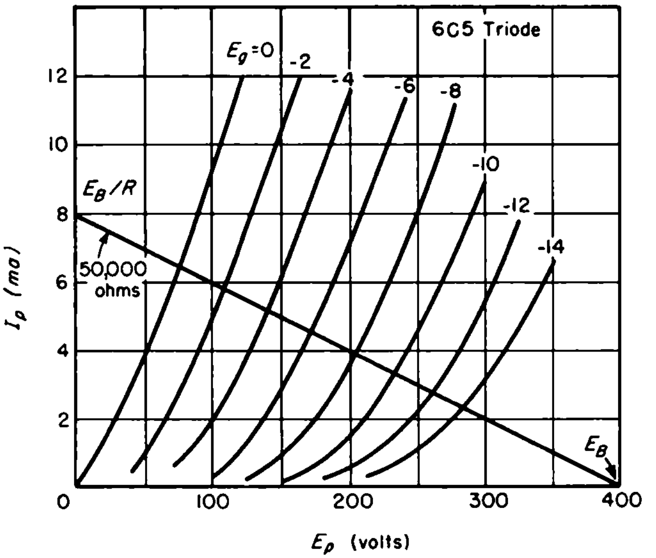


FIG. 13.5.

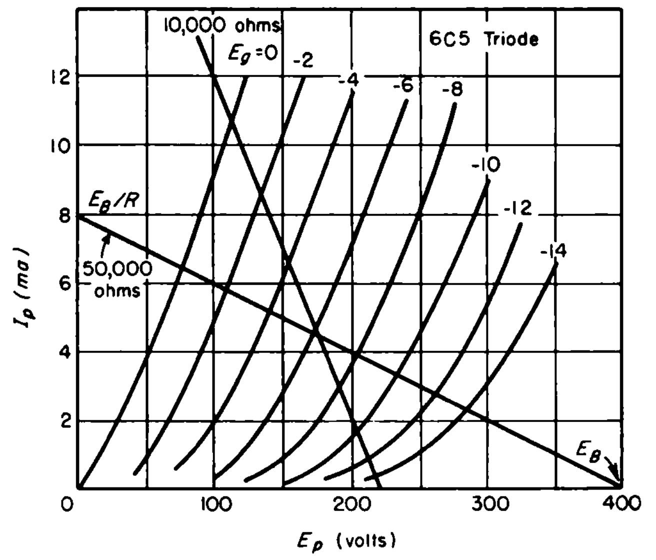


FIG. 13.6.

changes in E_g have μ times as large an effect as equal changes in E_p; μ is therefore called the *amplification factor* of the tube.

13-2 Partial Derivatives. The relation (13-1) is a particular case of the more general relation (13-2), which simply expresses the fact that I_p *is a function of two variables,* E_g *and* E_p.

$$I_p = f(E_g, E_p) \tag{13-2}$$

This means that if we know *both* E_g and E_p, we have a formula for computing I_p. If we vary *either* E_g or E_p by itself, there will be a corresponding variation in I_p. For sufficiently small changes in E_g and E_p, their effects on I_p will be essentially independent; in the limit we have the differential relationship

$$dI_p = g_m dE_g + g_p dE_p \tag{13-3}$$

where the coefficients g_m and g_p are themselves functions of the initial values of E_g and E_p that are varied by dE_g and dE_p. For changes of E_g *alone*, we have $dE_p = 0$, and

$$g_m = \left(\frac{dI_p}{dE_g}\right)_{E_p} \tag{13-4}$$

where the subscript indicates that E_p is held constant. In calculus books, such a derivative with respect to *one variable alone* is called a *partial derivative*, and written

$$g_m = \left(\frac{\partial I_p}{\partial E_g}\right)_{E_p}$$

or even just

$$g_m = \left(\frac{\partial I_p}{\partial E_g}\right)$$

where there is no ambiguity as to which variable is being held constant. The partial derivative g_m is the *mutual conductance*; "conductance" because it is the ratio of a current to a voltage, "mutual" because the current is in one circuit, the voltage in another.

Similarly,

$$g_p = \left(\frac{dI_p}{dE_p}\right)_{E_g} = \frac{\partial I_p}{\partial E_p} \tag{13-5}$$

and g_p is the *plate conductance*. Its reciprocal is the *dynamic plate resistance*:

$$r_p = 1/g_p = \left(\frac{dE_p}{dI_p}\right)_{E_g} \tag{13-6}$$

For I_p constant ($dI_p = 0$), Eq. (13-3) yields

$$\left(\frac{dE_p}{dE_g}\right)_{I_p} = -g_m/g_p \equiv -\mu \tag{13-7}$$

where μ is dimensionless (the ratio of a voltage to a voltage) and is called the *voltage amplification factor*. Note the relation implied by Eq. (13-7):

$$\mu = g_m r_p \tag{13-8}$$

13-3 Small-Signal Parameters. The preceding partial derivatives, μ, g_m, r_p, are called the small-signal parameters of the tube. They describe the response to small signals superimposed on the fixed supply voltages of the tube. For if the plate current is expanded as a Taylor series in *two*

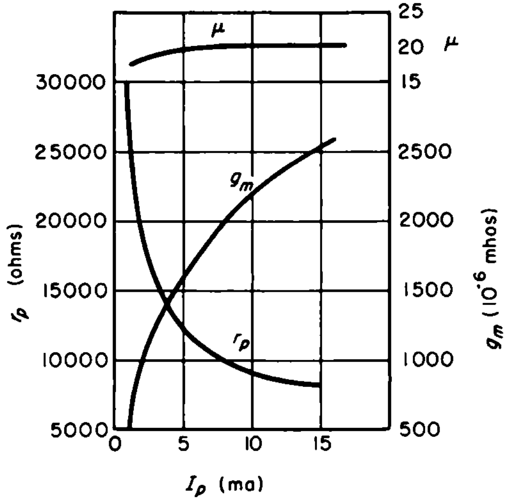


FIG. 13.7.

variables (similar to the Taylor series in one variable in Chapter XII), we have

$$I_p = I_{p0} + \left(\frac{\partial I_p}{\partial E_g}\right)_0 \Delta E_g + \left(\frac{\partial I_p}{\partial E_p}\right)_0 \Delta E_p + \frac{1}{2!}\left\{\left(\frac{\partial^2 I_p}{\partial E_g{}^2}\right)_0 (\Delta E_g)^2 + 2\left(\frac{\partial^2 I_p}{\partial E_g \partial E_p}\right)_0 (\Delta E_g)(\Delta E_p) + \left(\frac{\partial^2 I_p}{\partial E_p{}^2}\right)_0 (\Delta E_p)^2\right\} + \cdots \tag{13-9}$$

or

$$\Delta I_p = g_m \Delta E_g + g_p \Delta E_p + \tfrac{1}{2}\{\cdots\} + \cdots \tag{13-10}$$

If we let the small variations be the small alternating current and voltages, i_p, e_g, e_p, we have

$$i_p = g_m e_g + g_p e_p = g_m e_g + e_p/r_p \tag{13-11}$$

Equation (13-11) thus describes the linear approximation to the tube behavior for small signals.

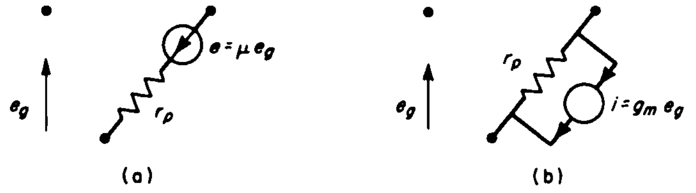


FIG. 13.8.

It must be remembered, however, that although the parameters μ, g_m, r_p, are constants insofar as the signals are concerned, they vary with the operating point of the tube. Their variation is shown, for a particular case, in Fig. 13.7. Note that the relation (13-8) is maintained.

The defining Eqs. (13-4), (13-6), and (13-7) show that the small-signal equivalent circuit of the triode is either of the (equivalent) representations of Fig. 13.8. Note the positive directions of the controlled sources.

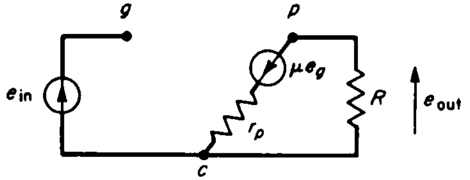


FIG. 13.9.

13-4 Basic Amplifier Circuits. The term "amplifier" implies the presence of *input* and *output*, hence a two-port network. Since our basic amplifying element is a three-terminal device, there are three pairs of terminals: *gc*, *gp*, and *pc*. Any two of these can be used as input and output ports, and these ports must have one terminal in common. This common terminal is usually called "ground." Hence our basic circuits will be *grounded cathode*, *grounded plate*, and *grounded grid*.

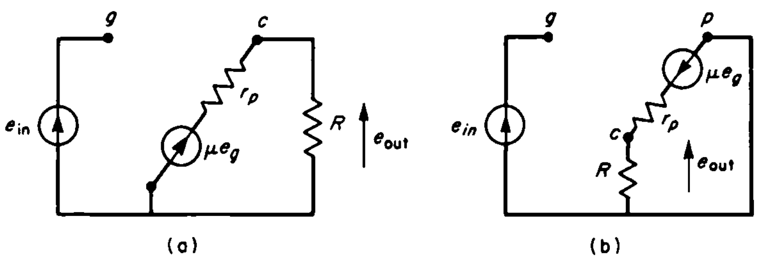


FIG. 13.10.

The grounded-cathode amplifier is shown in Fig. 13.9. The output signal is the voltage *rise* across the load resistance R. Since e_g is the voltage on the grid, *with respect to the cathode*, $e_g = e_{in}$, and the plate current is

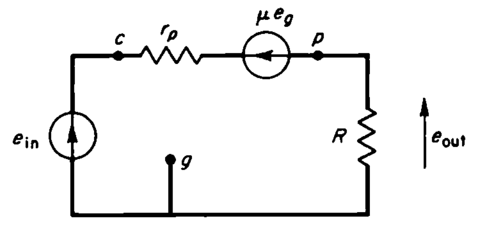


FIG. 13.11.

given by

$$i_p = \frac{\mu e_g}{r_p + R} = \frac{\mu e_{in}}{r_p + R} \tag{13-12}$$

and the output voltage is

$$e_{out} = -Ri_p = -\frac{\mu R}{r_p + R} e_{in}$$

The *amplification is*

$$\frac{e_{out}}{e_{in}} = -\frac{\mu R}{r_p + R} \tag{13-13}$$

with the sign indicating a reversal of phase in the amplifier. For $R \gg r_p$, the magnitude of the amplification approaches μ, the amplification factor of the tube.

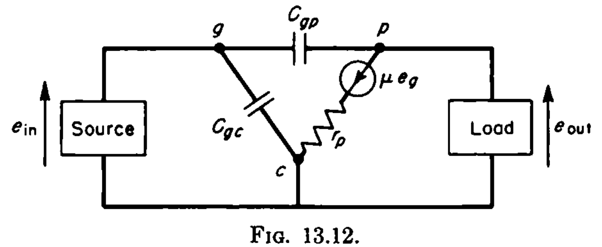


FIG. 13.12.

The grounded-plate amplifier (or *cathode follower*) of Fig. 13.10 is governed by the equations:

$$e_{out} = Ri_p$$

$$e_g = e_{in} - e_{out}$$

$$i_p = \frac{\mu e_g}{r_p + R}$$

giving

$$e_{out} = \frac{\mu R}{r_p + R}(e_{in} - e_{out})$$

which is readily solved for

$$e_{out} = \frac{\mu R}{r_p + (\mu + 1)R} e_{in} \tag{13-14}$$

For $R \gg r_p$ and $\mu \gg 1$, this reduces to $e_{out} \doteq e_{in}$, making $e_g \doteq 0$: the cathode potential "follows" changes in the grid potential. More precisely,

$$e_g = e_{in} - e_{out} = e_{in}\left[1 - \frac{\mu R}{r_p + (\mu + 1)R}\right]$$

$$= e_{in}\frac{r_p + R}{r_p + R + \mu R} \doteq \frac{e_{in}}{\mu}$$

If we consider the effect of the unavoidable internal capacitance between grid and cathode, we see that this "follower" action reduces the voltage across this capacitance by the factor μ. Hence the current through the capacitance is not $\omega C e_{in}$ (as in the case of the grounded cathode amplifier) but $\omega C e_{in}/\mu$. This is, of course, the current that would be drawn by a capacitance μ times smaller than C_{gc}. The cathode follower is therefore used to provide a large input impedance.

The grounded-grid amplifier is shown in Fig. 13.11. In this arrangement we have:

$$e_g = -e_{in}$$

$$e_{out} = -Ri_p = -R(\mu e_g - e_{in})/(r_p + R)$$

$$= \frac{R(\mu + 1)}{r_p + R} e_{in} \tag{13-15}$$

Since the input current is forced to be $-i_p$ (ignoring capacitances), the input impedance is

$$\frac{e_{in}}{-i_p} = \frac{r_p + R}{\mu + 1} = \frac{\mu}{\mu + 1}\left(\frac{1}{g_m} + \frac{R}{\mu}\right) \tag{13-16}$$

which is relatively low because of the μ dividing the R. The arrangements of Figs. 13.10 and 13.11 have very useful impedance step-down and step-up properties.

13-5 Internal Capacitances. Since the grid acts as an electrostatic shield between the plate and the cathode, the direct plate-cathode capacitance is small, compared to the plate-grid and grid-cathode capacitances. These capacitances draw current from the input source, i.e., they are responsible for an *input admittance* of the amplifier. (At very high frequencies, additional factors enter, and the input admittance becomes very complicated.) As seen from Fig. 13.12, C_{gc} is subject directly to the input voltage; it is a simple shunt capacitance across the input. The voltage across C_{gp}, however, is $e_{in} - e_{out}$, which is usually much larger than e_{in} alone. For a pure resistance load, r_l, we have

$$e_{out} = -\frac{\mu r_l}{r_p + r_l} e_{in}$$

making

$$e_{in} - e_{out} = e_{in}\left[1 + \frac{\mu r_l}{r_p + r_l}\right]$$

In terms of the *amplification* achieved by the tube, $e_{out} = -Ae_{in}$, and the voltage across C_{gp} is $e_{in}(1 + A)$. The current drawn from the source by C_{gp} is $i = e_{in}(1 + A)j\omega C_{gp}$, so that the amplification makes C_{gp} appear *larger* by the factor $(1 + A)$. This is known as the Miller effect. The resulting capacitive loading of the input circuit tends to lower the gain of the upper frequency range of an audio amplifier.

At radio frequencies, the load is not usually pure resistance, and A is therefore complex, say $a + jb$, where $b > 0$ for an inductive load and $b < 0$ for a capacitive load. The current through C_{gp} will be

$$i = e_{in}(a + jb)j\omega C_{gp} = e_{in}\omega C_{gp}(ja - b)$$

or C_{gp} presents the impedance

$$\frac{e_{in}}{i} = \frac{1}{\omega C_{gp}} \frac{1}{ja - b} = \frac{-b - ja}{b^2 + a^2} \frac{1}{\omega C_{gp}} \tag{13-17}$$

For $b > 0$, the resistive component is *negative* and tends to cancel the positive resistance in the source. For certain tuning conditions, the amplifier becomes unstable and oscillates. In early radio receivers, before the day of screen-grid tubes and pentodes, special circuit arrangements were used to *neutralize* C_{gp} and restore stability. Neutralizing circuits are still used in many transmitters.

To avoid the need for neutralizing, the grounded-grid amplifier is sometimes used. In this arrangement, C_{gc} appears directly across the input, and C_{gp} directly across the output. The capacitance from input to output is C_{pc}, which is small. Since the input impedance of the grounded-grid amplifier is low, it is useful only in special circumstances.

13-6 Tetrodes and Pentodes. The direct capacitance from grid to plate can be eliminated by separating these two electrodes with a grounded shield. This would, of course, increase the capacitance from grid to ground, but would eliminate the capacitance that is increased by the Miller effect. Since a solid shield would prevent the electrons from reaching the plate, we must be content with a perforated shield. Luckily, the shield can be almost entirely perforations (i.e., a very open grid) and still be an effective shield. Since the physics of shielding is to prevent the potential of the plate from contributing to the electric field beyond the shield, the plate potential no longer has an appreciable effect on the electron current. The equivalent diode, Eq. (13-1), is now approximated by

$$I_p = k\left(E_g + \frac{E_{sg}}{\mu_{sg}}\right)^{3/2} \tag{13-18}$$

where

$$\mu_{sg} \equiv \left(\frac{dE_s}{dE_g}\right)_{I_s}$$

By operating the screen at a *fixed* voltage (i.e., grounded for signals), there is no Miller effect between screen grid and control grid. An additional advantage is that variations of I_p are controlled by e_g alone; the output

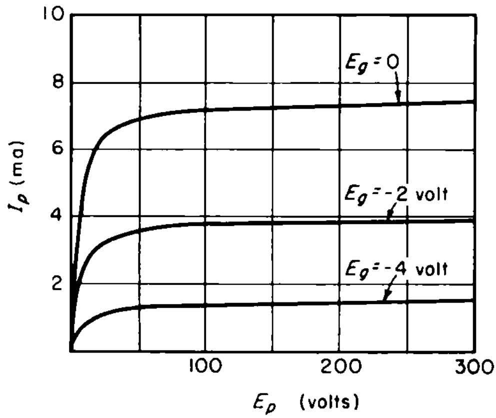


FIG. 13.13.

current I_p is essentially independent of e_p and the plate circuit acts like a *current generator*. This effect can be seen by letting r_p become very large in Eq. (13-12):

$$i_p = \frac{\mu e_g}{r_p + R} \doteq \frac{\mu e_g}{r_p} = g_m e_g \tag{13-19}$$

The early screen-grid tubes did some unexpected things. It turned out that the electrons impinging upon the plate were "knocking out" *secondary* electrons. In the triode, these secondaries returned to the plate and caused no trouble. But in a screen-grid tube, when the instantaneous plate potential is *below* the screen potential, these electrons go to the screen. This reduces the plate current and may even make it negative! By itself, secondary emission is not necessarily an evil, but it is difficult to make this behavior reproducible. The difficulty was cured by inserting still another

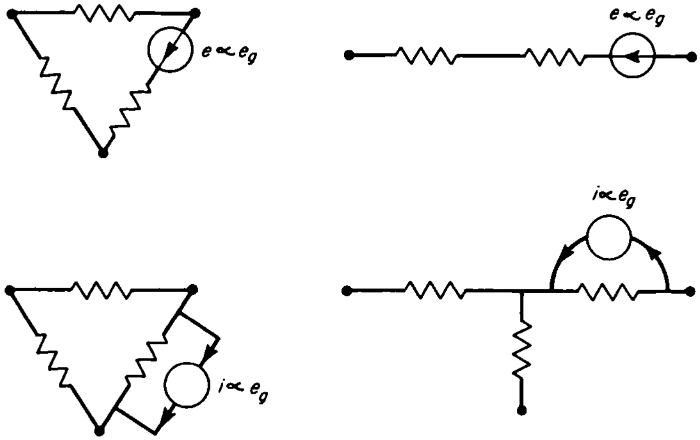


FIG. 13.14.

grid (at cathode potential) between the screen and plate. This "suppressor" grid repels the secondary electrons and returns them to the plate. Such a tube is called a pentode (five elements). Typical plate characteristics are shown in Fig. 13.13.

13-7 Transistors. The analysis of thermionic amplifiers is greatly simplified by their lack of input conductance. In Figs. 13.8 and 13.9, for example, the grid terminal is isolated. The equivalent circuit of a transistor, on the other hand, comprises *three* internal resistances and a controlled source. In some respects it is similar to a triode with added re-

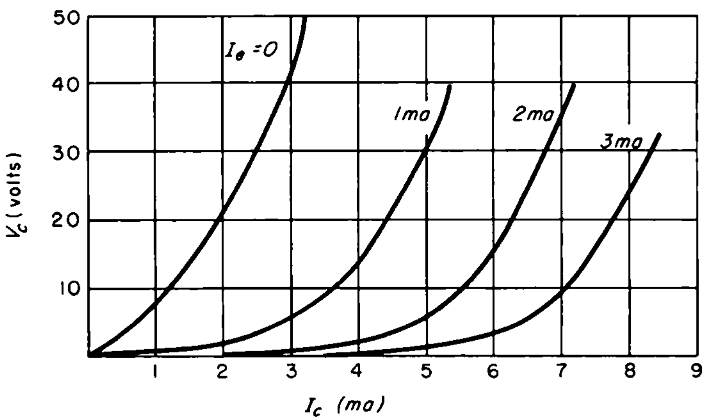


FIG. 13.15.

sistance between terminals. Such a shunted triode can be represented as in Fig. 13.14, where the four circuits are all equivalent. Note, however, that in all cases, the output of the controlled source is proportional to a *voltage*. These same circuits, with the controlled source proportional to a *current*, would represent a transistor. This duality is exhibited by the characteristic curves.

