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This issue of *The Bell System Technical Journal* is devoted to a selection of articles dealing with various phases of mathematical statistics and quality control. The Editorial Board and Editorial Staff of the *Journal* present this "all statistics" issue in the belief that the growing importance of statistics to communication technology warrants the simultaneous publication of these articles.

The Editors are pleased to include in this series of papers on statistical subjects one by Dr. Walter A. Shewhart whose pioneering work in statistical quality control has served as an impetus to wider use of statistical methods in the Bell System. This paper, which dates back to 1935, was one of a series of internal technical memoranda of the Quality Assurance Department of the Bell Telephone Laboratories, Inc. It was prepared by Dr. Shewhart in the course of a series of departmental group discussions having to do with the development of the fundamental philosophies of quality control and quality assurance.

Nature and Origin of Standards of Quality

By W. A. SHEWHART

(Manuscript received September 25, 1957)

This paper discusses the importance, from the viewpoint of judging quality, of: the end to be served by a standard of quality; the nature of the accepted binding force of the standard upon the acts of those interested in the standard; and the role of the judge of quality in shaping the standard in terms of natural law, authority, specification, custom, and precedent.

I. OBJECT

The control of quality of manufactured product involves three coordinate functional steps: the specification of the aimed-at standard of quality; the production of pieces of product that will be of standard quality; and the determination of whether or not product thus made is of standard quality. These three steps are respectively legislative, executive, and judicial in character. The object of this paper is to consider the nature and origin of standards of quality from the viewpoint of judging the quality of product.

Such a judgment as herein considered is made the basis of one or the other of two kinds of action: (1) the acceptance or rejection of a piece of a given kind of product for service; and (2) the adjudication of a complaint about the quality of a piece of product in service. The two judgments are of the type: J_A — this piece of product (or this lot of N pieces of product) is (or is not) of standard quality, and J_B — this piece of product (or this lot of N pieces of product) was (or was not) of standard quality. In either case, it should be noted that the judgment is rendered in respect to the quality of a piece of product that is already in existence at the time the judgment is rendered — it is a judgment after the act of specifying and after the act of making the piece of product in question. This problem of judging the quality of a piece of product after it is made is definitely different from the legislative problem of specifying prior to the making of a piece of product what its quality should be in the light of information then available; and different from the coordinate executive problem of making a piece of product that will have the standard quality.

Judgment, in the sense here used, implies a comparison of the quality of a piece of the given kind of product at some particular time with the standard for the piece at that time in the light of the evidence then available. If it were possible to specify completely and in an operationally definite and verifiable sense the standard of quality for things of a given kind, and if it were possible to specify the operational technique that would determine with certainty whether or not the quality of a given thing was that specified, the problem of judging would be routine in nature. But neither of these operations is possible. Hence in judging the quality of product, we must take account of the fact that a standard cannot be specified in this rigorous sense and that the practical standard of quality is determined not alone by written specifications of the quality characteristics prior to the making of a particular piece of

product but also by natural law, authority, custom, and precedent, existing at the time the particular piece of product is being judged. In other words, the quality judge is not, as it were, handed a standard of quality already made with which to compare the quality of a given piece of product. Instead he is only handed the stones with which to build such a standard. Through his interpretation of specifications, custom, precedent, natural law, and authority, the quality judge in a sense gives operational meaning to the standard of quality in much the same way that a judge gives operational meaning to the law of the land, whether it be statutes, custom, precedent, or constitution.

Obviously, therefore, before a quality judge may render a judgment of either type J_A or J_B, he must "determine" the standard that is to be used. But what is there to guide such determination? It goes without saying that he is not free to act as he pleases. In what follows we shall see how the acts of the quality judge in determining the standard depend upon: (a) the intent of the standard; (b) the nature of the binding force that the standard is presumed to have upon those concerned; and (c) the available source or sources from which a standard must be derived.

To begin with, we shall consider the nature of a standard of quality as a means to an end, as this will give us a background for considering in turn the binding or constraining force of a standard upon the acts of those making use of it and then the origin of a standard in natural law, authority, specifications, custom, and precedent.

II. STANDARD AS MEANS TO AN END

Dr. Gaillard of the American Standards Association defined a standard as: "A formulation established verbally, in writing or by any other graphical method, or by means of a model, sample or other physical means of representation, to serve during a certain period of time for defining, designating, or specifying certain features of a unit or basis of measurement, a physical object, an action, a process, a method, a practice, a capacity, a function, a performance, a measure, an arrangement, a condition, a duty, a right, a responsibility, a behavior, an attitude, a concept, or a conception."