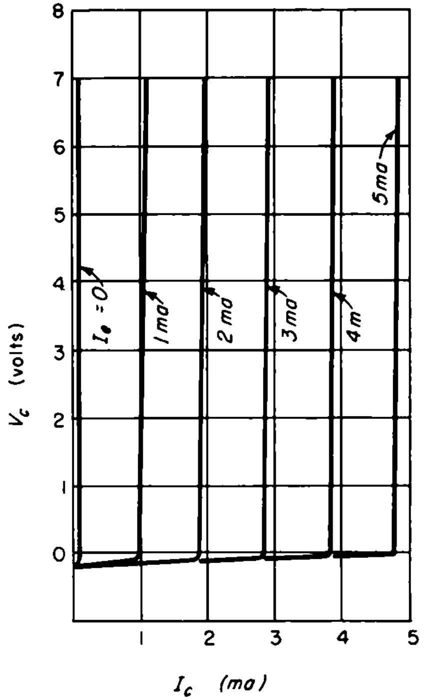


FIG. 13.16.

For the triode, we plotted "plate" *current* against "plate" *voltage* for various grid voltages (Fig. 13.3). For a *point-contact transistor*, plotting "collector" *voltage* against "collector" *current* for various "emitter" *currents* yields the similar curves of Fig. 13.15. The *junction transistor*, plotted in the same manner, yields the curves of Fig. 13.16. These resemble pentode characteristics, with the vertical and horizontal axes interchanged. If we interchange I_c and V_c (Fig. 13.17), the direct resemblance to pentode curves shows that the complete duality implied by Fig. 13.15 is misleading. The essential difference between triodes and transistors is the change from voltage control to current control.

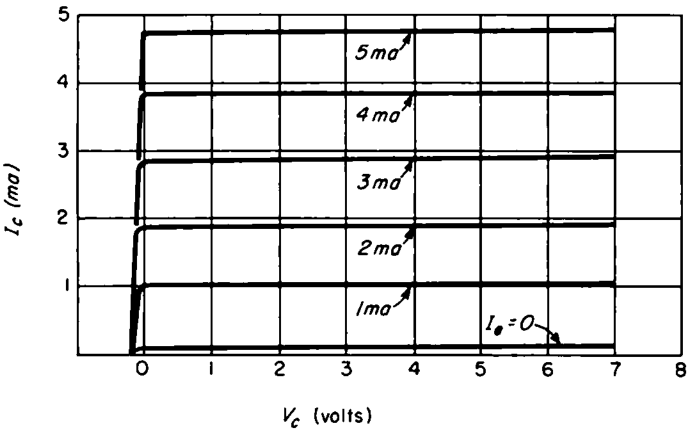


FIG. 13.17.

In the triode, the interesting parameters were the partial derivatives with respect to the controlling parameter E_g:

$$\left(\frac{dE_p}{dE_g}\right)_{I_p} = \mu,$$

the voltage amplification factor, and

$$\left(\frac{dI_p}{dE_g}\right)_{E_p} = g_m,$$

the mutual conductance. For transistors, the control parameter is I_e, and we have

$$-\left(\frac{dI_c}{dI_e}\right)_{V_c} = \alpha,$$

the current amplification factor, and

$$\left(\frac{dV_c}{dI_e}\right)_{I_c} = r_m,$$

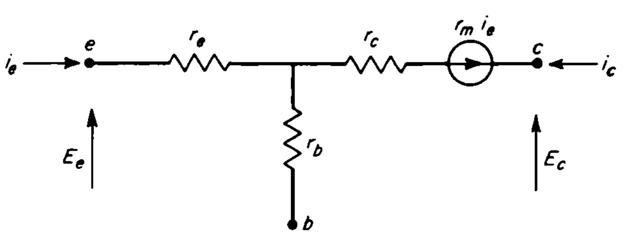


FIG. 13.18.

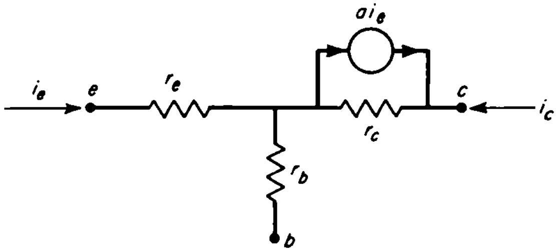


FIG. 13.19.

the mutual resistance. For our basic representation, we take the T-net of Fig. 13.18, where the resistances have the following names:

r_e = emitter resistance
r_b = base resistance
r_c = collector resistance

Changing the voltage source to the equivalent current source by Thevenin's theorem yields Fig. 13.19, where a has been written for r_m/r_c. The presence of the base resistance (r_b) prevents a from being precisely equal to α, the current amplification factor. For by definition,

$$\alpha = -\left(\frac{dI_c}{dI_e}\right)_{E_c} = -\left(\frac{i_c}{i_e}\right)_{e_c=0}$$

i.e., α is (the negative of) the short-circuit current amplification. In Fig. 13.20, the input current divides between r_b and r_c, thus contributing $-i_e r_b/(r_b + r_c)$ to the collector current. By superposition, the total collector current is

$$i_c = -\frac{i_e r_b}{r_b + r_c} - \frac{r_m i_e}{r_b + r_c} = -i_e \frac{r_m + r_b}{r_c + r_b}$$

yielding

$$\alpha = \frac{r_m + r_b}{r_c + r_b} \doteq \frac{r_m}{r_c} = a \qquad (13\text{-}20)$$

for r_b small.

13-8 Amplifier Configurations. We again have three choices for

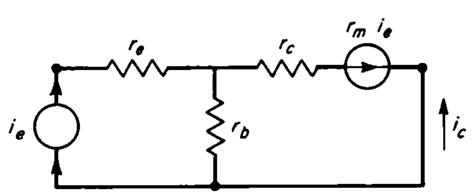


FIG. 13.20.

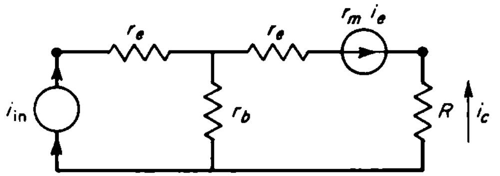


FIG. 13.21.

the ground, or terminal, common to the input and output ports. For simplicity, we shall again ignore the internal impedance of the signal source, and consider the signal to come from an ideal generator.

In the grounded-base amplifier (Fig. 13.21) we have

$$i_e = i_{in}$$

$$i_{out} = i_c = -i_{in}\frac{r_b}{r_b + r_c + R} - \frac{r_m i_e}{r_b + r_c + R}$$

yielding

$$\frac{i_{out}}{i_{in}} = -\frac{r_m + r_b}{r_c + r_b + R} \doteq -\alpha \tag{13-21}$$

(for small R).

In the grounded-emitter configuration (Fig. 13.22), the dependence of the internal generator upon the emitter current is awkward, since the emitter current is not a convenient variable. We know, however, that $i_e = -i_b - i_c$ making $r_m i_e = -r_m i_b - r_m i_c$ so that Fig. 13.22 is equivalent to Fig. 13.23 (note the reversal of the generator sense). Adding a load resistance R, gives the equations

$$i_b = i_{in}$$

$$i_{out} = i_c = -i_{in}\frac{r_e}{R + r_e + r_c - r_m} + \frac{r_m i_b}{R + r_e + r_c - r_m}$$

so that

$$\frac{i_{out}}{i_{in}} = \frac{r_m - r_e}{r_c - r_m + R + r_e} \doteq \frac{r_m}{r_c - r_m} = \frac{a}{1 - a} \tag{13-22}$$

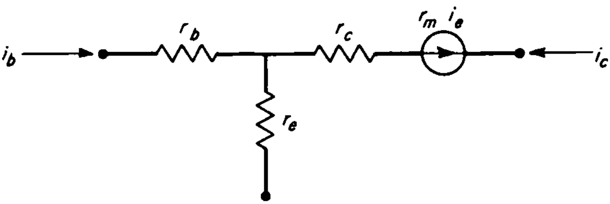


FIG. 13.22.

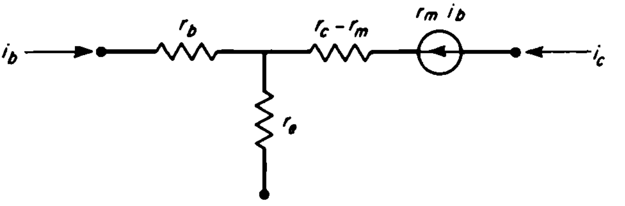


Fig. 13.23.

For the grounded-collector amplifier, it is still convenient to use the internal configuration of Fig. 13.23, giving Fig. 13.24. (We could use the internal configuration of Fig. 13.21, but then the internal generator would be expressed in terms of the *output* current instead of the input *current.* This would lead to the same results, but lacks intuitive appeal.) The equations are:

$$i_{\text{in}} = i_b$$

$$i_{\text{out}} = i_e = -\frac{i_{\text{in}}(r_c - r_m)}{R + r_e + r_c - r_m} - \frac{r_m i_b}{R + r_e + r_c - r_m}$$

yielding

$$\frac{i_{\text{out}}}{i_{\text{in}}} = -\frac{r_c}{r_c - r_m + R + r_e} \doteq \frac{1}{a - 1} \qquad \text{(13-23)}$$

The complete equations for voltage amplification, current amplification, power gain, input and output resistance, are given by Shea[1] for the general case where the signal source has internal resistance. The relative magnitudes of r_e, r_b, and r_c are also discussed by Shea, and the various resultant approximations to the exact formulas.

13-9 Generalized Amplifiers. Although the triode amplifier was simple because of the lack of a full network of interconnecting resistances, the presence of this complication in transistors makes it worthwhile to discuss a general two-port amplifier.

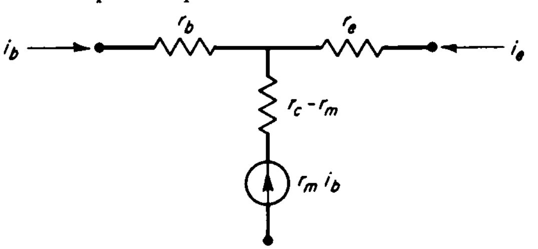


Fig. 13.24.

[1] *Principles of Transistor Circuits*, R. F. Shea, Editor (John Wiley & Sons, New York), 1953.

The behavior of the "black box" of Fig. 13.25 can be described in various ways, as was found for passive two-port networks. We can choose the voltages as independent variables, giving the equations

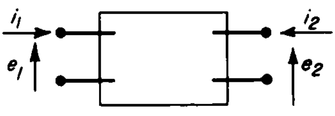


FIG. 13.25.

$$i_1 = g_{11}e_1 + g_{12}e_2$$
$$i_2 = g_{21}e_1 + g_{22}e_2 \tag{13-24}$$

or the currents can be the independent variables

$$e_1 = r_{11}i_1 + r_{12}i_2$$
$$e_2 = r_{21}i_1 + r_{22}i_2 \tag{13-25}$$

or the *input* variables can be the independent variables

$$e_2 = Ae_1 - Bi_1$$
$$i_2 = Ce_1 - Di_1 \tag{13-26}$$

In the first case, Eq. (13-24), the cross-resistances g_{12} and g_{21} were equal for passive networks. For active networks, $g_{12} \neq g_{21}$. Consider the circuit of Fig. 13.26. The principle of superposition tells us that the current

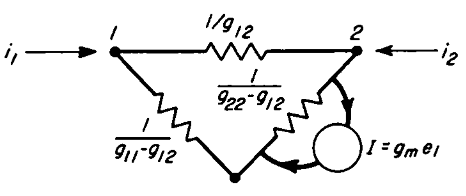


FIG. 13.26.

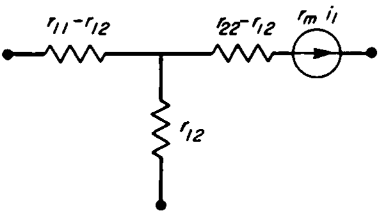


FIG. 13.27.

i_2 is that current we would find *without* the source I, *plus* the current $g_m e_1$. The circuit equations are therefore

$$i_1 = g_{11}e_1 + g_{12}e_2$$
$$i_2 = (g_{12} + g_m)e_1 + g_{22}e_2 \tag{13-27}$$

or

$$g_m \equiv g_{21} - g_{12}$$

Similarly, for Fig. 13.27, the controlled source increases the voltage e_2 by the amount $r_m i_1$; the circuit is described by Eq. (13-25), with $r_m = r_{21} - r_{12}$.

In the input-output relations, Eqs. (13-26), we have $AD - BC = 1$ for a passive network. In the active case, where $AD - BC \neq 1$, we can attribute this inequality to any one of the four parameters. For example,

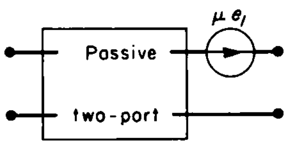


FIG. 13.28.

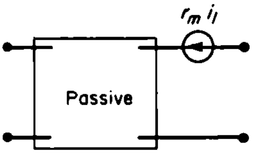


FIG. 13.29.

we can let

$$A = A_0 + \mu$$

with

$$A_0D - BC = 1$$

making

$$A_0 = \frac{1 + BC}{D}$$

and

$$\mu = A - \frac{1 + BC}{D}$$

In this case, the inequality is attributed to an additional component of e_2, given by μe_1, as in Fig. 13.28.

Similarly, we can take

$$B = B_0 + r_m$$

$$AD - B_0C = 1$$

with the equivalent circuit of Fig. 13.29, where r_m is a *transresistance* like the "mutual" resistance of a transistor.

Choosing

$$C = C_0 - g_m$$

$$AD - BC_0 = 1$$

yields the arrangement of Fig. 13.30 having the *transconductance* g_m (like a triode).

Finally,

$$D = D_0 + \alpha$$

$$AD_0 - BC = 1$$

yields Fig. 13.31 with a current amplification factor.

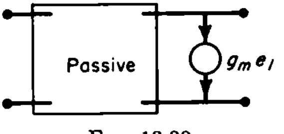


FIG. 13.30.

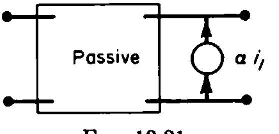


FIG. 13.31.

It is apparent from Eq. (13-26) that the short-circuit current amplification ($-i_2/i_1$ with $e_2 = 0$) can be found from

$$0 = Ae_1 - Bi_1, \quad e_1/i_1 = B/A$$

$$-\frac{i_2}{i_1} = -C\frac{e_1}{i_1} + D = \frac{-CB}{A} + D = \frac{1}{A} + \alpha$$

Hence $1/A$ is the current "amplification" due to impedance transformation in the passive network, and α is the active current amplification. The other three cases can be analyzed similarly.

In fact, if we define the four generalized amplifications:

$$\text{Short-circuit current amplification} \equiv -\left(\frac{i_2}{i_1}\right)_{e_2=0} = \alpha_s$$

$$\text{Short-circuit transconductance} \equiv -\left(\frac{i_2}{e_1}\right)_{e_2=0} = g_s$$

$$\text{Open-circuit voltage amplification} \equiv \left(\frac{e_2}{e_1}\right)_{i_2=0} = \mu_o$$

$$\text{Open-circuit transresistance} \equiv \left(\frac{e_2}{i_1}\right)_{i_2=0} = r_o$$

we have

$$AD - BC = A\alpha_s = Bg_s = Cr_o = D\mu_o \tag{13-28}$$

for a general two-port network. For a passive network, $AD - BC \equiv 1$, giving $\alpha_s = 1/A$, etc. For an active network, these "amplifications" are changed to:

$$\begin{aligned} \alpha_s &= \frac{1}{A} + \alpha \\ g_s &= \frac{1}{B} + g_m \\ r_o &= \frac{1}{C} + r_m \\ \mu_o &= \frac{1}{D} + \mu \end{aligned} \tag{13-29}$$

where α, g_m, r_m, and μ refer to the equivalent generators in Figs. 13.28–13.31. Thus any active (linear) two-port can be expressed as a passive two-port *plus* any one of the four types of controlled source.

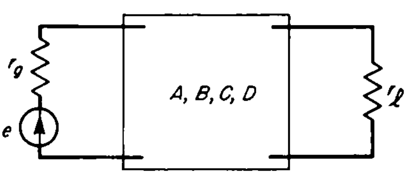


FIG. 13.32.

The active two-port can also be compared with a passive two-port in terms of its input and output resistances. This is useful since the input and output resistances of an amplifier are often required to be known for load and source matching. Without exhibiting any active elements explicitly, we take amplifier, with source and load, as in Fig. 13.32. In Chapters IV and XI, we found the input, output, and image impedances of a *passive* two-port. We shall now consider a two-port for which all the impedances are resistive, and A, B, C, D are *independent*. The same procedures as before, without using $AD - BC = 1$, yield:

$$\text{Input resistance} \equiv r_1 = \frac{B + Dr_l}{A + Cr_l}$$
$$\text{Output resistance} \equiv r_2 = \frac{Ar_g + B}{Cr_g + D} \tag{13-30}$$

For the special cases of $r_l = 0, r_l = \infty, r_g = 0, r_g = \infty$, we have four "open-circuit" and "short-circuit" input (output) resistances:

$$r_{1s} = B/A \qquad r_{2s} = B/D$$
$$r_{1o} = D/C \qquad r_{2o} = A/C$$

Since $A/D = r_{2s}/r_{1s} = r_{2o}/r_{1o}$, these four resistances are not independent. As independent parameters of the network, we can choose any *three* input resistances (from r_{1o}, r_{1s}, r_{2o}, r_{2s}) *plus* any *one* member of the set: A, B, C, D, μ, α, g_m, r_m. There is indeed a great deal of flexibility in the choice of independent parameters to describe a complete amplifier. This great freedom of choice is the reason so many writers, particularly on transistors, seem to have different approaches to circuit theory. The choice of parameters is one of convenience for the type of problem of immediate interest.

The image impedances are:

$$R_1^2 = r_{1s}r_{1o}$$
$$R_2^2 = r_{2s}r_{2o}$$

The input and output impedances under loaded conditions, Eq. (13-30), are readily expressed in terms of the open-circuit and short-circuit resistances:

$$r_1 = \frac{D}{C}\frac{r_l + B/D}{r_l + A/C} = r_{1o}\frac{r_l + r_{2s}}{r_l + r_{2o}}$$
$$r_2 = \frac{A}{C}\frac{r_g + B/A}{r_g + D/C} = r_{2o}\frac{r_g + r_{1s}}{r_g + r_{1o}} \tag{13-31}$$

13-10 Gain. The *power gain* of an amplifier is the ratio of the power output to the power input, under the particular operating conditions considered. This is not a figure-of-merit, because mismatch losses may be such that the power in the load is less than could be obtained with a passive network that matched the source and load. That is, the *insertion gain* of the amplifier may be less than the insertion gain of an image-matching passive network. Now the best passive network can deliver to the load only the *available power* of the source:

$$P_a = e^2/4r_g = i^2r_g/4$$

where e is the open-circuit voltage of the source, i is the short-circuit current of the source, and r_g is the internal (generator) resistance of the source. To be useful, the amplifier power output, P_o, should exceed the available source power, P_a. In fact, a commonly used measure of amplifier capability, into a *given* load, is the ratio P_o/P_a. This ratio is called the *transducer gain* of the two-port.

Since

$$P_o = i_2^2r_l$$

$$P_a = e^2/4r_g = i_1^2(r_1 + r_g)^2/4r_g$$

we have

$$G_t = \frac{P_o}{P_a} = \frac{4r_gr_l}{(r_1 + r_g)^2}\left(\frac{i_2}{i_1}\right)^2 \tag{13-32}$$

and i_2/i_1 is the current amplification under the load conditions considered.

When the source is matched to the amplifier input resistance, the input power to the amplifier *is* the available source power, P_a. The transducer gain G_t is therefore the actual power gain when the amplifier is *input-matched* (we have said nothing about matching the amplifier at the output).

Now if we output-match the amplifier (ignoring the input conditions), the amplifier will deliver *its* available output power, P_{oa}. We have $P_{oa} = i_{os}^2r_o/4$, where i_{os} is the short-circuit output current. But $i_{os} = -i_{1s}\alpha_s$, where α_s is the short-circuit current amplification. Therefore

$$P_{oa} = \tfrac{1}{4}r_oi_{1s}^2\alpha_s^2$$

Now the available source power can also be expressed in terms of i_{1s}, the input current for output shorted:

$$P_a = \frac{e^2}{4r_g} = \frac{i_{1s}^2(r_g + r_{1s})^2}{4r_g}$$

yielding the *available power gain:*

$$G_a \equiv \frac{P_{oa}}{P_a} = \frac{r_or_g}{(r_g + r_{1s})^2}\alpha_s^2 \tag{13-33}$$

The greatest power gain that can be had from the amplifier is obtained when both input and output are simultaneously matched, i.e., when the amplifier is image-matched. The resulting gain is called the *maximum available* *power gain*; a formula is readily found by setting $r_o = R_2$, $r_g = R_1$, in G_a:

$$G_{am} = \frac{R_1R_2}{(R_1 + r_{1s})^2}\alpha_s^2 \tag{13-34}$$

In terms of the A, B, C, D parameters, this can be readily found to be:

$$G_{am} = \frac{-\dot{e}_2\dot{i}_2}{\dot{e}_1\dot{i}_1}\text{ (image matched)} = (\sqrt{AD} - \sqrt{BC})^2$$

13-11 Feedback. Consider first the abstract case of an amplifier of amplification A_0: $e_2 = A_0e_1$. Let a fraction β of the output voltage be fed back to the input, i.e., a connection is made such that the input voltage is $e_1 = e + \beta e_2$, where e is the source voltage. These two equations together give

$$e_2 = A_0e_1 = A_0(e + \beta e_2)$$

$$e_2(1 - A_0\beta) = A_0e$$

and the amplification is

$$A \equiv \frac{e_2}{e} = \frac{A_0}{1 - A_0\beta} \tag{13-35}$$

If β is small ($|A_0\beta| < 1$), A is greater than A_0 if $\beta > 0$, and A is less than A_0 for $\beta < 0$. These two cases are called *positive* and *negative* feedback, respectively. In complicated feedback amplifiers, selective circuits are used, making β change with frequency. In fact β commonly changes sign so that the feedback may be positive at some frequencies and negative at others. In general, β becomes a complex number, so that positive and negative lose their simple meaning; a complete discussion of feedback amplifiers is outside the scope of this book.

Note that for A_0 very large, making $|A_0\beta| \gg 1$, the amplification with feedback is

$$A \doteq -1/\beta$$

and is therefore independent of A_0 for A_0 sufficiently large. This is of advantage in vacuum tube voltmeters and similar devices where we wish the amplification to be independent of tube age and replacement.

Now for a specific case (Fig. 13.33). An unbypassed cathode bias resistor is added to a simple triode amplifier. The grid-cathode voltage is

$$e_g = e - r_c\dot{i}_p$$

making

$$\dot{i}_p = \frac{\mu e_g}{r_p + r_c + R} = \frac{\mu(e - r_c\dot{i}_p)}{r_p + r_c + R}$$

Solving for i_p:

$$i_p = \frac{\dfrac{\mu e}{r_p + r_c + R}}{1 + \dfrac{\mu r_c}{r_p + r_c + R}}$$

and the output voltage becomes

$$Ri_p = e\,\frac{\dfrac{\mu R}{r_p + r_c + R}}{1 + \dfrac{\mu R}{r_p + r_c + R}\cdot\dfrac{r_c}{R}} = e\,\frac{A_0}{1 + A_0\dfrac{r_c}{R}}$$

where A_0 is the amplification without feedback—*not* the amplification for $r_c = 0$ but for the source connected directly between grid and cathode. Hence $\beta = -r_c/R$, and there is negative feedback.

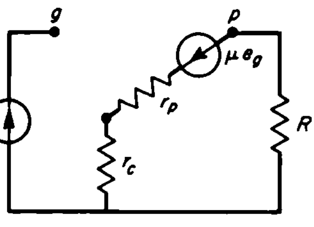


Fig. 13.33.

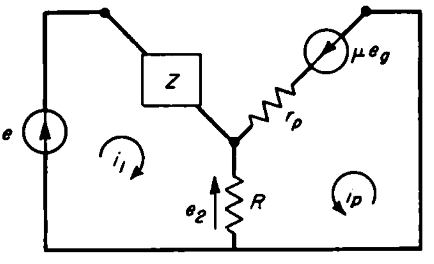


Fig. 13.34.

13-12 Cathode Follower. If in Fig. 13.33 we let $R = 0$ and r_c be the load resistor (grounded-plate amplifier) we have 100 percent negative feedback ($\beta = -1$). This time we shall make allowance for capacitance and possible conductance between grid and cathode (Fig. 13.34).

The equations are:

$$e = i_1(Z + R) + i_pR$$

$$\mu e_g = i_1R + i_p(R + r_p)$$

with

$$e_g = e - e_2 = e - Ri_p$$

Solving for i_1 and i_p (Problem: Do this) yields:

$$i_1 = e\,\frac{R + r_p}{(\mu R + r_p)(R + Z) + RZ}$$

$$i_p = e\,\frac{\mu(R + Z) - R}{(\mu R + r_p)(R + Z) + RZ}$$

For μ very large, these results are approximated by:

$$i_1 \doteq e \frac{R + r_p}{\mu R(R + Z)}$$

$$i_p \doteq e/R$$

making

$$e_2 = Ri_p \doteq e \tag{13-36}$$

and the input impedance

$$e/i_1 \doteq \mu R(R + Z)/(R + r_p)$$

becomes large with μ.

The output impedance, as seen across the terminals of R, is, by Thevenin's theorem, the ratio of the open-circuit voltage to the short-circuit current. Here, open-circuit and short-circuit refer to external connections across R. The OCV is therefore e (approximately), and the SCC is found by shorting R (which of course eliminates the feedback), giving

$$i_{sc} = \mu e/r_p = g_m e$$

and

$$r_o = e/g_m e = 1/g_m$$

so that the output resistance is small for a high-g_m tube, and essentially independent of R. The physical clue to this reduction of output impedance is to be found in the parenthetical remarks above; the output impedance is the ratio of the output voltage *with* feedback and the short-circuit output current *without* feedback. This situation arose because the output voltage was the feedback voltage. In the case of Fig. 13.33, shorting the output resistance R would *increase* the feedback, and therefore we should find that feedback *increases the output resistance.*

Problem.

Show that, in the absence of feedback, the amplifier of Fig. 13.33 has the output resistance $r_o = R(R_p + r_c/)(R + r_p + r_c)$, being the resistance of R in parallel with the series combination of r_p and r_c, and that with feedback, for μ very large,

$$r_o \doteq R$$

13-13 Uses of Feedback. Positive feedback is sometimes used to "peak" an amplifier; i.e., to make its amplification greater, by selective feedback, at some particular frequency. In fact, if positive feedback with $\beta = 1/A_0$ is used, the amplification $A = A_0/(1 - A_0\beta)$ becomes infinite, and we have an oscillator.

Negative feedback is commonly employed for the converse effects, namely to make the amplification essentially independent of frequency over a given band. In addition, nonlinear distortion of large signals can be reduced, for nonlinear distortion is due to the variation of A_0 with signal amplitude. Consider an amplifier with negative feedback, $\beta = -1/10$, and an amplification (A_0) varying between 90 and 100. The amplification with feedback is:

$$A_0 = 100, \quad A = \frac{100}{1 + 100/10} = \frac{100}{11} = 9.09$$

$$A_0 = 90, \quad A = \frac{90}{1 + 90/10} = \frac{90}{10} = 9.00$$

so that the original 10 percent variation in A_0 is reduced to a 1% variation in A. In general, for $A = A_0/(1 - A_0\beta)$, changes in A_0 give:

$$\delta A = \frac{\delta A_0}{1 - A_0\beta} + \frac{A_0\beta\delta A_0}{(1 - A_0\beta)^2} = \frac{\delta A_0}{(1 - A_0\beta)^2}$$

$$\frac{\delta A}{A} = \frac{\delta A_0/A_0}{(1 - A_0\beta)} \tag{13-37}$$

so that the fractional change of amplification is reduced by the factor $(1 - A_0\beta)$.

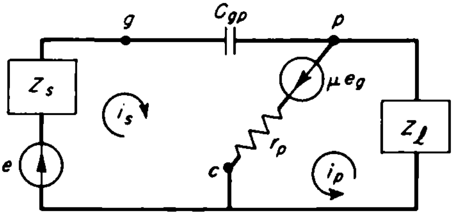


FIG. 13.35.

Feedback circuits are not always easy to analyze in terms of β, for the feedback circuit may appreciably change the load impedance presented to the amplifier. The amplification without feedback is not the same as the amplification with the feedback circuit removed.

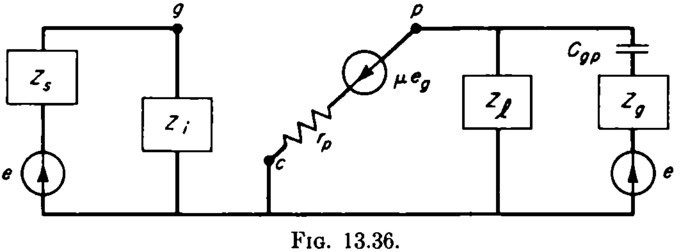


FIG. 13.36.

To illustrate this point, and some others, let us analyze the effect of C_{gp} on a triode amplifier. Ignoring the other internal capacitances, we have the equivalent circuit of Fig. 13.35, where Z_s is the source impedance, and Z_l the intended load impedance. The circuit equations are

$$\begin{aligned} i_s(Z_s + 1/j\omega C_{gp} + r_p) + i_l r_p &= e + \mu e_g \\ i_s r_p + i_l(Z_l + r_p) &= \mu e_g \end{aligned} \qquad (13\text{-}38)$$

plus the relation

$$e_g = e - Z_s i_s$$

The source sees the (unknown) input impedance Z_i and develops the voltage e_g across the input impedance. The circuit can be redrawn as in Fig. 13.36, where it becomes apparent that the feedback effect of C_{gp} can be expressed in terms of what the amplification of the triode does to Z_i, since the other circuit impedances are independent of μ.

We can evaluate Z_i by returning to Fig. 13.35 and Eqs. (13-38), and letting $Z_s = 0$. This gives $e_g = e$ and $Z_i = e/i_s$:

$$\begin{aligned} i_s(1/j\omega C_{gp} + r_p) + i_l r_p &= (1 + \mu)e \\ i_s r_p + i_l(Z_l + r_p) &= \mu e \end{aligned}$$

These readily yield

$$Z_i \equiv \frac{e}{i_s} = \frac{r_p/j\omega C_{gp} + r_p Z_l + Z_l/j\omega C_{gp}}{r_p + (1 + \mu)Z_l}$$

Separating Z_l into its real and imaginary parts ($Z_l \equiv r_l + jx_l$) gives

$$Z_i = \frac{r_p r_l + x_l/\omega C + j\{r_p x_l - r_p/\omega C - r_l/\omega C\}}{r_p + (1 + \mu)r_l + j(1 + \mu)x_l}$$

Multiplying numerator and denominator by the conjugate of the denominator yields a real positive denominator and a complex numerator:

$$Z_i = \frac{N_r + jN_x}{[r_p + (1 + \mu)r_l]^2 + (1 + \mu)^2 x_l^2}$$

The real part of the numerator is

$$N_r = (1 + \mu)r_p x_l^2 - x_l \mu r_p/\omega C + r_p r_l[r_p + (1 + \mu)r_l]$$

which is obviously positive for all negative (capacitive) x_l. For positive x_l (inductive Z_l), N_r is negative for values of x_l lying between the two roots of $N_r = 0$. The two roots of the quadratic equation are real if

$$\left(\frac{\mu r_p}{\omega C}\right)^2 > 4(1 + \mu)r_p^2 r_l[r_p + (1 + \mu)r_l]$$

or

$$\left(\frac{\mu}{\omega C}\right)^2 > 4(1 + \mu)r_l[r_p + (1 + \mu)r_l]$$

which can be satisfied by making r_l small, hence Z_l a low-loss inductor. The roots of the quadratic are

$$x_l = \frac{\mu r_p/\omega C \pm \sqrt{(\mu r_p/\omega C)^2 - 4(1+\mu)r_p{}^2 r_l[r_p + (1+\mu)r_l]}}{2(1+\mu)r_p} \tag{13-39}$$

Since the radical is smaller than the first term, both roots are less than $\mu/(1+\mu)\omega C \leq 1/\omega C$. This relation will be used below.

The imaginary part of the numerator is

$$N_x = -(1+\mu)x_l{}^2/\omega C + r_p{}^2 x_l - (r_p + r_l)[r_p + (1+\mu)r_l]/\omega C$$

which is obviously negative for x_l negative. For x_l positive, but smaller than $1/\omega C$, the positive term $r_p{}^2 x_l$ is more than compensated by the last term, so N_x is still negative. Thus if x_l lies between the two values of (13-39), the input impedance has a *negative resistance* component, and a negative (capacitive) reactance component. Thus Z_s can be a real inductor, i.e., have positive reactance *and resistance* and still, under suitable circumstances, we can have $Z_s + Z_i = 0$. Referring back to Fig. 13.36, we see that this implies an infinite current in the input circuit, and an infinite voltage e_g at the grid. The resulting amplification is infinite; the amplifier will oscillate, or produce a finite output for zero input voltage.

This effect is utilized in the tuned-plate tuned-grid oscillator, wherein the impedances Z_l and Z_s are parallel tuned circuits, operated *below* resonance frequency, so as to present an inductive reactance. The grid impedance Z_g is often a quartz-crystal resonator with Z_l a tuned circuit for feedback control and coupling to a useful load.

When the circuit conditions are such that the amplifier does not oscillate, the amplification is finite, but high for a narrow range of frequencies. This selective increase of amplification by feedback is called regeneration. Before the days of pentodes, regeneration (positive feedback) was frequently employed to increase amplification. But just as negative feedback *reduces* the sensitivity of an amplifier to tube replacement, supply voltage changes, etc., positive feedback increases this sensitivity. Regenerative amplifiers are therefore potential trouble makers.

Modern amplifiers, particularly the very elaborate ones developed for long-line telephony, use very complicated feedback arrangements to achieve a fine control of a complex β over a wide frequency range. The techniques involve elegant applications of mathematics too advanced to discuss in this book.

13-14 Transistors. The grounded-base transistor amplifier (Fig. 13.37) has a similarity with the feedback amplifiers of Figs. 13.33 and 13.34 because of the internal base resistance r_b. This base resistance, common

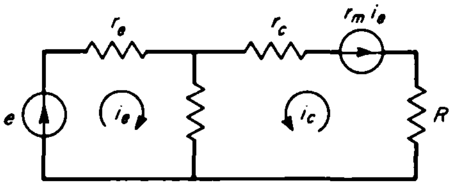


FIG. 13.37.

to input and output meshes, provides a built-in feedback. The circuit equations are readily set up and solved, yielding

$$e_2 = -Ri_2 = e\frac{R(r_m + r_b)}{r_e(r_b + r_c + R) + r_b(r_c + R) - r_m r_b}$$

To interpret the effect of r_b in terms of feedback, compare the amplifications

$$A_0 = \left(\frac{e_2}{e}\right)_{r_b=0} \quad \text{and } A = \frac{e_2}{e}$$

with the feedback formula, $A = A_0/(1 - A_0\beta)$. Solving the feedback formula for β yields

$$\begin{aligned}\beta &= \frac{1}{A_0} - \frac{1}{A} \\ &= \frac{r_e(r_c + R)}{Rr_m} - \frac{r_e(r_b + r_c + R) + r_b(r_c + R) - r_m r_b}{R(r_m + r_b)} \\ &= r_b\frac{r_m^2 - r_m(r_c + r_e + R) + r_e(r_c + R)}{Rr_m(r_m + r_b)}\end{aligned}$$

which, for $r_m \to \infty$, becomes $\beta \doteq r_b/R$ which is *positive* feedback, or *regeneration*. (Negative feedback is often called *degeneration*.) This unexpected reversal of the sign of feedback as compared with the triode case is due to the reversed sense of the controlled source. If r_b becomes very large, the controlled source will obviously *increase* the emitter input current, i_e. In fact, point-contact transistors have $r_m > r_c$, and if improperly loaded may become unstable and oscillate. Junction transistors under normal operation have $r_c > r_m$; ignoring the other resistances our β formula is approximated by

$$\beta = -r_b(r_c - r_m)/Rr_m < 0$$

Chapter XIV

NOISE

The word "noise" originally referred to unpleasant or meaningless sounds. In modern science and technology, the meaning of the word has been both broadened and narrowed: broadened in the sense that electrical signals and even experimental numerical data are often called "noisy," and narrowed in the sense that "noise" implies a statistically random fluctuation of the quantity being observed.

The current through a diode, for example, is not actually a constant current. It is a stream of *many* electrons per second; the dc current is the average rate of arrival of charge at the anode. But if we take a "time microscope" and observe the charge arrivals during a very short interval, we can see the individual pulses that make up the current. There is a pulse for each individual electron arrival. It is the statistical effect of averaging such a large number of electrons that make the current appear steady.

Another example is offered by radioactive decay. We know that the individual radioactive atoms "pop off" at random, but there are so many doing this each second that we ordinarily notice only the *average* rate. With a Geiger counter, we can observe the individual decays and notice the fluctuations.

Radioactive decay and thermal emission are statistically similar. The photoemission of electrons from the cathode of a photoelectric cell is another example of the same type of random process. The number of electrons emitted in a given time interval, say one second, is rarely equal to the average of this number over many intervals; any particular second is apt to have either too many or too few emissions. Let N be the actual number of emissions in a unit time interval, and $\bar{N}$ the average number, i.e., the average of the individual N's of a large number of intervals. The *deviation* from the average, $(N - \bar{N})$, is a number which is positive for some intervals, negative for others. The *average deviation* is zero, by our definition of $\bar{N}$. However, the squared deviation, $(N - \bar{N})^2$, is positive for *all* intervals that do not have precisely the average behavior, and is larger the larger the deviation. Hence the *average value* of $(N - \bar{N})^2$ (called the *mean square deviation*) is a convenient measure of the variation among individual intervals. The square root of this mean square deviation (called the root-mean-square or rms deviation) can be directly compared with the average rate $\bar{N}$ to express the fluctuations as a percentage.

For the random processes mentioned above, the average emission rate $\bar{N}$ is proportional to the length (time) of a unit interval, whereas the rms deviation turns out to be proportional to the *square root* of the length of the unit interval. Thus the rms deviation is a smaller *fraction* of the average rate, the longer the unit interval. Percentagewise, then, the "current" is more nearly constant from minute to minute than from second to second. That is, the *total emission per minute* fluctuates less than that *per second*, on a *percentage* basis (although not on an actual count basis).

14-1 Diode Noise. The emission current in a diode can therefore be interpreted as a steady current, with a superimposed "random noise" current (the deviation). The noise current has a zero average value (like a sinusoidal ac), and a non-zero rms value (again like a sinusoidal ac). This noise current, however, does not have a definite frequency, but has components of all frequencies. It has a continuous "smear" for a frequency distribution, and the noise power that will pass through any filter is proportional to the bandwidth of that filter and independent of the center frequency of the filter, within reasonable limits. This fundamental and unavoidable noise in an emission current is called "shot noise," since it is due to the discrete particle make-up of the current. Its circuit equivalent is a noise-current generator, i_n, (Fig. 14.1) such that

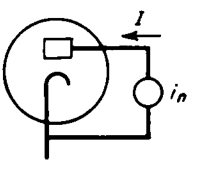


FIG. 14.1.

$$\langle i_n^2 \rangle = 2eIB \tag{14-1}$$

where

$\langle i_n^2 \rangle$ is the mean square noise current (amperes^2),
e is the charge on the electron (1.59×10^{-19} coulombs),
I is the average current (amperes),
B is the bandwidth of interest (cps).

The full shot noise, Eq. (14-1), is observed in temperature-limited diodes, where *all* the emitted electrons are captured at the anode. When the current is space-charge limited, there is a "cushioning" effect by the virtual cathode, and the shot noise is reduced.

There are other sources of noise in a diode, but they are of less importance than the shot noise. The "flicker effect" is produced by low-frequency variations in the emissive properties of various areas of the cathode. Other minor effects are variations in the secondary emission at the anode, small ionization currents due to residual gas traces, etc.

14-2 Triode Noise. A triode is found to possess two apparent noise generators (Fig. 14.2). The plate-cathode terminals are shunted by a shot-noise generator, as is to be expected. The additional noise source appears in the grid circuit and indicates that there is a noise current induced in the grid circuit by the random motions of the electrons flowing from cathode to anode. The major contribution to this grid noise current appears to be generated as a potential induced on the grid by the passage of electrons through the grid. The experimentally observed grid noise is several times greater than can be attributed to the electrons producing plate current. The excess noise has been attributed[1] to the electrons that get through the grid, are reflected back through the grid, then pass through the grid again on their way to the anode. This effect is exaggerated in pentodes, where the suppressor grid operated at cathode potential gives rise to considerable "reflection" noise. At high frequencies, say above 15 mc, the induced grid noise is the limiting factor in amplifier design.

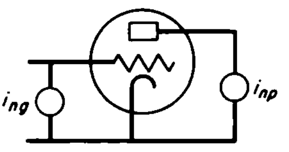


FIG. 14.2.

As additional grids are added (pentodes, etc.) new noise sources appear. Induction effects like that in the triode appear on each additional grid. In addition, the division of the total emission current between screen and plate is subject to random variations; the mean square deviation of this is a noise source (partition noise). Also, there is some secondary emission from each electron-collecting electrode. Random variation in the secondary emission is another source of noise. In general, every complication added to a tube makes it more noisy.

14-3 Thermal Noise. Tubes are not the only important source of noise. Any conductor contains many electrons; they are so dense as to remind one of the molecules in a gas, bumping each other and bouncing hither and yon. Indeed, the kinetic energy of the particles in an "electron gas" is subject to the same fundamental thermodynamic effects as the kinetic energy of any other gas particles. If there is no average current, there is no *net* flow of electrons through the conductor, but there is a *random* flow of *zero average value*, but having a non-zero mean square value. Hence any conductor inherently contains a noise-current generator. The thermal agitation energy is proportional to the absolute temperature of the conductor, and the resulting *available noise power* (watts), in a frequency bandwidth B, turns out to be kTB, where T is the absolute temperature, B is the

[1] T. E. Talpey and A. B. Macnee, *Proc. I.R.E.* *43*, 449–454, April 1955.

bandwidth in cycles per second, and $k = 1.374 \times 10^{-23}$ joules/°K is Boltzmann's constant.

A generator having open circuit voltage e and internal resistance R has the available power $e^2/4R$, hence the thermal noise of a resistance can be

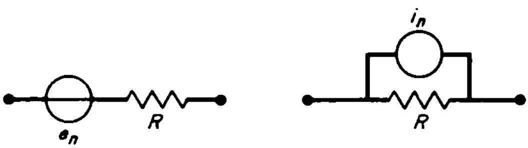


FIG. 14.3.

described (Fig. 14.3) as being due to a noise voltage source whose *mean square voltage* is given by

$$\langle e_n^2 \rangle = 4RkTB \tag{14-2}$$

By Thevenin's theorem, this effect can also be represented by a current generator; the mean square short-circuit noise current from the resistance is

$$\langle i_n^2 \rangle = 4kTB/R \tag{14-3}$$

A noise source, such as a tube, can be evaluated in terms of an *equivalent resistance*, i.e., that resistance which, at room temperature, would have an available noise power equal to that of the actual noise source. For this purpose, a standard nominal room temperature of 290°K (63°F) is used. This gives $kT = 4.00 \times 10^{-21}$ watt-seconds.

A parallel-tuned circuit presents a high resistance at resonance, but much lower resistance at frequencies away from resonance. Thus even over a relatively narrow frequency band, the resistance is a function of frequency. The available noise power is still kT, so that the mean-square open-circuit voltage across the tuned circuit is found by integrating over the frequency range:

$$\langle e_n^2 \rangle = 4kT \int R(f)df \tag{14-4}$$

This is the general formula. For R constant, Eq. (14-4) reduces to Eq. (14-2).

14-4 Equivalent Resistance. The noise output of a tube can usefully be expressed in terms of that resistance which, at room temperature, would provide the same available noise power. Direct comparison of tube noise with the thermal noise of a circuit is then readily made. If, for example, we equate the shot noise of a temperature-limited diode, Eq. (14-1) to the thermal noise of a resistance, Eq. (14-3), we find

$$\langle i^2 \rangle = 2eIB = 4kTB/R_{eq}$$

yielding

$$R_{eq} = \frac{2kT_o}{eI} = \frac{0.05}{I} \tag{14-5}$$

for the equivalent resistance. For I in amperes, R is in ohms. For T we have used the standard T_o(290°K), which gives

$$kT_o/e = 0.025 \text{ volt} \tag{14-6}$$

In a diode operated at its rated conditions, only a small portion of the cathode emission reaches the anode. The heavy space-charge "cushioning" of the shot noise reduces it by a factor of about 25, reducing the equivalent resistance to

$$R_{eq} \doteq \frac{0.002}{I}$$

For a triode, we consider the combination of a perfectly quiet triode (having the same parameters) with a resistance R_{eq} at temperature T_o across the input (Fig. 14.4). The equivalent resistance (R_{eq}) is that resistance whose thermal noise amplified by the tube gives the same available output noise power as would the actual noisy tube by itself. For a space charge limited triode (normal operating condition), R_{eq} is given approximately by

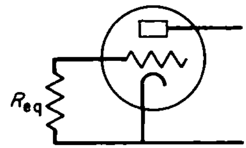


FIG. 14.4.

$$R_{eq} = \frac{3}{g_m} \text{ (ohms)}$$

where g_m is the mutual conductance in mhos.

The noise from a pentode is of the order of three to five times as great as that from a triode, under operating conditions to give the same amplification. As we shall see, this puts pentodes at a severe disadvantage for operation in the input stage of a sensitive receiver.

14-5 Noise Addition. When well-defined signals, such as sinusoidal voltages, are combined by a series connection of the sources, the resulting signal has a voltage given by the sum of the individual voltages. A similar statement holds for sinusoidal current generators connected in parallel. Consider the case of two voltages having a common frequency and amplitude. If they are *in phase*, the resulting sum has twice the amplitude and therefore delivers *four* times the power to a given resistance. If the voltages are 180° out of phase, the net voltage is zero, and no power is delivered.

In the case of noise sources, the equivalent generators can be represented by a combination of a very large number of sinusoidal generators having completely random amplitudes and phases. In combinations of noise sources, some generators are in phase, some out of phase, etc., with a uniform distribution of phase difference between pairs of generators. The net result is that the separate noise sources neither help nor hinder each other and the output *power* is the sum of the separate output *powers*. Thus if *independent* noise-voltage sources are connected in series, the *mean square voltage* of the combination is given by the sum of the *individual mean square voltages*. A similar statement is valid for noise-current sources in parallel. ("Independent" means that there is no statistical correlation among the sources, i.e., no component of separate generators attributable to a common cause. Unless otherwise stated, all noise sources discussed in this chapter will be assumed to have statistical independence.)

14-6 Noise Figure. One of the most important characteristics of any signal is its signal-to-noise ratio, S/N, where S is the power in the "quiet" signal, and N is the accompanying noise power. Amplifying the "noisy" signal increases the signal power and the noise power (per unit of bandwidth) by the same factor. We therefore restrict the band pass of the amplifier to that needed for the use to which the signal is being put, to avoid amplifying the noise power at other frequencies. For a fixed pass band, amplification with quiet tubes has no effect on the signal-noise ratio. Tube noise is, however, added to the incoming S/N, or degrades the signal. An important design objective is to minimize this effect, which is often measured by the ratio of $(S/N)_{in}$ to $(S/N)_{out}$. To be explicit, we use the *available* signal-noise ratios, i.e., the ratio of available signal power to available noise power.

We define the *noise figure* as the ratio of (available) input signal-noise to (available) output signal-noise:

$$F \equiv \frac{(S/N)_i}{(S/N)_o} \geq 1 \tag{14-7}$$

This can be developed as

$$F = \frac{S_i}{S_o}\frac{N_o}{N_i} = \frac{N_o}{GN_i} \tag{14-8}$$

since $S_o/S_i = G$, the available power gain. Note that N_o is the available output power, while GN_i is that part of N_o that is due to the source alone (quiet amplifier). Dividing N_o into its two parts

$$N_o = N_{amp} + GN_i$$

we have

$$F = \frac{N_{amp} + GN_i}{GN_i} = 1 + \frac{N_{amp}}{GN_i} \tag{14-9}$$

or F is unity plus the ratio of (amplifier-produced noise) to (amplified input noise).

Since the input noise will be the thermal noise of the source, $N_i = kTB$, and Eq. (14-8) yields

$$N_o = kTBGF \tag{14-10}$$

for the available output noise.

Consider two amplifiers in cascade, with available gains G_1 and G_2, and a common bandwidth B. The noise figures F_1 and F_2 are assumed known; what is the noise figure F_{12} of the combination? From Eq. (14-10) we have, for the noise output of the overall amplifier,

$$N_{12} = F_{12}G_{12}kTB = F_{12}G_1G_2kTB \tag{14-11}$$

and for the first amplifier alone:

$$N_1 = F_1G_1kTB \tag{14-12}$$

Now N_1G_2 is the available noise out of amplifier 2 due to noise sources *ahead* of amplifier 2, i.e., source noise and noise of the first amplifier. The available noise *produced* by amplifier 2 is, by Eq. (14-9): $(F_2 - 1)G_2kTB$. The total output noise of the overall amplifier is the sum of these two parts:

$$N_{12} = N_1G_2 + (F_2 - 1)G_2kTB \tag{14-13}$$

Substituting N_1 from Eq. (14-12) and comparing the result with Eq. (14-11) give

$$F_{12} = F_1 + \frac{F_2 - 1}{G_1} \tag{14-14}$$

This important relation shows that for G_1 large (i.e., high gain first stage), the noise figure of the amplifier is approximately that of its first stage alone. Thus the first stage is the critical part of an amplifier, from the standpoint of signal-noise ratio. This is the reason for using a triode in the first stage of a super-quiet receiver.

For a three-stage amplifier, the overall noise figure is

$$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1G_2} \tag{14-15}$$

The general formula is obvious.

14-7 Input Impedance. Consider a triode, whose internal noise is represented by sources at both input and output (Fig. 14.2), driven by a signal source of internal resistance r_g (Fig. 14.5). At the output, the tube looks like a current generator $i = e_g g_m$, so the load sees the parallel current generators of Fig. 14.6. By Eq. (14-9) and the interpretation following the equation, the noise figure is

$$F = 1 + \frac{\dfrac{\langle i_{pn}^2\rangle}{4} r_p + \dfrac{\langle i_{gn}^2\rangle r_g^2 g_m^2}{4} r_p}{\dfrac{4kTBr_g g_m^2}{4} r_n} \tag{14-16}$$

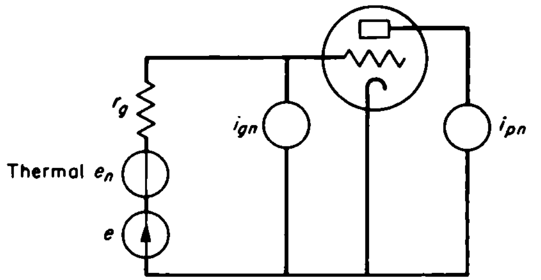


FIG. 14.5.

since the thermal noise voltage of the source is given by $\langle e_n^2 \rangle = 4kTBr_g$. Rewriting Eq. (14-16) as

$$F = 1 + \frac{1}{4kTBg_m^2}\left\{\frac{\langle i_{pn}^2 \rangle}{r_g} + r_g g_m^2 \langle i_{gn}^2 \rangle\right\}$$

we see that the optimum value of r_g is that which minimizes the quantity

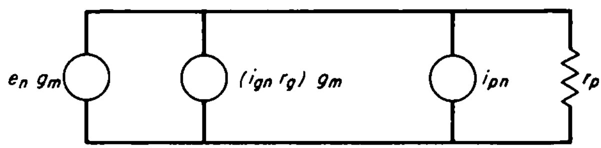


FIG. 14.6.

in braces. This is readily found to be

$$r_{go}^2 = \frac{\langle i_{pn}^2 \rangle}{g_m^2 \langle i_{gn}^2 \rangle}$$

and the resulting minimum is

$$\frac{1}{r_{go}}\{\langle i_{pn}^2 \rangle + \langle i_{pn}^2 \rangle\}$$

showing that the minimum is yielded by that input resistance which makes

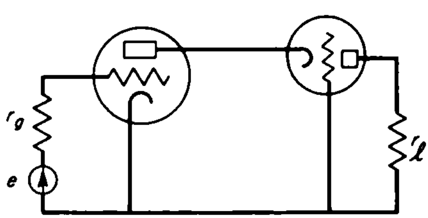


FIG. 14.7.

the *two internal* noise sources of *equal* importance. Consider an arbitrary source coupled to the grid through an ideal transformer. The relation of r_{go} to actual source resistance specifies the optimum transformer ratio. A greater voltage step-up presents a higher impedance to the noise current i_{gn}, and increases the noise. A lesser voltage step-up reduces the signal voltage at the grid.

This matching for equal grid and plate noise is a strong effect. Ordinarily, we would expect to match a source, such as an antenna, to the input impedance of the first stage of a receiver, for maximum signal power. Where signal-noise ratio is more important than saving tubes, the antenna-matching is chosen for minimum noise figure instead of maximum power.

14-8 Cascode Amplifier. We have seen that for low noise, the first tube of an amplifier should be a triode. Since the gain of a triode is not high, the second stage may contribute appreciably to the overall noise figure, Eq. (14-14). This suggests using triodes for each of the first two stages. A triode can be used in any of three ways: grounded cathode, grounded grid, or grounded plate. This makes nine different combinations available for two triodes in cascade. The most advantageous scheme appears to be a grounded-cathode triode followed by a grounded-grid triode.[2] This combination (Fig. 14.7), called a *cascode amplifier*, has a number of intriguing properties. Coupling resistors, etc., are not shown, as only the principles are to be explained.

The second stage needs no neutralizing (Chapter XIII). The input resistance of the second stage is low, Eq. (13-16), so that the load seen by the first stage is small compared to the plate resistance. This implies a low voltage amplification for the first stage, hence although it must be neutralized, the neutralization is not critical. The small first stage load resistance makes the output of the first stage approximately eg_{m1}, where g_{m1} is the transconductance of the first triode. Since the second stage is a "current follower," the load current is also approximately eg_{m1} and is only slightly affected by r_1. Thus the combination amplifies like a pentode of transconductance g_{m1}, that of the first triode. The transconductance of the second triode does not affect the overall gain, yet high g_{m2} is desirable. By Eq. (13-16), the input resistance of the second triode is approximately $1/g_{m2}$, so that increasing g_{m2} reduces the amplification of the first tube. This makes the system more stable, i.e., the first stage neutralization is improved.

Problem.

Show that the "bare" amplifier of Fig. 14.7 is equivalent to a pentode

[2] Wallman, Macnee, and Gadsden. *Proc. I.R.E. 36*, 700–708, June, 1948.

having the parameters:

$$r_p = (\mu_2 + 1)r_{p1} + r_{p2}$$
$$\mu = \mu_1(\mu_2 + 1)$$
$$g_m = \frac{\mu_1(\mu_2 + 1)}{(\mu_2 + 1)r_{p1} + r_{p2}} \doteq \frac{\mu_1}{r_{p1}} = g_{m1}$$

14-9 Transistor Noise. The behavior of transistors was discussed in Chapter XIII on a phenomenological basis. To understand transistor

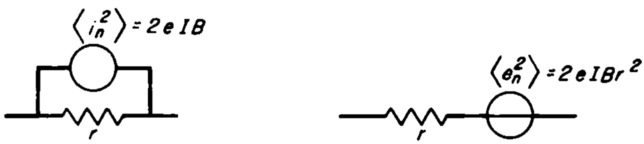


FIG. 14.8.

noise, we must delve lightly into the physics of transistor behavior. For simplicity, only junction transistors will be discussed.

Recall the junction diode, whose current is given theoretically by

$$I = I_o\{\epsilon^{eV/kT} - 1\} \tag{14-17}$$

The dynamic resistance is given by

$$\frac{1}{r} \equiv \frac{\partial I}{\partial V} = \frac{e(I + I_o)}{kT} \doteq \frac{eI}{kT} \tag{14-18}$$

since for a conducting diode, $I \gg I_o$. The diode current is an emission current and is subject to shot noise—the equivalent noise generator is

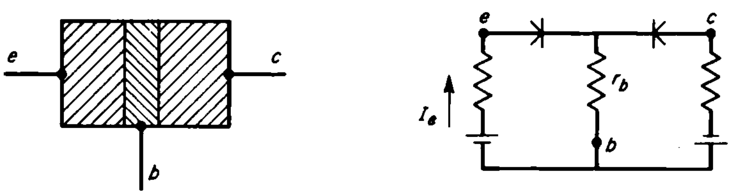


FIG. 14.9.

shown in Fig. 14.8, where the current source is given by Eq. (14-1), and the voltage source deduced by Thevenin's theorem. The equivalent resistance is found by relating this shot noise to thermal noise:

$$\langle e_n^2 \rangle = 2eIBr^2 = 4kTBR_{eq} \tag{14-19}$$

Substituting the expression for r given by Eq. (14-18) yields

$$R_{eq} = r/2$$

so that the diode junction noise is one-half the thermal noise expected from a resistance equal to the dynamic junction resistance, and operating at the same temperature as the diode.

From Eq. (14-17), a diode operating under reverse bias carries a current I_o. (The equation for large negative V gives a *forward* current of $-I_o$.) This current I_o is subject to shot noise.

A junction transistor is essentially a pair of diode junctions, arranged *back-to-back* against a common base section (Fig. 14.9). The emitter junction operates under *forward* bias ($V > 0$ in Eq. (14-17)) and the collector junction under *reverse* bias. For $I_e = 0 (V_e = 0)$, the collector current is the reverse saturation current I_{co}. For $I_e > 0$, the emitted charge carriers enter the middle (base) region and diffuse through it. Most of them (the fraction α) reach the collector junction and are captured by the strong electric field across this junction. The total collector current is then made up of the steady (useless) diode current I_{co} and the variable (controllable) emitted charges that drift across:

$$I_c = I_{co} + \alpha I_e \tag{14-20}$$

Both I_e and I_{co} are subject to shot noise, and the ohmic base resistance r_b is a source of thermal noise. The equivalent circuit of the transistor, includ-

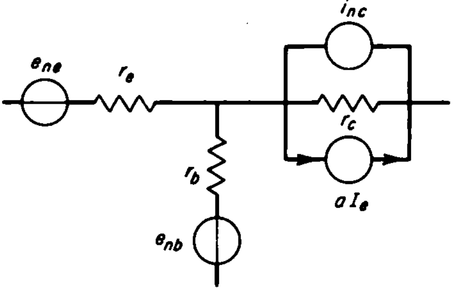


FIG. 14.10.

ing its noise sources, is therefore approximated by Fig. 14.10. The noise sources have the characteristics:

$$\langle e_{ne}^2 \rangle = 2eI_eBr_e^2$$

$$\langle i_{nc}^2 \rangle = 2eI_{co}B$$

$$\langle e_{nb}^2 \rangle = 4kTBr_b$$

For a typical junction transistor, the thermal noise in the base resistance is the most important noise source.

There are, of course, additional sources of noise in transistors. Since

$\alpha \neq 1$, the emitter current can be interpreted as dividing between collector and base, much as the cathode current in a pentode divides between anode and screen grid. Random fluctuations of the instantaneous division ratio produce "partition" noise. In addition, at low frequencies there is a "flicker" effect inversely proportional to frequency. (This also occurs in thermionic tubes.) The $1/f$ flicker noise is more prominent in point-contact type transistors than in junction transistors.

Chapter XV

MODULATION, DEMODULATION, AND DISTORTION

A completely ideal amplifier would have an output whose instantaneous value (voltage or current) was proportional to the instantaneous value of the input. Thus the output would be a "scale model" of the input; the amplification would be the same for all signals, including dc. The addition of a *fixed* time delay between input and output would do no harm.

In practice, electric networks and electronic devices are not ideal, and signals are distorted. Nonlinearity in a device will distort a simple sine wave, and the output will not be a "scale model" of the input. More complicated signals, such as the superposition of two sine waves (of different frequencies) may be distorted because of different amplification at these two frequencies. And finally, even with the same amplification, a change of time delay with frequency can distort a signal.

These various distortion effects are not completely "evil." Although they often occur when not wanted, they can be made useful, for they make *modulation* possible. After all, the goal of a communication system is to transmit information, not just sine waves.

15-1 Nonlinearity. If a device has an output (y) that is *not* a linear function of input (x), but follows some such relationship as

$$y = ax + bx^2 + cx^3 + dx^4 + \cdots \tag{15-1}$$

then if x is $A \cos \omega t$, the output is

$$y = aA \cos \omega t + bA^2 \cos^2 \omega t + cA^3 \cos^3 \omega t + dA^4 \cos^4 \omega t + \cdots \tag{15-2}$$

We are not accustomed to thinking in terms of signals such as $\cos^2 \omega t$; we prefer sinusoidal signals—we can compute impedances and responses for simple sine waves. We shall therefore re-express Eq. (15-2) as a sum of sinusoidal terms.

Now

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

so that

$$\cos^2 x = \frac{e^{2jx} + 2 + e^{-2jx}}{4} = \frac{1 + \cos 2x}{2}$$

We can find a similar identity for any power of $\cos x$. The first few are

$$\cos^2 x = (1 + \cos 2x)/2$$
$$\cos^3 x = (3 \cos x + \cos 3x)/4$$
$$\cos^4 x = (3 + 4 \cos 2x + \cos 4x)/8$$

Equation (15-2) becomes:

$$y = \left(\frac{bA^2}{2} + \frac{3dA^4}{8} + \cdots\right) + \left(aA + \frac{3cA^3}{4} + \cdots\right) \cos \omega t$$
$$+ \left(\frac{bA^2}{2} + \frac{dA^4}{2} + \cdots\right) \cos 2\omega t + \left(\frac{cA^3}{4} + \cdots\right) \cos 3\omega t + \cdots \quad (15\text{-}3)$$

The output contains harmonics of the input. The *even* power terms in Eq. (15-1) contribute *even* harmonics (and dc); the *odd* terms contribute *odd* harmonics. The nth power term contributes harmonics up to the nth, but not higher. The magnitude of the contribution from the nth power term is proportional to the nth power of the amplitude of the input. Thus the distortion increases rapidly with signal amplitude, but is correspondingly small for small amplitude.

The dc term in Eq. (15-3) is useful. A *square-law detector* ($y = ax + bx^2$) yields a dc output that varies with the amplitude of the input.

The term x^2 is useful in another way. Let the signal be the superposition of a high-frequency signal (carrier) and a low-frequency signal (modulating signal). The output is

$$y = a(A \cos 2\pi f_c t + B \cos 2\pi f_s t) + b(A \cos 2\pi f_c t + B \cos 2\pi f_s t)^2 \quad (15\text{-}4)$$
$$= A(a + 2bB \cos 2\pi f_s t) \cos 2\pi f_c t$$
$$+ \{aB \cos 2\pi f_s t + bA^2 \cos^2 2\pi f_c t + bB^2 \cos^2 2\pi f_s t\} \quad (15\text{-}5)$$

The term in braces represents unwanted distortion products of frequencies $0, f_s, 2f_s, 2f_c$; the first term is a carrier frequency oscillation whose amplitude varies with the modulating signal.

15-2 Amplitude Modulation. The useful first term of Eq. (15-5) can also be produced by using a carrier-frequency amplifier whose gain can be varied, say by varying the plate voltage. The simplest case is

$$y = x_c(1 + mx_m) \quad (15\text{-}6)$$

where $x_c = A \cos 2\pi f_c t$ is the carrier signal input, and $x_m = \cos 2\pi f_s t$ is the modulating signal input:

$$y = A(1 + m \cos 2\pi f_s t) \cos 2\pi f_c t \quad (15\text{-}7)$$

The coefficient m is the *degree of modulation*; 100 m is the *modulation percentage*. For $m \leq 1$, the instantaneous amplitude varies between $A(1 - m)$ and $A(1 + m)$. For $m > 1$, the apparently negative amplitude $A(1 - m)$ represents an amplitude $|A(1 - m)|$ plus a reversal of polarity (or phase) of the oscillation. This introduces distortion (*overmodulation*).

Again we resort to trigonometric identities and rewrite Eq. (15-7) as

$$y = A\{\cos 2\pi f_c t + \frac{m}{2} \cos 2\pi(f_c - f_s)t + \frac{m}{2} \cos 2\pi(f_c + f_s)t\} \qquad (15\text{-}8)$$

showing that the modulated carrier can be interpreted as the sum of three sinusoids:

(1) A carrier, frequency f_c,
(2) a lower side-frequency, $f_c - f_s$,
(3) an upper side-frequency, $f_c + f_s$.

For $f_s \ll f_c$, these three frequencies are close together. The additional frequencies of Eq. (15-5) are very different from these and can readily be removed by filters. (In fact, any normal coupling circuit for f_c would not pass these extra terms.) When f_s is made up of a band (range) of frequencies, as in modulation by speech or music, the terms $f_c \pm f_s$ occupy *sidebands* above and below the carrier.

Since the amplitude-modulated signal, Eq. (15-7) is equivalent to a carrier and sidebands, Eq. (15-8), any transmission link must pass the range of frequencies $f_c - f_s$ to $f_c + f_s$ if the signal is not to be changed (distorted). A sharp filter passing f_c only will remove the modulation. This is physically equivalent to saying that we have a tuned circuit of such high Q that the amplitude of oscillation cannot be changed appreciably in the time $1/f_s$ that the modulation goes through a complete cycle.

If this modulated signal $f_c, f_c \pm f_s$, is now used to modulate a frequency f_o (supplied by a *local oscillator* in a *superheterodyne* receiver), we again find sum and difference frequencies. We now have super-sidebands at $f_o \pm f_c$ (Fig. 15.1). These are easily separated by filters (simple tuned

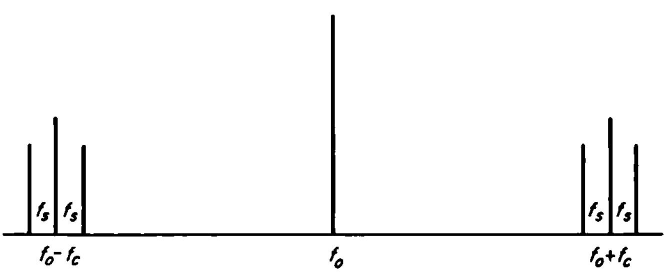


FIG. 15.1.

circuits), and *either* one can be a modulated intermediate frequency (*IF*) signal. By varying f_o, *any* carrier f_c can be "shifted" to a predetermined intermediate frequency for amplification by a fixed-tuning amplifier. This is very convenient, for now the major amplification can be obtained in a fixed amplifier of proper bandpass characteristics. Only the local oscillator and input circuits need be tuned to select the desired signal.

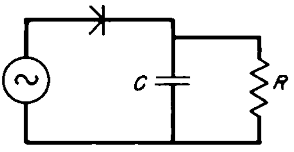


FIG. 15.2.

15-3 Demodulation. We have previously encountered diode rectifiers producing a dc output proportional to the amplitude of a radio-frequency input (Fig. 15.2). The capacitance C "by-passes" the load resistance and allows full voltage to be impressed across the diode. If the source is suddenly turned off, the output voltage does not drop to zero instantaneously, for the capacitance must discharge through the resistance. The voltage decays exponentially; its rate of decay depends upon the time constant RC. If we use this "linear" detector to recover the variable modulation envelope $(1 + m \cos 2\pi f_s t)$ of an amplitude modulated wave, a rapid decrease of input amplitude may produce the same situation. The instantaneous reverse bias on the diode (produced by the output signal) may exceed the amplitude of the modulated wave and cut off the diode for a short while. This will certainly produce distortion. The problem is similar to that encountered in inductance-input filters for rectifier power supplies.

In the present case, we desire an output voltage

$$V_o = a(1 + m \cos 2\pi f_s t) \tag{15-9}$$

If the output load has dc resistance R, the output current will have a dc component a/R. The signal frequency component of the output current will have the magnitude am/Z, where Z is the impedance of the load at signal frequency. The load current will therefore be

$$I = a\left[\frac{1}{R} + \frac{m}{Z}\cos(2\pi f_s t + \theta)\right] \tag{15-10}$$

where θ is the phase of the signal current in the load. Since the rectifier cannot supply negative current, we must have

$$\frac{1}{R} > \frac{m}{Z}$$

or

$$Z/R > m \tag{15-11}$$

is a necessary condition for distortionless demodulation. If this condition is violated, there will be *negative-peak clipping*, i.e., the diode will be biased beyond cutoff when the signal reaches its minimum amplitude.

15-4 Frequency Modulation and Phase Modulation. A sine wave is a special case of the signal $A \sin \theta$, with $\theta = \omega t$, or $d\theta/dt = \omega = 2\pi f =$ constant. We have considered the case of varying A from an otherwise constant value (amplitude modulation). If we now deviate θ from its constant rate of increase, we have a modulated signal

$$y = A \sin \theta(t), \quad d\theta/dt \neq \text{constant}$$

We have modulated the phase of the signal. In particular, if we cause θ to vary sinusoidally about its average value:

$$\theta = \omega t + a \sin pt \tag{15-12}$$

where the maximum phase deviation a is independent of modulating frequency p, we have the signal

$$y = A \sin [\omega t + a \sin pt] \tag{15-13}$$

The constant a is called the modulation index, m_p, of the phase-modulated (PM) signal Eq. (15-13).

On the other hand, if we vary the rate of increase of θ:

$$\frac{d\theta}{dt} = \omega + a \cos pt \tag{15-14}$$

we find

$$\theta = \omega t + \frac{a}{p} \sin pt \tag{15-15}$$

by integrating. The signal is

$$y = A \sin [\omega t + \frac{a}{p} \sin pt] \tag{15-16}$$

and is called a frequency-modulated (FM) signal with modulation index $m_f = a/p$. For a *given* modulation frequency, comparison of Eq. (15-13) and Eq. (15-16) shows that there is no difference between phase modulation and frequency modulation. The difference between these two types of modulation is purely a question of how the modulation index varies with modulating frequency. If the modulation index is independent of frequency, the signal is said to be phase-modulated. If the modulation index is inversely proportional to frequency, the signal is said to be frequency-modulated. It is a question of whether the maximum *phase* excursion is unaffected by modulating frequency, or whether the maximum excursion of the instantaneous *frequency* ($d\theta/dt$) is unaffected. A phase-modulated signal is a frequency-modulated signal whose modulation index *varies* directly with modulating frequency. Conversely, an FM signal is a PM signal whose modulation index *varies* inversely with modulating frequency.

If the modulating signal passes through a high-pass network (having an amplitude response proportional to frequency) on its way to an FM transmitter, a PM signal will be generated. Conversely a $1/f$ low-pass audio amplifier input will convert a PM transmitter to an FM transmitter.

The signal Eq. (15-13) can be expanded as a sum of sinusoids. The mathematics involved is beyond the scope of this book, but the result is:

$$\begin{aligned} y = A\{&J_0(a) \sin \omega t \\ &+ J_1(a) [\sin (\omega + p)t - \sin (\omega - p)t] \\ &+ J_2(a) [\sin (\omega + 2pt) + \sin (\omega - 2p)t] \\ &+ J_3(a) [\sin (\omega + 3pt) - \sin (\omega - 3p)t] + \cdots\} \end{aligned} \tag{15-17}$$

The coefficients giving the amplitude of the side-frequency pairs are Bessel functions of the phase-modulation index a. These functions are available

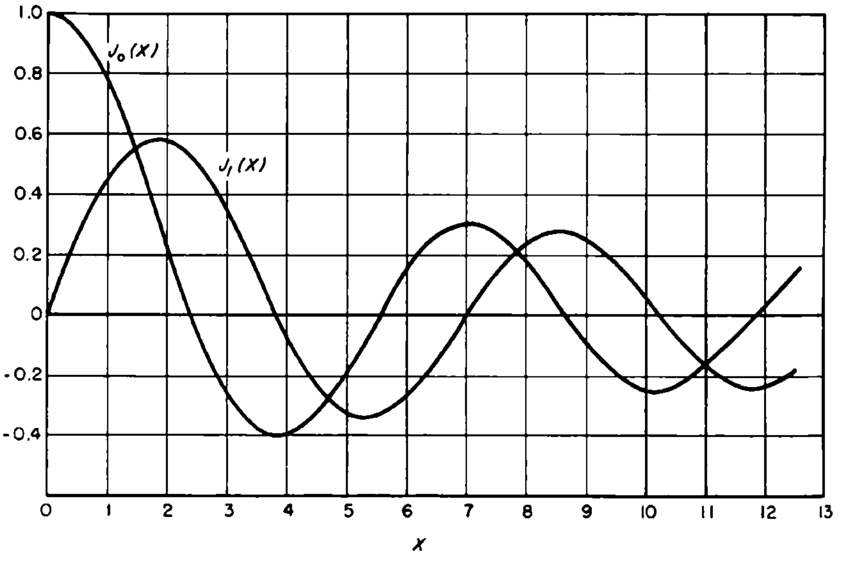


Fig. 15.3.

in various tables; they can be described for our purposes as decaying oscillations. The functions $J_0(a)$ and $J_1(a)$ are plotted in Fig. 15.3; $J_2(a)$ and $J_3(a)$ are shown in Fig. 15.4. Note that for a modulation index of 3.832, the first sidebands are missing; for an index of 2.405, the carrier is missing!

Increasing the modulating frequency spreads the sidebands apart, but for FM, *reduces* the modulation index, hence the amplitude of the higher order sidebands. The net result is that the side frequencies of appreciable order occupy approximately the same total bandwidth for any modulating frequency (Fig. 15.5). For PM (and AM) transmission, the bandwidth varies with modulating frequency and must accommodate the maximum modulation frequency, even though this maximum bandwidth is only occasionally utilized. The FM signal utilizes the entire bandwidth for *all* modulation frequencies (ignoring variations of modulation amplitude).

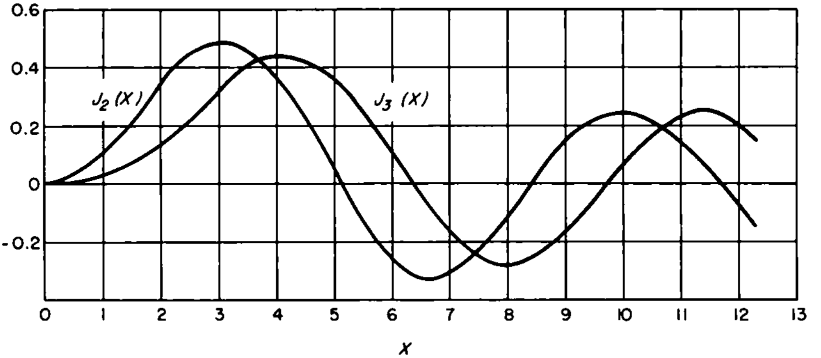


Fig. 15.4.

When the modulating signal is at low amplitude, the modulation index is correspondingly reduced for both FM and PM, and the bandwidth used is also reduced.

Again, distortion can be useful. In this case we consider the generation of harmonics by distortion. If the signal of Eq. (15-13) is passed through a frequency multiplier, the output is

$$\begin{aligned} y &= A \sin n[\omega t + a \sin pt] \\ &= A \sin [n\omega t + (na) \sin pt] \end{aligned}$$

The carrier frequency is increased by the factor n, and so also is the modulation index. Hence in practice, FM and PM transmitters are often modulated at low level (small a) for linearity, using a relatively low frequency carrier. A chain of frequency multipliers then produces the desired high carrier frequency with a large modulation index.

15-5 Fourier Series. We have several times mentioned the expansion of a signal as a sum of simple sinusoidal signals. The basic problem is

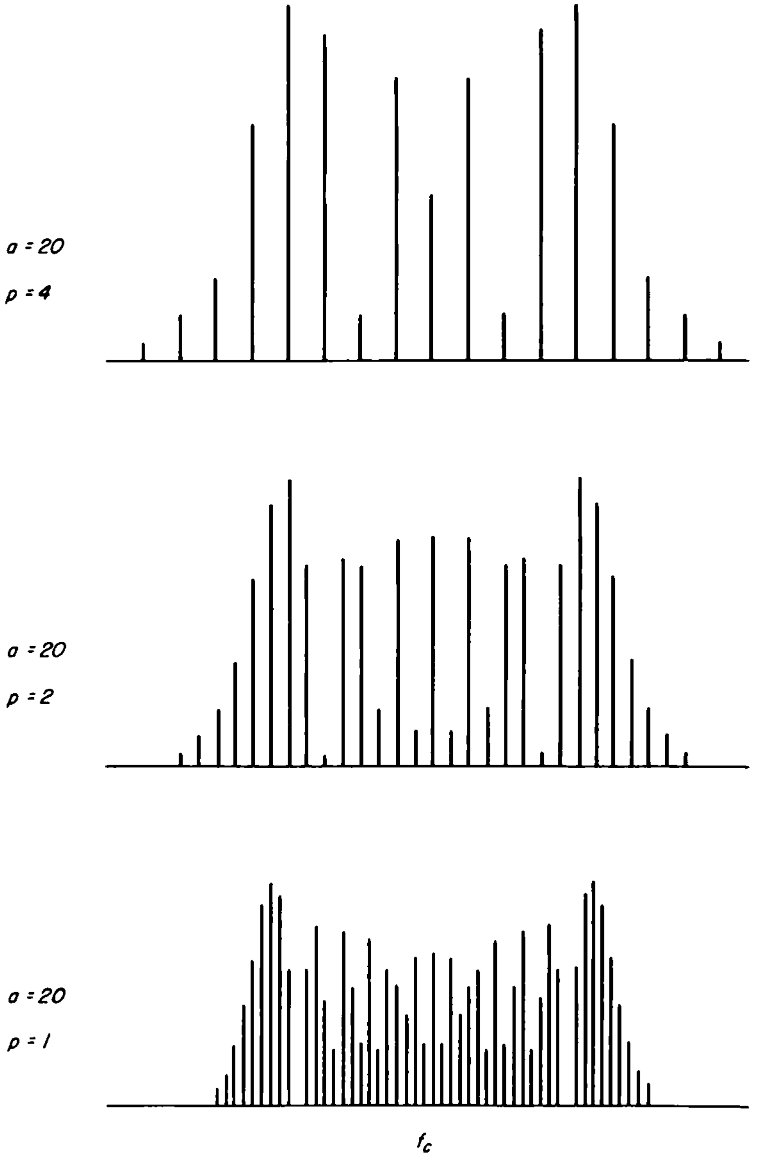


FIG. 15.5.

the expansion of a periodic signal as a *Fourier series*. A function $f(t)$ is said to be periodic, with period T, if $f(t + T) = f(t)$ for all t. This implies $f(t + nT) = f(t)$ where n is any integer, positive or negative (hence nT is also a period). The values of $f(t)$ over any interval of length T, say from $t = a$ to $t = a + T$, completely determine $f(t)$ by the above relations.

The basic periodic functions are *sine* and *cosine*. Since $\sin\theta$ has the period 2π, $\sin\omega t$ ($= \sin 2\pi ft$) has the period $2\pi/\omega = 1/f$. For an arbitrary period T, setting $f_o = 1/T$ gives $\sin 2\pi t/T$ as a *fundamental* oscillation having the desired period. The *harmonics*, $\sin n2\pi t/T = \sin n2\pi f_o t$, have the *least* period T/n, but also possess T as a period. It is shown in advanced mathematical texts that any periodic function, subject to certain restrictions, can be expressed as a sum of harmonic oscillations whose fundamental period is the period of the given function:

$$f(t) = a_0 + a_1 \cos\omega_o t + a_2 \cos 2\omega_o t + \cdots + b_1 \sin\omega_o t + b_2 \sin 2\omega_o t + \cdots \tag{15-18}$$

where

$$f(t) = f(t + T)$$

and

$$\omega_o = 2\pi/T$$

The Fourier series, Eq. (15-18), can be written in a more compact notation as

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos n\omega_o t + b_n \sin n\omega_o t] \tag{15-19}$$

and can also be expressed in complex form:

$$f(t) = \sum_{-\infty}^{+\infty} A_n e^{jn\omega_o t} \tag{15-20}$$

In this last form, the coefficients A_n are complex numbers, combining with the real and imaginary parts of the exponential to give a real series of sines and cosines. Note that in the complex representation, both positive and negative values of n are used.

For a given $f(t)$, the coefficients a_n and b_n are readily found, in *principle*. The formulas follow from the *orthogonality* of the sines and cosines:

$$\int_{\theta_0}^{\theta_0+2\pi} \sin n\theta \sin m\theta \, d\theta = 0, \quad n \neq m$$

$$\int_{\theta_0}^{\theta_0+2\pi} \cos n\theta \cos m\theta \, d\theta = 0, \quad n \neq m \tag{15-21}$$

$$\int_{\theta_0}^{\theta_0+2\pi} \sin n\theta \cos m\theta \, d\theta = 0$$

For $n = m$, we have

$$\int_{\theta_0}^{\theta_0+2\pi} \sin^2 n\theta \, d\theta = \tfrac{1}{2}\int_{\theta_0}^{\theta_0+2\pi} (1 - \cos 2\theta)d\theta = \pi, \quad n \neq 0 \quad (15\text{-}22)$$

and

$$\int_{\theta_0}^{\theta_0+2\pi} \cos^2 n\theta \, d\theta = \tfrac{1}{2}\int (1 + \cos 2\theta)d\theta = \pi, \quad n \neq 0 \quad (15\text{-}23)$$

while for $n = 0$, $\sin n\theta = 0$ and $\cos n\theta = 1$, giving

$$\int_{\theta_0}^{\theta_0+2\pi} d\theta = 2\pi \quad (15\text{-}24)$$

Thus, if we assume $f(t)$ expanded in the form Eq. (15-18), multiply both sides by $\cos n\omega t$, and integrate over *one* period, we find

$$\int_{t_0}^{t_0+T} f(t) \cos n\omega t \, dt = a_n \int_{t_0}^{t_0+T} \cos^2 n\omega t \, dt = a_n T/2$$

This procedure yields:

$$\begin{aligned} a_n &= \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \cos n\omega t \, dt, \quad n \neq 0 \\ a_0 &= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt \\ b_n &= \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \sin n\omega t \, dt \end{aligned} \quad (15\text{-}25)$$

It is convenient to choose $t_o = -T/2$ so that the integration is over the range $-T/2$ to $+T/2$. Since *cosine* and *sine* are respectively *even* and *odd* functions, an even $f(t)$ will have no sine components, and an odd $f(t)$ will have no cosine components, by Eq. (15-25). It is therefore convenient to shift the *origin* of t to take advantage of any symmetry that $f(t)$ may have. For example, if $f(t)$ is a periodic sequence of isosceles triangles, we let $t = 0$ be the time of a peak, and the function is even. It is also often convenient to change the *scale* of t to make the period 2π. That is, we let $\theta \equiv 2\pi t/T$, so that $f(\theta)$ has the period 2π, and the coefficients are given by:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) \, d\theta \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \end{aligned} \quad (15\text{-}26)$$

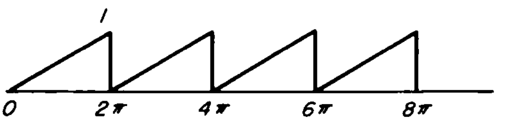


FIG. 15.6.

Note that the dc component a_0 is simply the average value of the function. The other coefficients are given by *twice* the average value of the product of the function and the harmonic term of interest.

Periodic functions that can readily be expressed analytically can be expanded analytically by Eq. (15-26). Functions that are specified only by a numerical tabulation of values require numerical integration of Eq. (15-26). This procedure will not be discussed, but some typical useful wave forms will be expanded.

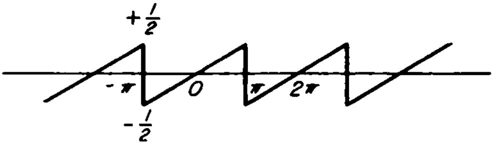


FIG. 15.7.

Figure 15.6 shows a simple sawtooth wave. Its average value is obviously $a_0 = 1/2$. By subtracting this dc component and shifting the origin, the remainder of the function is *odd* and can be expanded as a sine series (Fig. 15.7). We have

$$f(\theta) = \frac{1}{2} + \frac{\theta}{2\pi}, \quad -\pi < \theta < \pi \tag{15-27}$$

The coefficients are given by

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\theta}{2\pi} \sin n\theta \, d\theta$$

which can be integrated *by parts* (or looked up in tables), giving

$$b_n = -\frac{2 \cos n\pi}{2\pi n} = -\frac{1}{n}(-1)^n$$

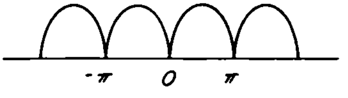


FIG. 15.8.

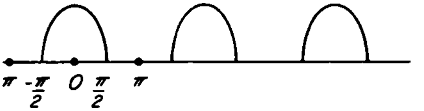


FIG. 15.9.

since $\cos n\pi = (-1)^n$. The Fourier series for the sawtooth of Fig. 15.6 is therefore:

$$f(\theta) = \tfrac{1}{2} + \sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \cdots \qquad (15\text{-}28)$$

The full-wave rectifier output of Fig. 15.8 is even, hence can be expanded in cosines alone. Here it is convenient to keep the period of the *original* sine wave, rather than to change time scale. The harmonics found will then be the appropriate harmonics of the original power supply frequency (usually 60 cps). We have

$$f(\theta) = |\sin\theta|, \qquad -\pi < \theta < \pi$$

and

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin\theta|\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin\theta\, d\theta = -\frac{1}{\pi}\cos\theta\Big]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin\theta| \cos n\theta\, d\theta = \frac{2}{\pi}\int_0^{\pi} \sin\theta\cos n\theta\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)\theta - \sin(n-1)\theta]\, d\theta$$

$$= -\frac{1}{\pi}\frac{\cos(n+1)\theta}{n+1}\Big]_0^{\pi} + \frac{1}{\pi}\frac{\cos(n-1)\theta}{n-1}\Big]_0^{\pi}$$

$$= \begin{cases} 0, & n \text{ odd} \\ -\dfrac{4}{\pi(n^2-1)}, & n \text{ even} \end{cases}$$

The series is

$$f(\theta) = \frac{2}{\pi}\left\{1 - \frac{2}{3}\cos 2\theta - \frac{2}{15}\cos 4\theta - \frac{2}{35}\cos 6\theta - \cdots\right\} \qquad (15\text{-}29)$$

If $\theta = 0$ were chosen to be at a peak, rather than as shown, θ would be different by $\pi/2$. Since

$$\cos 2n(\theta + \pi/2) = \cos 2n\theta \cos n\pi = (-1)^n \cos 2n\theta$$

we also have

$$f(\theta) = \frac{2}{\pi}\left\{1 + \frac{2}{3}\cos 2\theta - \frac{2}{15}\cos 4\theta + \frac{2}{35}\cos 6\theta - \cdots\right\} \qquad (15\text{-}30)$$

The *amplitudes* of the various harmonics are unaffected by a time shift, but the *phases* are changed.

The output of a half-wave rectifier (Fig. 15.9) is

$$f(\theta) = \begin{cases} \cos\theta, & -\pi/2 < \theta < \pi/2 \\ 0, & \pi/2 < |\theta| < \pi \end{cases}$$

The coefficients are given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta = 1/\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos\theta\cos n\theta\,d\theta$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n+1)\theta}{n+1} + \frac{\sin(n-1)\theta}{n-1}\right]_{-\pi/2}^{\pi/2}$$

$$= \begin{cases} 0, & n \text{ odd}, \quad n \neq 1 \\ -\dfrac{2(-1)^{n/2}}{\pi(n^2-1)}, & n \text{ even} \end{cases}$$

For $n = 1$,

$$a_1 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = \frac{1}{2}$$

The series is:

$$f(\theta) = \frac{1}{\pi}\left\{1 + \frac{\pi}{2}\cos\theta + \frac{2}{3}\cos 2\theta - \frac{2}{15}\cos 4\theta + \frac{2}{35}\cos 6\theta - \cdots\right\} \tag{15-31}$$

Finally, the so-called "square" wave (Fig. 15.10) can be taken as the *even* function

$$f(\theta) = \begin{cases} 1, & 0 < |\theta| < \pi/2 \\ -1, & \pi/2 < |\theta| < \pi \end{cases}$$

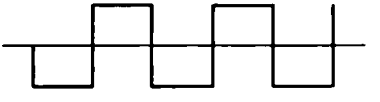


FIG. 15.10.

or as the *odd* function

$$f(\theta) = \begin{cases} 1, & 0 < \theta < \pi \\ -1, & -\pi < \theta < 0 \end{cases}$$

Using the odd-function representation, we have a sine series.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} \sin n\theta\,d\theta + \frac{1}{\pi}\int_{0}^{\pi} \sin n\theta\,d\theta$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin n\theta\,d\theta = -\frac{2}{\pi n}[(-1)^n - 1]$$

$$= \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{\pi n}, & n \text{ odd} \end{cases}$$

The series is:

$$f(\theta) = \frac{4}{\pi}\left\{\sin \theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots\right\} \tag{15-32}$$

For a square wave of arbitrary period T, the change of scale $\theta = 2\pi t/T \equiv 2\pi f_o t$ changes the period to 2π as we assumed in deriving Eq. (15-32). Using this expression for θ gives

$$f(t) = \frac{4}{\pi}\left\{\sin 2\pi f_o t + \frac{\sin 6\pi f_o t}{3} + \frac{\sin 10\pi f_o t}{5} + \cdots\right\} \tag{15-33}$$

as the Fourier series of a square wave of period $T \equiv 1/f_o$. If we let T increase to a very large value, the harmonics $f = (2n + 1)f_o = (2n + 1)/T$ become very close together; the Fourier series approaches a *continuous* spectrum with amplitude proportional to $1/f$. Hence a step function $(T \rightarrow \infty)$ has a $1/f$ spectrum amplitude.

15-6 Truncation of Series. An infinite series, such as a Fourier series, raises many practical questions. If we make numerical computations, we must obviously truncate the series, or cut it off after some finite number of terms. How much error is involved in using only the first term, the first five, or the first ten? If we use only a few terms, can we correct somewhat for the missing terms by slightly changing the coefficients of the terms we use? What is the physical significance of using only a few terms?

Since successive terms of a Fourier series represent increasingly higher frequencies, the series can be truncated physically by the upper frequency cutoff of a filter. In general, passage of a signal through a network will affect all the terms differently: the *amplitude-frequency characteristic* of the network alters the magnitudes of the coefficients, and the *phase-frequency characteristic* changes the relative phases of the terms (i.e., changes the ratio between the sine and cosine terms of each harmonic). It is apparent that if we modify the amplitudes and phases of the various harmonics, we will have the Fourier series of a *different* signal. Hence even a linear network will, in general, distort a complex signal (one having more than one sinusoidal component). Whether the distortion is appreciable or not depends upon the importance of the harmonics affected.

If most of the power of the signal is in the first few terms, i.e., if the harmonic amplitudes decrease rapidly with the order of the harmonics, then "good" characteristics of the filter are needed only over a relatively narrow band. That is, the response of the network to very weak terms is of minor interest. For example, Fig. 15.5 indicates graphically that the circuit response is important in a certain band, but unimportant outside this band, where the component amplitudes are too small to be shown in the figure.

If the Fourier series for a signal $f(t)$ is truncated after n terms, the resulting short series represents a different signal, say $g(t)$. The correction signal, $f(t) - g(t)$, which could be added to $g(t)$ to restore the original signal, offers a measure of the error or distortion introduced by the truncation. The *power* needed for the correction signal is proportional to the time average of $[f(t) - g(t)]^2$. The question arises as to how to choose the n coefficients of $g(t)$ to minimize this power. In mathematical treatises, it is shown that the best coefficients for $g(t)$ are the first n coefficients in the Fourier series for $f(t)$. In other words, *any* change in the terms remaining after truncation will *increase* the distortion (if we use the above power criterion as a measure of distortion).

15-7 Time Delay. If a signal $f(t)$ undergoes a pure time delay τ, the resulting signal is $f(t + \tau)$. The same signal at a later time is considered to be undistorted. Now $f(t + \tau)$ has the same Fourier series as $f(t)$ with the argument $t + \tau$ substituted for t in each term. A typical term, $\cos n\omega_o t$, becomes

$$\cos n\omega_o(t + \tau) = \cos (n\omega_o t + \varphi)$$

where

$$\varphi = n\omega_o\tau = \omega_n\tau$$

with ω_n the frequency of the particular term. Hence a time delay corresponds to a phase shift which is proportional to frequency. A network with such a phase-frequency characteristic will therefore delay a signal but will not distort it.

Consider the general expression for a Fourier component, $\cos (\omega t + \varphi)$, with φ a function of ω. A Taylor series expansion of φ about the frequency ω_1 yields:

$$\varphi = (\varphi)_{\omega_1} + (\omega - {}_{\omega_1}) \left(\frac{d\varphi}{d\omega}\right)_{\omega_1} + \frac{(\omega - \omega_1)^2}{2} \left(\frac{d^2\varphi}{d\omega^2}\right)_{\omega_1} + \cdots \qquad (15\text{-}34)$$

where the subscript ω_1 indicates the frequency at which the various derivatives are evaluated. The argument, $\omega t + \varphi$, becomes

$$\omega t + \varphi = \omega \left[t + \left(\frac{d\varphi}{d\omega}\right)_{\omega_1}\right] + \left[(\varphi)_{\omega_1} - \omega_1 \left(\frac{d\varphi}{d\omega}\right)_{\omega_1}\right] + \cdots \qquad (15\text{-}35)$$

so that all components of frequency near ω_1 are delayed by the time $(d\varphi/d\omega)_{\omega_1}$. Hence the time delay of a network, at any frequency ω, is *defined* as $d\varphi/d\omega$. If this delay is the same for all frequencies, say equal to T, then φ is a *linear* function of frequency:

$$\varphi = \varphi_0 + \omega T \qquad (15\text{-}36)$$

If the phase is not linear over a frequency range of interest, the network possesses *phase distortion.*

The linear phase of Eq. (15-36) makes Eq. (15-35) become

$$\omega t + \varphi = \omega(t + T) + \varphi_0$$

If the *phase intercept* φ_0, the phase shift at zero frequency indicated by Eq. (15-36), is not zero, the delayed signal is distorted by the addition of this constant phase shift to each term of the series. The resulting distortion is often called *phase intercept distortion*.

15-8 Differentiation and Integration. The Fourier series for df/dt can be found by differentiating Eq. (15-19), term-by-term. (This is not always legitimate, the resulting series may not converge; lack of convergence indicates that $f(t)$ has places where it cannot be rigorously differentiated, such as at steps.) The resulting series is:

$$\frac{df}{dt} = \sum [-n\omega_o a_n \sin n\omega_o t + n\omega_o b_n \cos n\omega_o t] \tag{15-37}$$

The effect of differentiation has primarily been the increase of each amplitude by $n\omega_o$, i.e., the amplitudes have been increased in proportion to the frequency of the term. Thus differentiation is essentially the same as passing the signal through a high-pass network whose response is proportional to frequency. Differentiation has also added $\pi/2$ to the phase of each term. We can more readily observe these effects simultaneously by differentiating the exponential form of the series (15-20):

$$\frac{df}{dt} = \sum (jn\omega_o)A_n e^{jn\omega_o t} \tag{15-38}$$

Let us examine the RC voltage divider of Fig. 15.11. We have

$$\frac{E_{out}}{E_{in}} = \frac{R}{R + 1/j\omega C} = \frac{jR\omega C}{1 + jR\omega C} \doteq jR\omega C \text{ for } R\omega C \ll 1 \tag{15-39}$$

Passing a signal through this divider multiplies each component by $jR\omega C = (jn\omega_o)RC$, for components having $\omega \ll 1/RC$. Comparison with

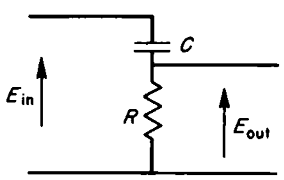


FIG. 15.11.

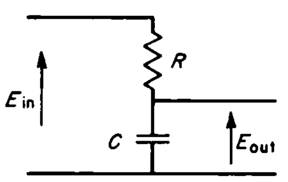


FIG. 15.12.

Eq. (15-38) shows that the output is $RC\ df/dt$, within the approximation in Eq. (15-39). Note that since RC must be small, the output of this "differentiating circuit" is small. Conversely, integrating Eq. (15-20) term-by-term yields $\Sigma A n e^{jn\omega_o t}/jn\omega_o$. This operation is approximated by the "integrating circuit" of Fig. 15.12. We have

$$\frac{E_{out}}{E_{in}} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + jR\omega C} \doteq 1/jR\omega C \text{ for } R\omega C \gg 1$$

Note that RC must now be *large*, and the output is the integral of $f(t)$, *divided* by RC.

We can interpret the differentiator as a low-frequencies-missing circuit, and the integrator as a high-frequencies-missing circuit. Thus any circuit wherein the low frequency response is deficient tends to differentiate a signal, or emphasize the "edges" of a pulse and lose the constant part. A TV receiver so badly mistuned that the low-frequency picture modulation components fall outside the IF amplifier pass band, gives a picture showing the outlines of objects, but lacking black in what should be large black areas. Conversely, too narrow an IF band pass loses the high frequency modulation components, and the edges of objects are blurred. Integration of a curve "smooths out" the little kinks and steps.

PROBLEMS

Chapter I

1-1. In scientific units, resistivity is expressed in ohm-cm; for copper at 20°C (68°F), $\rho = 1.724 \times 10^{-6}$ ohm-cm. In wire tables, the cross-sectional area of wire is usually given in "circular mils." A circular mil is the area of a circle 1 mil (0.001 inch) in diameter. Recalling $A = \pi r^2$ and 1 inch = 2.54 cm, compute the relation, 1 circular mil = 5.07 cm². Show that the resistivity of copper is $\rho =$ 10.37 ohm-circular mils per foot. (The advantage of circular mil units is that the factor π is incorporated into the resistivity; the area of a circle in circular mils is simply the square of the diameter in mils.)

1-2. Number 10 copper wire has a diameter of 101.9 mils. Using the result of Problem *1-1*, show that this wire has a resistance of approximately 1 ohm per 1000 ft. The diameter of a wire is halved for every 6 sizes increase; the cross-sectional area is halved every 3 sizes increase. Hence the resistance of 1000 ft of #13 copper wire is 2 ohms; of #16, 4 ohms, etc. The convenient 1 ohm per 1000 ft for #10 is an easily remembered starting point that allows ready estimation of copper wire resistance when no tables are at hand.

1-3. A certain dry cell has an OCV of 1.5 volts and delivers 20 amperes into a short-circuit. Compute its internal resistance.

1-4. What voltage would the cell of Problem *1-3* develop across a 0.1-ohm load? What current would it deliver? How much power is dissipated in the load? What fraction of the total power is delivered to the load?

1-5. Same as Problem *1-4*, but with a 1-ohm load. Compare the percentage of total power in the load in these two cases.

1-6. A 1-ohm resistance is connected in series with the parallel combination of 2 ohms and 3 ohms. Compute the resistance of the whole combination.

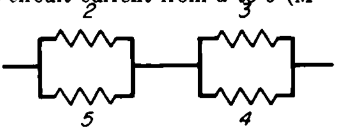
1-7. For 2 amperes through the combination of Problem *1-6*, compute the current through each individual resistance, the voltage across each, and the power dissipated in each. Check the individual values against the total voltage drop and power for consistency.

1-8. Let the series combination of 1 ohm and 3 ohms be connected in parallel with 2 ohms. Compute the quantities corresponding to those asked for in Problem *1-7*.

Chapter II

2-1. In Fig. 2.1, let $R_1 = 2$ (ohms), $R_2 = 3$, $R_3 = 5$, $R_4 = 4$. Let a 20-volt source be connected across the bridge (at points c,d).

(*a*) Compute the OCV between *a* and *b* (i.e., $M = \infty$).
(*b*) Compute the short-circuit current from *a* to *b* ($M = 0$). Hint: With $M = 0$, the bridge becomes



Find the current in each resistance. Comparison with Fig. 2.1 yields the current in the zero resistance "cross-over."

(c) Compute the equivalent simple source seen by *M* (OCV and internal resistance).

(d) Compute the current through *M* for $M = 1$ ohm, and the voltage across *M*.

2-2. For the network of Problem *2-1*, with $M = 1$, write the mesh equations for the currents. Solve these if you already know how. (Simultaneous equations are discussed in Chapter III.) Compute the voltage drop across each resistance.

2-3. Assume the network of Problem *2-1* excited by a 1-ampere *current* source, instead of a 20-volt voltage source. Write the nodal equations for the unknown voltages. Find the voltage drop across the whole bridge (between *c* and *d*). By simple proportion, find the current generator required to make this drop 20 volts. This is the current that would be drawn from a 20-volt source, and should check with the appropriate mesh current of Problem *2-2*.

2-4. Find the various voltage drops and currents in Problem *2-3* and check with those found by mesh analysis in Problem *2-2*.

Chapter III

3-1. Solve for *x*, *y*, *z* by triangularizing:

$$3x - y + 2z = 1$$

$$2x + y - z = 2$$

$$3x + 4y - 3z = 2$$

Check your answer by substitution into the original equations.

3-2. Same as Problem *3-1* except with the coefficient of *z* in the third equation changed to -5. When you get in trouble, compute the determinant of the coefficients and explain the difficulty.

3-3. Solve:

$$2x + y + z = 1$$

$$x - y + z = 0$$

$$x + y - z = 0$$

3-4. Evaluate the determinant:

$$\begin{vmatrix} 2 & 1 & -5 & 4 \\ 3 & 2 & 4 & 5 \\ -1 & -4 & 2 & -3 \\ 1 & 2 & 3 & 4 \end{vmatrix}$$

Chapter IV

4-1. Consider a T-network having series elements of 1 ohm and 2 ohms, and a 3-ohm shunt element. Find the equivalent Π-network.

4-2. Find the A, B, C, D two-port parameters of the network of Problem *4-1*.

4-3. Find the image impedances of the network of Problem *4-1*.

4-4. Let the T-pad of Fig. 4.19 be connected between a source of internal resistance R and a load resistance R. Let the OCV of the source be 12 volts. Find the values of R_1 and R_2 to make the output voltage 6 volts, preserving the image match. Repeat for outputs of 4, 3, 2, and 1 volt.

Chapter V

5-1. A capacitor comprising two plates has a capacitance of 10 $\mu\mu$f. It is charged to 100 volts, and the charging source disconnected. The plates are now mechanically separated until the capacitance is halved.

(*a*) How much mechanical work has been done?
(*b*) What is the final potential difference?

5-2. If in Problem *5-1*, the 100-volt source had been left connected, what would the final p.d. and charge have been? Would more or less mechanical work have been done?

5-3. In Problem *5-2*, how much charge would flow through the battery? How much work was required to force this charge through the battery? What became of this work?

5-4. In Problem *5-3*, what is the final stored energy in the capacitor? How much total mechanical work was done?

5-5. Compute the capacitance of

(*a*) 1 μf in series with 2 μf.
(*b*) 1 μf in parallel with 2 μf.
(*c*) 2 μf in series with (*a*); in parallel with (*a*).
(*d*) 2 μf in series with (*b*); in parallel with (*b*).

5-6. A 100-volt battery is connected to a 1-μf capacitor through a 1000-ohm resistor. What is the final p.d. across the capacitor? Find the time constant. How long will it take for the p.d. to reach 50 volts?

5-7. A 100-volt battery with an internal resistance of 10 ohms is connected across the parallel combination of 50 ohms and 2 μf. Compute the final p.d. and the time constant.

5-8. A 2 μf capacitor is charged to 100 volts and then connected to 1 μf through 1000 ohms. What is the final p.d. on each and the time constant?

5-9. The following measurements were made on a triode:

(*a*) With plate and grid tied together, the capacitance between this combination and the cathode was 2.6 μμf.

(*b*) The capacitance between grid-and-cathode and plate was 2.0 μμf.

(*c*) The capacitance between plate-and-cathode and grid was 3.8 μμf. Find C_{gp}, C_{pc}, and C_{gc}.

Chapter VI

6-1. Compute the inductance of:

(*a*) 1 henry in series with 2 henries.
(*b*) 1 henry in parallel with 2 henries.
(*c*) 2 henries in series with (*a*); in parallel with (*a*).
(*d*) 2 henries in series with (*b*); in parallel with (*b*).

6-2. A coil surrounds a removable iron core. With the core in position, the inductance is 2 henries; without the core it is 0.1 henry. If the coil is carrying a current of 10 amperes, how much work is required to remove the core? To replace it?

6-3. Compare the phenomenon of Problem *6-2* with the operation of a relay. Does current through a relay coil tend to increase or decrease the inductance? The stored magnetic energy? Note that the force on the relay armature is related to the rate of increase of inductance with armature movement.

6-4. A 1-henry coil carries 10 amperes. If this current is reduced to zero at a constant rate, in a time of 0.01 second, how much back-voltage is developed?

Chapter VII

7-1. A 1-henry lossless inductance is connected in parallel with a 0.01 μf capacitance. There is a steady current of 10 amperes through the coil. The supply circuit is then broken. Compute the frequency of the resulting oscillation and its peak voltage and current.

7-2. The parallel combination of 1 henry and 100 ohms is subject to 115 volts rms at 60 cps. Compute the current in the inductance and in the resistance, the total current, and the power.

7-3. The series combination of 1 henry and 100 ohms is connected across the 115-volt 60-cps supply line. Compute the current, the voltage across each element, and the power.

7-4. An extension cord of 1-ohm resistance connects the 115-volt 60-cps supply to an inductance. The current is 10 amperes. What is the voltage across the inductance?

7-5. Same as Problem *7-2* with a 10-μf capacitance replacing the inductance.

7-6. Problem *7-3* with the substitution of Problem *7-5*.

7-7. Problem *7-4* with the substitution of capacitance for inductance.

7-8. The series combination of 25 $\mu\mu$f, 1 mh (millihenry), and 10 ohms is connected across 10 volts (peak) at 1 megacycle per second. Compute the current and the voltage across each element.

7-9. Same as Problem *7-8*, but with R and L in series, across C.

7-10. Same as Problem *7-8*, with R, L, and C in parallel.

7-11. Same as Problem *7-8*, with L and C in parallel; the combination in series with R.

7-12. Same as Problem 7-8, with the parallel combination of R, C in series with L.

7-13. From Eq. (7-14), we can write

$$I_0^2 = \frac{E_0^2}{\omega_0^2 L^2} \frac{1}{\dfrac{1}{Q^2} + \left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)^2}$$

where

$$\omega_0^2 = 1/LC$$
$$Q = \omega_0 L/R$$

Show that detuning to half-power occurs at

$$\omega \doteq \omega_0\left(1 \pm \frac{1}{2Q}\right)$$

so that ω_0/Q is the approximate half-power bandwidth of a simple resonant circuit.

7-14. In the solution (7-5) of Eq. (7-1), let $\dfrac{1}{LC} = \dfrac{R^2}{4L^2}$, giving $\omega = 0$. This is the case of "critical damping." Show that the complete solution of Eq. (7-1) in this case is $i = Ae^{-Rt/2L} + Bte^{-Rt/2L}$ where A and B are arbitrary constants. Hint: Substitute this answer into Eq. (7-1). See also Problem *8-18*.

Chapter VIII

8-1. Find:

(*a*) The sum of $2 + 3j$ and $3 + 4j$.
(*b*) Their product.
(*c*) Reduce the ratio $(2 + 3j) \div (3 + 4j)$ to standard form.

8-2. Find the square root of $3 + 4j$.

8-3 through *8-13*. Compute the complex impedances of the combinations of Problems *7-2* through *7-12*.

8-14. Express the impedances of Problems *8-3* through *8-13* as magnitudes and phase angles.

8-15. Compute the power drawn from the generator in Problems *7-4* through *7-12*.

8-16. Consider a T, Y, or "star" of three elements. Let these be $j\omega L_1$, $j\omega L_2$, and R. Find the equivalent "delta." (The formulas are given in Chapter IV, but watch the complex impedances.)

8-17. Convert a delta of R, L, C to its equivalent T.

8-18. In the differential equation (7-1), let $i = e^{pt}$, converting Eq. (7-1) to

$$\left(p^2L + pR + \frac{1}{C}\right)e^{pt} = 0$$

which becomes an algebraic equation

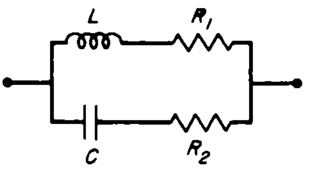
$$p^2L + pR + 1/C = 0$$

after removing the nonvanishing factor e^{pt}. Show that

$$i = Ae^{p_1t} + Be^{p_2t}$$

is the general solution of Eq. (7-1) if p_1 and p_2 are distinct roots of the above algebraic equation. What is the significance of the imaginary part of p_1? Of p_2? What happens when p_1 and p_2 are both real? What relation among L, R, C brings this about? Compare Problem *7-14* and explain what happens when $p_1 = p_2$.

8-19. Show that the impedance of the combination in the sketch is independent of frequency if $R_1 = R_2 = \sqrt{L/C}$.



CHAPTER IX

9-1. In Fig. 9.1, let

$$L = 1 \text{ mh}$$

$$C_1 = 10\ \mu\mu\text{f}$$

$$C_2 = 10\ \mu\mu\text{f}$$

$$R_1 = 1000 \text{ ohm}$$

$$R_2 = 100 \text{ ohm}$$

$$E = 10 \sin 2\pi \text{ ft}$$

$$f = 10^6 \text{ (1 mc)}$$

Compute I_1, I_2, and the voltages across L and C_1.

9-2. In Fig. 9.5, let R_1, R_2, C_1, and C_2 have the values of Problem *9-1*. Let $L_1 = 1$ mh and $L_2 = 0.5$ mh, with a coefficient of coupling of 0.3, and M positive. Compute I_1, I_2, and the voltage across C_2.

9-3. Same as Problem *9-2*, but with M negative.

9-4. Same as Problem *9-2*, but with $k = 1$.

9-5. A certain transformer has a secondary OCV of 11.5 volts when used on a 115-volt, 60-cps line. With the secondary shorted, the secondary current is limited to 50 amperes by leakage inductance. Assuming zero resistance, compute the leakage inductance as referred to (*a*) the primary, (*b*) the secondary. With a 1-ohm resistive load, and a supply line resistance of 5 ohms, compute (*c*) the line drop, (*d*) the voltage across the transformer primary, (*e*) the voltage across the load, (*f*) the magnitude and phase of the load current.

Chapter X

10-1. In Fig. 10.2, let

$$E = \sin 2\pi\, 10^6 t$$

$$L_1 = 0.1 \text{ mh}$$

$$L_2 = 2 \text{ mh}$$

$$k = 0.5$$

$$R = 1 \text{ ohm}$$

Compute the value of C for maximum voltage across C. Compute this voltage.

10-2. In the Wien bridge of Fig. 10.15, let $R_4 = 30{,}000$ ohms, $R_1 = R_2 = R_3 =$ 10,000 ohms. Find C_1 and C_3 to make the bridge balance at 1000 cps.

10-3. For the bridge of Problem *10-2*, with C_1 and C_3 as found, compute the fraction of generator voltage appearing across the bridge output for (*a*) $f = 500$ cps, (*b*) $f = 2000$ cps.

10-4. In the bridge of Fig. 10.13, find the equivalent T-network between source and output, and explain why the output voltage vanishes at 1000 cps.

10-5. Same as Problem *10-4*, but with the equivalent Π-network.

10-6. Find the T-network equivalent to Fig. 10.19 and relate the conditions for zero output to the balance conditions given for Fig. 10.20.

10-7. (*a*) Find values of R and C for Fig. 10.23 suitable for a 1000 cps oscillator.
(*b*) Adjust R and C to make the input impedance have a magnitude of 10,000 ohms.

Chapter XI

11-1. Find the Π-network equivalents of Fig. 11.6.

11-2. In Fig. 11.6a let the operating frequency and components be such that the left-hand coil has the impedance $1000\,j$ ohms, the right-hand coil, $2000\,j$ ohms, and the capacitor, $-3000j$ ohms. Compute the image impedances and the image transfer factor.

11-3. With the section of Problem *11-2* operated under image match conditions, compute (*a*) the voltage transformation ratio, (*b*) the phase shift, and (*c*) the insertion loss.

11-4. Let the load resistance of Problem *11-2* be increased to 2500 ohms. Compute (*a*) the reflection coefficient R_2; (*b*) the percentage change of input current (due to changing the load); (*c*) the percentage power loss due to the reflection mismatch; and (*d*) the insertion loss of the network (inserted between the 2500-ohm load and the 3742-ohm source).

11-5. Let the source resistance in Problem *11-4* be decreased to 3000 ohms. Compute the reflection coefficient R_1, the interaction factor σ, and the insertion loss.

Chapter XII

12-1. Let a certain diode have the current, $I = 10^{-2} V^{3/2}$, where V is in volts and I in milliamperes (ma). Assume this diode connected to a 180-volt battery through a 10,000-ohm resistor. Find the current, the voltage across the diode, and the voltage across the resistor. Hint: Do this graphically.

12-2. An inductance input filter power supply is to be designed for an output of 1000 volts at 200 ma. How large an input inductance is needed?

12-3. How large a transformer is needed for the power supply of Problem *12-2*? (Use the approximation for infinite input inductance.) With lossless inductors, what transformer output voltage is needed? (Neglect rectifier voltage drop.)

Chapter XIII

13-1. Consider a triode having $\mu = 10$, $g_m = 1000$ μ-mhos. (1 μ-mho $= 10^{-6}$ mho.) Compute the amplification in the grounded-cathode configuration with a load resistance of 25,000 ohms.

13-2. Same as Problem *13-1*, but connected as a cathode follower.

13-3. Same as Problem *13-1*, but connected as grounded grid. Compute also the input impedance.

13-4, 5, 6. Same as Problems *13-1, 2, 3*, but with a pentode having $\mu = 100$, $g_m = 1000$ μ-mho.

13-7. Using Fig. 13.22 and Eq. (13-22), show that the input impedance of a grounded-emitter transistor amplifier is approximately

$$r_b + \frac{r_e}{1 - a}$$

13-8. Using Fig. 13.21 and Eq. (13-21), show that the input impedance of a grounded-base transistor amplifier is approximately

$$r_b(1 - \alpha) + r_e$$

Chapter XIV

14-1. Compute the rms noise voltage in a 25-kc band developed across a 20,000 ohm load by a temperature-limited diode carrying an average current of 10 ma. Also for a space-change-limited diode with the same operating conditions.

14-2. Compute the thermal noise generated in the load resistance of Problem *14-1*.

14-3. A certain amplifier has a noise figure of 20 (=13 db). If a pre-amplifier with a gain of 5 is to be added, how low must be its noise figure, to improve the noise figure of the complete amplifier? How low to give 3 db improvement?

Chapter XV

15-1. An FM transmitter has been designed to operate with a peak frequency deviation of 25 kc. We wish to adjust the gain of the modulator. An audio oscillator is adjusted to supply a maximum-amplitude modulating signal. If this oscillator is set to the appropriate frequency, the modulator gain can be set by adjusting it for zero carrier output (detected through a sharp filter). What is an appropriate modulating frequency for this scheme of adjustment? Note: There is a multiplicity of answers, corresponding to the various zeros of $J_0(m)$. The highest modulating frequency allows the easiest carrier filtering.

15-2. A certain FM transmitter is modulated by a constant amplitude audio oscillator. The carrier is found to disappear for modulation frequencies of 4000 and 1750 cps. What is the peak frequency deviation for this amplitude of modulation?

15-3. A full-wave rectifier, fed at 60 cps, is connected to a resistance load with no filter. The voltage across the load is fed to a high-impedance differentiating circuit (series RC). In the output voltage of the differentiator, the 120-cps component is found to have 3 times the amplitude of the 240-cps component. Compute the time constant (RC) of the differentiator.

ANSWERS TO PROBLEMS

1-3. 0.075 ohm

1-4. 0.857 volt; 8.57 amp; 7.34 watts; 57%

1-5. 1.40 volts; 1.40 amp; 1.96 watts; 93%

1-6. 2.2 ohms

1-7. For the 1-ohm resistance: 2 amp, 2 volts, 4 watts. For the 2-ohm resistance: 1.2 amp, 2.4 volts, 2.88 watts. For the 3-ohm resistance: 0.8 amp, 2.4 volts, 1.92 watts.

1-8.

0.67 amp	0.67 volt	0.44 watt
1.33	2.67	3.55
0.67	2.00	1.33

2-1. (*a*) $\frac{28}{9} = 3.11$ (volts).
(*b*) The currents in the 2, 3, 4, 5 ohm resistances are, resp., $\frac{50}{11}$, $\frac{40}{11}$, $\frac{30}{11}$, $\frac{20}{11}$. The current through the short-circuit cross-over is $\frac{10}{11}$.
(*c*) $\frac{154}{45} = 3.42$ (ohms).
(*d*) $\frac{140}{199} = 0.703$.

2-2.
$$9I_1 - 5I_2 - 4I_3 = 20$$
$$-5I_1 + 8I_2 - I_3 = 0$$
$$-4I_1 - I_2 + 8I_3 = 0$$
$$I_1 = 6.33 \qquad I_2 = 4.42 \qquad I_3 = 3.72$$
Voltage drops

$2I_2 = 8.84$ $\qquad$ $5(I_1 - I_2) = 9.55$
$3I_3 = 11.16$ $\qquad$ $4(I_1 - I_3) = 12.44$
$1(I_2 - I_3) = 0.70$

2-3.
$$\tfrac{11}{6}V_a - V_b - \tfrac{1}{3}V_c = 0$$
$$-V_a + \tfrac{29}{20}V_b - \tfrac{1}{4}V_c = 0$$
$$-\tfrac{1}{3}V_a - \tfrac{1}{4}V_b + \tfrac{7}{12}V_c = 1$$
$$V_c = 3.16$$
$$I = \frac{20}{3.16} = 6.33$$

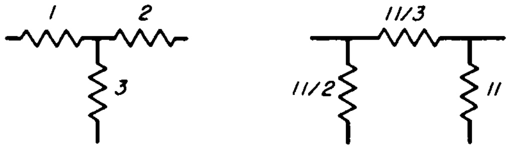
3-1. $x = 0.9$; $y = -1.3$; $z = -1.5$.

3-2. The determinant is zero; the equations are inconsistent. The first and third together (added) require $2x + y - z = 1$, which conflicts with the second equation.

3-3. Add second and third equations.
$$x = 0;\ y = \tfrac{1}{2};\ z = \tfrac{1}{2}$$

3-4. 168.

4-1.



4-2. Using $I_2 = 0$; $I_1 = 0$; $E_1 = 0$ successively, yields

$$A = \tfrac{4}{3};\ C = \tfrac{1}{3};\ D = \tfrac{5}{3};\ B = \tfrac{11}{3}.$$

$AD - BC = \frac{20}{9} - \frac{11}{9} = 1$ gives a check.

4-3. $Z_{I_1} = 2\sqrt{\tfrac{11}{5}}$; $Z_{I_2} = \tfrac{1}{2}\sqrt{55}$.

4-4. Under matched conditions, the load sees a source of R ohms internal resistance, having an OCV of 12, 8, 6, 4, 2 volts for the five output cases. From Figure 4.20 (include the source resistance!)

$$\text{OCV} = 12\frac{R_2 + R^2/(R_1 + 2R)}{\left(R_2 + \dfrac{R^2}{R_1 + 2R}\right) + \left(\dfrac{RR_1}{R_1 + 2R}\right) + R}$$

Substitute the matching condition $R_1R_2 = R^2$ and simplify to

$$\text{OCV} = \frac{12}{1 + R_1/R}.$$

This yields

$$\frac{R_1}{R} = \frac{12}{\text{OCV}} - 1$$

$$= 0, \tfrac{1}{2}, 1, 2, 5 \text{ for the five cases}$$

and

$$\frac{R_2}{R} = \frac{R}{R_1} = \infty, 2, 1, \tfrac{1}{2}, \tfrac{1}{5}$$

5-1. (*a*) $W_1 = 5 \times 10^{-8}$ joule
$W_2 = 10^{-7}$ joule
$\Delta W = W_2 - W_1 = 5 \times 10^{-8}$ joule $= 5 \times 10^{-8}$ watt-second
$= \tfrac{1}{2}$ erg.
(*b*) 200 volts

5-2. $V_2 = 100$; $Q_2 = 5 \times 10^{-10}$
Less work, because potential difference (hence also force) is less during separation.

5-3. $Q_1 - Q_2 = 5 \times 10^{-10}$ coulombs
$W = QV = 5 \times 10^{-8}$ joules
Charges battery.

5-4. $W_2 = 2.5 \times 10^{-8}$
$W_1 = 5 \times 10^{-8}$
$W_1 + W_{mech.} = W_2 + W_{batt}$
$W_{mech.} = 2.5 \times 10^{-8} + 5 \times 10^{-8} - 5 \times 10^{-8} = 2.5 \times 10^{-8}$

5-5. (*a*) $\tfrac{2}{3}$ μf.
(*b*) 3 μf.
(*c*) $\tfrac{1}{2}$ μf, $2\tfrac{2}{3}$ μf
(*d*) $\tfrac{6}{5}$ μf, 5 μf

5-6. 100 volts
$10^3 \times 10^{-6} = 10^{-3}$ second = 1 millisecond.

$$e^{-10^3 t} = \frac{1}{2},\ t = \frac{\ln 2}{1000} = \frac{0.693}{1000} = 0.693 \text{ millisecond.}$$

5-7. (*a*) 83.33 volts.
(*b*) 16.67 microseconds.
(The capacitor sees 50 ohms and 10 ohms in parallel.)

5-8. $Q = 2 \times 10^{-6} \times 10^2 = 3 \times 10^{-6}V$
$\therefore V = 66.67$ volts.
The time constant is 0.67 millisecond (the resistor sees the capacitors in series.)

5-9. $C_{pc} = 0.4$; $C_{gc} = 2.2$; $C_{gp} = 1.6$

6-1. (*a*) 3 henries.
(*b*) $\frac{2}{3}$ henries.
(*c*) 5 henries, 1.2 henries.
(*d*) 2.67 henries, 0.5 henries.

6-2. $W = \frac{1}{2}LI^2$
$W_{in} = 100$ joules
$W_{out} = 5$ joules
95 joules to remove core, −95 to replace it. The solenoid can do 95 joules of mechanical work.

6-3. The closing of the relay increases the inductance and the stored magnetic energy.

6-4. 1000 volts.

7-1. $\omega^2 = 10^8$
$\omega = 10^4$
$f = 1591$ cps.
$I_p = 10$ amp.
$V_p = \omega L_p I_p = 10^5$ volts

7-2. $I_R = 1.15$ amp (r.m.s.)
$I_L = 0.305$ amp, 90° lagging
$I = \sqrt{(1.15)^2 + (0.305)^2} = 1.19$ amp
$P = 115 \times 1.15 = 132.25$ watts (all in R)

7-3. $|Z| = \sqrt{(100)^2 + (\omega L)^2} = 390$ ohms
$|I| = \frac{115}{390} = 0.295$ amp.
$V_R = 29.5$ volts
$V_L = 111$ volts (90° leading)
$P = I^2R = 8.7$ watts

7-4. $V_L{}^2 + (10)^2 = (115)^2$
$V_L = 114.56$

7-5. $I_R = 1.15$
$I_C = 0.435$ (90° leading)
$I_t = 1.23$
$P = 132.25$ watts

7-6. $|Z| = 283$
$|I| = 0.406$
$V_R = 40.6$
$V_C = 108$
$P = 16.5$

7-7. Same as **7-4.**

7-8. $Z = 10 + 6283\angle\ 90° + 6366\angle\ -90°$
$= 10 + 83\angle\ -90°$
$|Z| = 83.6$ ohms
$I = 0.119$ amp (peak)
$V_R = 1.19$
$V_L = 746$
$V_C = 757$

7-9. $V_C = 10$
$I_C = 1.57 \times 10^{-3}$ (leading)
Z (of R and L in series) $= 10 + 6283\angle\ 90° \doteq 6283\angle\ 90°$
$I_R = I_L = 1.59 \times 10^{-3}$ (lagging)
$I_{line} \doteq 0.02 \times 10^{-3}$
$V_R = 0.016$
$V_L \doteq 10$

7-10. $V_L = V_C = V_R = 10$
$I_R = 1$
$I_L = \frac{10}{6283}\angle\ -90° = .001592\angle\ -90°$
$I_C = \frac{10}{6366}\angle\ 90° = .001571\angle\ 90°$
$I_{line} = 1 + 0.000021\angle\ -90° \doteq 1$

7-11. Parallel L, C

$$\frac{1}{Z} = \frac{1}{6283}\angle\ -90° + \frac{1}{6366}\angle\ 90° = 0.000021\angle\ -90°$$

$Z = 47600\angle\ 90°$
$Z_{total} = 10 + 47600\angle\ 90° \doteq 47600\angle\ 90°$
$I = 0.00021\angle\ -90°$
$V_R = 0.0021\angle\ -90°$
$V_L = V_C \doteq 10$

7-12. $Z \doteq 6283\angle\ 90°$
$I = 0.00159\angle\ -90°$
$I_R = 0.00159\angle\ -90°$
$V_C = V_R \doteq 0.016\angle\ -90°$
$V_L \doteq 10$

$$I_C = \frac{0.016\angle\ -90°}{6366\angle\ -90°} \doteq 0.0000025$$

8-1. (*a*) $5 + 7j$
(*b*) $-6 + 17j$
(*c*) $\dfrac{18 + j}{25} = 0.72 + 0.04j$

8-2. $3 + 4j = 5e^{j\theta}$, $\theta = \tan^{-1} \frac{4}{3} = 53°$

$$\sqrt{3 + 4j} = \pm\sqrt{5}\, e^{j\theta/2} = \pm\sqrt{5}\,(\cos \theta/2 + j \sin \theta/2)$$
$$= \pm\sqrt{5}\,(0.895 + 0.446j)$$
$$= \pm(2 + j)$$

Also

$$(a + jb)^2 = 3 + 4j = a^2 - b^2 + 2abj$$
$$\therefore a^2 - b^2 = 3$$
$$ab = 2$$

Yielding $b = \pm 1$, $a = \pm 2$ (a and b same sign)

Hence $\sqrt{3 + 4j} = \pm(2 + j)$

8-3. $Z_L = 377j$

$$Z = \frac{100 \times 377j}{100 + 377j} = 93.5 + 24.8j$$

8-4. $Z = 100 + 377j$

8-5. $Z = 1 + j\omega L = 1 + 377Lj$

8-6. $Z_C = -265j$

$$Z = \frac{100(-265j)}{100 - 265j} = 100\frac{7.02 - 2.65j}{8.02} = 87.5 - 33j$$

8-7. $Z = 100 - 265j$

8-8. $Z = 1 - \dfrac{j}{\omega C} = 1 - \dfrac{0.00265}{C}j$

8-9. $Z = 10 + 6283j - 6366j = 10 - 83j$

8-10. $$Z = \frac{(10 + 6283j)(-6366j)}{10 - 83j} = \frac{40{,}000{,}000 - 63660j}{10 - 83j}$$

$$\doteq \frac{40{,}000{,}000}{10 - 83j} = 40{,}000{,}000\frac{10 + 83j}{6989}$$

$$\doteq 57{,}300 + 475{,}000j$$

8-11. $$\frac{1}{Z} = \frac{1}{10} + \frac{1}{6283j} - \frac{1}{6366j}$$

$$\begin{aligned} &= 0.1 - 0.0001592j + 0.0001571j \\ &= 0.1 - 0.0000021j \\ &\doteq 0.1 \\ Z &= 10 \end{aligned}$$

8-12. $$Z = 10 + \frac{(6283j)(-6366j)}{6283j - 6366j}$$

$$\begin{aligned} &= 10 + 482{,}000j \\ &\doteq 482{,}000j \end{aligned}$$

8-13. $$Z = \frac{10(-6366j)}{10 - 6366j} + 6283j \doteq 10 + 6283j \doteq 6283j$$

8-14.
(*8-3*) $96.7\angle\ 15°$
(*8-4*) $390\angle\ 75°$
(*8-6*) $93\angle\ -21°$
(*8-7*) $283\angle\ -69°$
(*8-9*) $83.6\angle\ -83°$
(*8-10*) $478{,}000\angle\ 83°$
(*8-11*) $10\angle\ 0°$
(*8-12*) $482{,}000\angle\ 90°$
(*8-13*) $6283\angle\ 90°$

8-15. For Problems 7-4 and 7-7,

$$P = |I|^2R = |I|^2 ReZ = 100 \text{ watts}$$

For the others

$$P = |I|^2R = |E|^2\frac{R}{|Z|^2}$$

Here, E is r.m.s. Note that $|E_{rms}|^2 = \frac{1}{2}|E_{peak}|^2$

(7-5) (8-6) $\frac{R}{Z^2} = Re\,\frac{1}{Z} = 0.01$

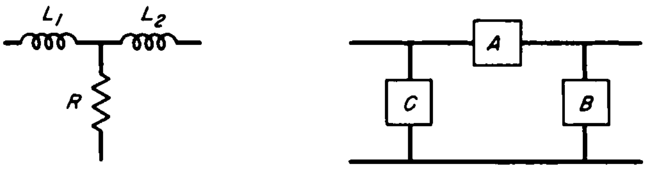
$P = 132$ watts

(7-6) (8-7) $\frac{R}{Z^2} = 0.00125$

$P = 16.5$

(7-8) (8-9) $P = 0.0715$
(7-9) (8-10) $P = 12.5 \times 10^{-6}$
(7-10) (8-11) $P = 5$
(7-11) (8-12) $P = 2 \times 10^{-9}$
(7-12) (8-13) $P = 12.6 \times 10^{-6}$

8-16.

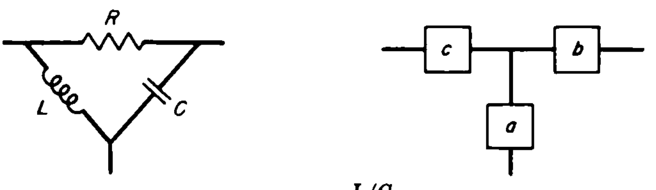


$$A = -\frac{\omega^2 L_1 L_2}{R} + j\omega(L_1 + L_2)$$

$$B = R\left(1 + \frac{L_2}{L_1}\right) + j\omega L_2$$

$$C = R\left(1 + \frac{L_1}{L_2}\right) + j\omega L_1$$

8-17.



$$a = \frac{L/C}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$b = \frac{R/j\omega C}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$c = \frac{j\omega LR}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

8-18. Real parts of p_1 and p_2 are damping factors. Imaginary parts are angular frequencies ($2\pi f$). When p_1 and p_2 are both real, decay without oscillation, called "overdamped case."

$$p = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

$R^2 > 4L/C$, both roots real, overdamped
$R^2 = 4L/C$, roots equal, critical damping
$R^2 < 4L/C$, roots complex, underdamped

9-1. Equations (9-4) give the answer in literal form with

$$Z_3 = j\omega L = 2\pi 10^6 10^{-3} j = 6283j$$

$$Z_2 = R_2 + \frac{1}{j\omega C_2} = 10^2 - \frac{j}{2\pi 10^6 10^{-11}} = 100 - 15910j$$

$$Z_1 = R_1 + \frac{1}{j\omega C_1} = 10^3 - \frac{j}{2\pi 10^6 10^{-11}} = 1000 - 15910j$$

$$V_L = (I_1 - I_2)j\omega L$$
$$V_{SS} = I_1/j\omega C_1$$

These yield (peak values)

$$I_1 = 10^{-3}(0.25 + 1.30j)$$
$$I_2 = -10^{-3}(0.17 + .85j)$$
$$V_L = -13.5 + 2.6j$$
$$V_{SS} = 20.7 - 4.0j$$

Alternatively, Equations (9-5) could be used.

9-2. $M = 0.21$

Use equivalent circuit Fig. 9.7 and Equations 9-4 or 9-5.

$$z_1 = R_1 + R_2 + j\omega L_1 + \frac{1}{j\omega C_1} = 1100 - 9627j$$

$$z_2 = R_2 + \frac{1}{j\omega C_2} + j\omega(L_1 + L_2 - 2M) = 100 - 9124j$$

$$z_c = R_2 + j\omega(L_2 - M) = 100 + 1823j$$
$$z_1 z_2 - z_c^2 = -10^6(84.4 + 11.4j)$$
$$(z_1 z_2 - z_c^2)^{-1} = -10^{-9}(11.6 - 1.57j)$$
$$I_2 = -(4.02 + 20.9j)10^{-5}$$
$$I_1 = (13.2 + 106j)10^{-5}$$
$$V_{C_2} = -3.32 + 0.64j$$

9-3. $z_1 = 1100 - 9627j$
$z_2 = 100 - 3846j$
$z_c = 100 + 503j$
$(z_1 z_2 - z_c^2)^{-1} = -10^{-9}(26.7 - 3.84j)$
$I_2 = -(4.59 + 13.0j)10^{-5}$
$I_1 = (12.1 + 103j)10^{-5}$
$V_{C_2} = -2.07 + 0.73j$

9-4. $M = 0.707$
$z_1 = 1100 - 9627j$
$z_2 = 100 - 15282j$
$z_c = 100 - 1301j$
$(z_1 z_2 - z_c^2)^{-1} = -10^{-9}(6.80 - 0.82j)$
$I_2 = (0.39 + 8.92j)10^{-5}$
$I_1 = (11.9 + 104j)10^{-5}$
$V_{C_2} = 1.42 - 0.06j$

9-5. $n = 10:1$

(a) $I_s = 50$, $I_p = 5$; $5\omega L_p' = 115$, $L_p' = 0.061$ henry

(b) $L_s' = L_p'/n^2 = 0.00061$ henry

(c) The loaded transformer presents the series combination of 100 ohms and 0.061 henry to the line.

$$Z = 105 + 23j$$
$$I = 1.045 - .23j$$
$$V\text{-Drop} = 5.225 - 1.15j; \ |\text{Drop}| = 5.35 \text{ volts}$$

(d) $|115 - (5.225 - 1.15j)| = 109.8$

(e) $\frac{1}{10}\{115 - (5.225 - 1.15j) - 23j(1.045 - 0.23j)\} = 10.45 - 2.3j$
or $\frac{1}{10}\{100(1.045 - 0.23j)\} = 10.45 - 2.3j$

(f) $I_s = 10.45 - 2.3j$
$I_s = 10.7\angle -12.5°$

10-1. Resonance is a satisfactory approximation for the maximum.

$$\omega^2 C L_2(1 + k^2) = 1$$
$$C = 9.9 \times 10^{-12} \doteq 10 \ \mu\mu\text{f}$$
$$V_0 = 35200 \text{ volts (peak)}$$

10-2. $C_1C_3 = \dfrac{1}{4\pi^2 10^6 10^8}$

$$\frac{C_1}{C_3} = \frac{30000}{10000} - 1 = 2$$

$C_3 = 0.011\ \mu$f
$C_1 = 0.022\ \mu$f

10-3. (a) $|0.06 - 0.11j| = 0.125$

(b) $|0.05 + 0.10j| = 0.112$

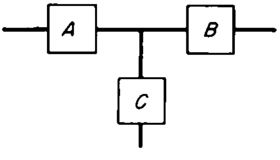
These answers are inaccurate, but typical of the effects of numerical round-off in calculations of balanced circuits. Algebraically, the general answer is $\dfrac{1}{4}\dfrac{1 - \rho^2}{1 - \rho^2 + 2\sqrt{2}\,j\rho}$; where $\rho = f/f_0$, and $f_0 = 1000$ cps, the frequency of balance.

The precise answers for $\rho = 2$ and $\rho = \frac{1}{2}$ are

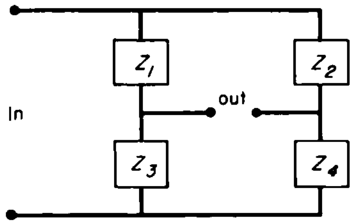
$$\left|\frac{9 \pm 12\sqrt{2}\,j}{164}\right| = 0.117$$

Note the symmetry in the response to frequencies ρf_0 and f_0/ρ.

10-4.



to be equivalent to



Open-circuit input impedances:

$$A + C = \frac{(Z_1 + Z_3)(Z_2 + Z_4)}{Z_1 + Z_3 + Z_2 + Z_4}$$

$$B + C = \frac{(Z_1 + Z_2)(Z_3 + Z_4)}{Z_1 + Z_2 + Z_3 + Z_4}$$

Open-circuit voltage transfer:

$$\frac{C}{A + C} = \frac{Z_3}{Z_1 + Z_3} - \frac{Z_4}{Z_2 + Z_4}$$

Yielding

$$C = \frac{Z_2 Z_3 - Z_1 Z_4}{Z_1 + Z_2 + Z_3 + Z_4}$$

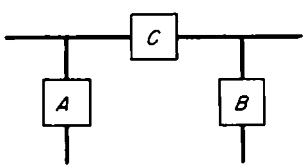
and

$$A = \frac{Z_1 Z_2 + Z_3 Z_4 + 2 Z_1 Z_4}{Z_1 + Z_2 + Z_3 + Z_4}$$

$$B = \frac{Z_1 Z_3 + Z_2 Z_4 + 2 Z_1 Z_4}{Z_1 + Z_2 + Z_3 + Z_4}$$

Note that $C = 0$ is the bridge balance condition.

10-5.



Voltage transfer ratios require

$$\frac{B}{B + C} = \frac{Z_3}{Z_1 + Z_3} - \frac{Z_4}{Z_2 + Z_4} = \frac{Z_2 Z_3 - Z_1 Z_4}{(Z_1 + Z_3)(Z_2 + Z_4)}$$

$$\frac{A}{A + C} = \frac{Z_2}{Z_1 + Z_2} - \frac{Z_4}{Z_3 + Z_4} = \frac{Z_2 Z_3 - Z_1 Z_4}{(Z_1 + Z_2)(Z_3 + Z_4)}$$

both of which vanish at balance, requiring either $A = B = 0$ or $C = \infty$.

The open-circuit input impedances are

$$\frac{A(B+C)}{A+B+C} = \frac{(Z_1+Z_3)(Z_2+Z_4)}{Z_1+Z_2+Z_3+Z_4}$$

$$\frac{B(A+C)}{A+B+C} = \frac{(Z_1+Z_2)(Z_3+Z_4)}{Z_1+Z_2+Z_3+Z_4}$$

which do not vanish at balance—hence the balance condition is not $A = B = 0$, but $C = \infty$.
Considerable algebraic manipulation yields

$$C = \frac{Z_2Z_3Z_4 + Z_1Z_3Z_4 + Z_1Z_2Z_4 + Z_1Z_2Z_3}{Z_2Z_3 - Z_1Z_4}$$

$$B = \frac{Z_2Z_3Z_4 + Z_1Z_3Z_4 + Z_1Z_2Z_4 + Z_1Z_2Z_3}{Z_1Z_2 + Z_3Z_4 + 2Z_1Z_4}$$

$$A = \frac{Z_2Z_3Z_4 + Z_1Z_3Z_4 + Z_1Z_2Z_4 + Z_1Z_2Z_3}{Z_1Z_3 + Z_2Z_4 + 2Z_1Z_4}$$

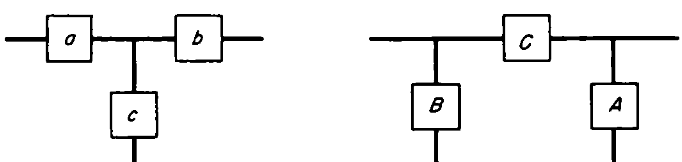
10-6. Close the third mesh with a load Z_L. Write the mesh equations and eliminate I_2. This yields a pair of equations for I_1 and I_3 which can be interpreted as the mesh equations for the equivalent T. The series arms are $\dfrac{Z_1Z_4}{Z_1+Z_2+Z_4}$ and $\dfrac{Z_2Z_4}{Z_1+Z_2+Z_4}$; the shunt arm is $Z_3 + \dfrac{Z_1Z_2}{Z_1+Z_2+Z_4}$. The balance condition, Equation (10-23), makes the shunt impedance zero.

10-7. (*a*) $RC = 65 \times 10^{-6}$ second $= 65$ microseconds.
(*b*) From Eq. (10-28), V_0/I_1 is readily expressed as a function of R and ωCR, which must have the value $1/\sqrt{6}$, yielding

$$\frac{V_0}{I_1} = \frac{29R}{3 + 4j\sqrt{6}}$$

whence $R = 3530$.

11-1. The transformation between the T and Π networks



is

$$aA = bB = cC = ab + ac + bc = \frac{ABC}{A+B+C}.$$

For case (a):

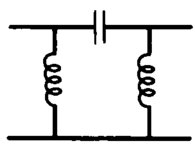
$$A = j\omega L_2 + \frac{1}{j\omega}\frac{L_1 + L_2}{L_1 C}$$

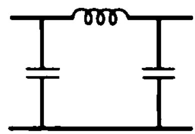
$$B = j\omega L_1 + \frac{1}{j\omega}\frac{L_1 + L_2}{L_2 C}$$

$$C = j\omega(L_1 + L_2) - j\omega^3 L_1 L_2 C$$

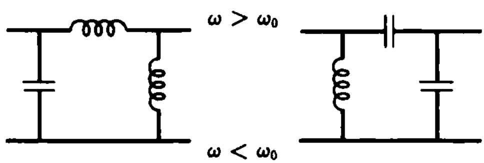
Hence A and B are series LC combinations, but C is not a simple network (note ω^3). At any *fixed* frequency, A, B, and C can be replaced by simple inductance or capacitance depending upon their signs. Note that A, B, and C all change sign at the same frequency, say ω_0. Thus at any fixed $\omega > \omega_0$, case (a) yields

while for $\omega < \omega_0$, we find





The same results follow in case (c). Similarly, for (b) and (d):



11-2. $Z_{I_1} = 3742$

$Z_{I_2} = 1871$

$r = -\frac{2}{7}$

$$e^{-2\theta} = \frac{\sqrt{14} - 7j}{\sqrt{14} + 7j} \text{ has magnitude unity}$$

$$e^{-2j\beta} = \frac{\sqrt{14} - 7j}{\sqrt{14} + 7j} = \frac{(\sqrt{14} - 7j)^2}{63}$$

$$e^{-j\beta} = \frac{\sqrt{14} - 7j}{3\sqrt{7}} = \frac{\sqrt{2} - j\sqrt{7}}{3}$$

$\tan\beta = \sqrt{\frac{7}{2}},\ \beta \doteq 62°$

11-3. (*a*) 0.707 in one direction
1.414 in the other direction
(*b*) 62°
(*c*) $k = \frac{2\sqrt{2}}{3} = 0.943$

$\log k = -0.0255$
$20 \log k = -0.51$
Insertion loss of $-\frac{1}{2}$db, or insertion gain of $\frac{1}{2}$db.

11-4. (*a*) $R_2 = \frac{2500 - 1871}{2500 + 1871} = 0.1439$

(*b*) From Eq. (11-25):
$$1 - Re^{-2\theta} \doteq 0.97 + 0.6j$$
$$\text{Magnitude} = 1.14$$
The input current is *increased* by 14%.
(*c*) $k_2^2 = 1 - R_2^2 = 1 - 0.021$
2.1% power loss
(*d*) $\text{Loss} = -20 \log k_2/k = -10 \log k_2^2 + 20 \log k$
$= -0.42$ db

11-5. $R_1 = -0.11$
$R_2 = 0.14$
$\sigma = \frac{1}{0.992 - 0.012j}$
$|\sigma| = 1.008$
$k_1^2 = 1 - 0.012$
$\text{Loss} = -20 \log \left|\frac{k_1k_2\sigma e^{-\theta}}{k}\right| = -0.44$ db.

12-1. $I = 8.8$ ma; $V_D = 92$ volts; $V_R = 88$ volts.

12-2. $R = 5000$ ohms; $L \geq 4.42$ henries.

12-3. $P_0 = 200$ watts; 314 va. secondary; 222 va. primary; 268 va. average; $E_s = 1110$ volts (rms).

13-1. $r_p = 10{,}000$ ohms; Amplification = 7.1

13-2. $A = 0.88$

13-3. $A = 7.86$; $Z = 3182$ ohms

13-4. $A = 20$

13-5. $A = 0.96$

13-6. $A = 20.2$; $Z = 1238$ ohms

14-1. From Eq. (14-1):
$$\langle i^2 \rangle = 2 \times 1.59 \times 10^{-19} \times 10^{-2} \times 25 \times 10^3$$
$$i_{rms} = 8.9 \times 10^{-8} \text{ amp}$$
$$e_{rms} = 17.8 \times 10^{-4} \text{ volt} = 1.78 \text{ mv.}$$
Space-charge limited, $\langle i^2 \rangle$ and $\langle e^2 \rangle$ reduced by factor of 25.
$$e_{rms} = 0.56 \text{ mv.}$$

14-2. $\langle e^2 \rangle = 8 \times 10^{-12}$
Rms open-circuit noise voltage, 8.9 μv

14-3. (*a*) $F_1 < 20 - \frac{19}{5} = 16.2$
(*b*) $F_1 = 10 - \frac{19}{5} = 6.2$

15-1. $a = 25$ kc

$$p = a/m = \frac{a}{2.405} = 10.4 \text{ kc.}$$

15-2. For vanishing carrier, $m = 2.4, 5.5, 8.7, \ldots$. The first two are consistent with

$$\begin{aligned} a = pm &= 4000 \times 2.4 = 9600 \\ &= 1750 \times 5.5 = 9625 \end{aligned}$$

Hence $a = 9.6$ kc (peak deviation).

15-3. From Eq. (15-29), the 120 cps component has 5 times the amplitude of the 240 cps component, hence the filter response at 120 cps must be $\frac{3}{5}$ of that at 240 cps. This yields $RC = 0.55$ millisecond.

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