This definition stresses one important characteristic which is commonly attributed to a standard, namely, that it is something fixed. The definition of standard here is very broad indeed; it would seem to include the rules of mathematics and formal logic, the rules of syntax of a language, and even legal statutes. In fact, it also includes social mores

assumed policy objective of production, any evidence which may have come to hand, particularly in the processes of production, inspection, and analysis of complaints, indicating the present specifications to be incomplete in that they do not include requirements on certain variables which it seems desirable to control. Quite naturally such requirements will sooner or later find their way into specifications, but the quality judge must, insofar as possible, act in accord with what he considers to be potential changes if the policy of accepting only quality that may reasonably be expected to be satisfactory, adequate, dependable, and economic is to be met. In other words, the quality judge must fill in the gaps in existing specifications in so far as new evidence obtained since such specifications were written would indicate to be reasonably desirable.

(b) If the quality judge is to accept the theory that a specification is but a means to an end and is to take account of the fact that the justification of a specification rests upon an ever-changing body of evidence, it is necessary for him to use discretion in judging quality of product to be either acceptable or rejectable upon the basis of specifications alone. In other words, certain non-conformance cases may arise in respect to specified quality characteristics which may have under certain conditions little effect upon the experienceable quality of such equipment in use. In such a case it may likely be uneconomical on the part of all concerned to reject such product. Such action on the part of the quality judge is not, as it were, ignoring a specification but rather making a judgment upon evidence which was not available at the time the specification was written.

(c) If any one of the four items in S_{X_i} and S_{Y_i} are omitted in the written specification, it is necessary that such be supplied by the quality

ment should lie if it is to be that which he believes will prove to be wanted. True enough, he is likely to give weight to the data constituting his previous experience of production methods which indicates limits within which variability may be expected under production. Obviously, however, such evidence is likely to be very meagre indeed as compared with the cumulative evidence obtained after production starts. Experience shows that there is an economic limit to the allowable variation in the quality of product turned out in a given process. In other words, it is often found that it is more economical to discover and eliminate assignable causes of variation of quality than it is to leave these in the production process and reject that portion of the product that does not meet the required limits. The quality judge has an important rôle to play in devising techniques which will indicate the presence of assignable causes and of using these in helping the production department to establish economic control limits which serve as standards for future production.

(e) We now come to what is perhaps the most important rôle of the judge of quality in giving operational meaning to a specification. Even though an operationally definite and verifiable meaning of quality is given in the specification, there are two reasons why it is often necessary to resort to sampling in order to determine whether or not quality meets the specification: (a) it is often uneconomical to give 100 per cent inspection, particularly where defective parts would be weeded out in final assembly or at the time of installation, and (b) it is often not feasible to give 100 per cent inspection because of the destructive nature of the method of verification of the quality, as, for example, in testing the tensile strength of materials and the blowing current for fuses. In such a case the quality judge must supply an inspection specification which will insure the following two things: (1) that a satisfactory amount of data or evidence will be accumulated upon which to render judgment as to the nature of the quality of the unsampled portion of the lot, and (2) that an operation will be indicated to determine whether or not it should be rejected whenever the degree of belief in the satisfactoriness of the unsampled portion of the lot upon the basis of evidence thus accumulated is insufficient to justify the acceptance of the lot. The question, How much data?, depends in general upon the degree of economic control of quality previously obtained and hence the inspection operation specified must be such that it keeps abreast of the continual supply of information obtained in the process of inspecting product if such an operation is to give adequate assurance of quality at a minimum of cost.

We are now in a position to turn our attention to a consideration of the nature of the binding force of specification. In the first place, a specifica-

tion may be made the basis of a contractual agreement between two parties, in which case it takes on certain legal as well as moral binding force characteristic of a contract. One of the conditions usually assumed for the validity of a contract is that the two parties to the contract be cognizant of the contents thereof. Of course, in many instances specifications of quality are extremely involved from a scientific and engineering viewpoint and hence it is to be expected that parties to a contractual agreement involving highly technical specifications of quality must be capable of arriving at a common meaning of such specifications. This limits the field in which technical specifications may be made the basis of valid contracts. The second source of binding is, of course, the requirement that the quality accepted as meeting the specifications be judged in the end as satisfactory by those making use of the product. In this case, however, we should note that the binding force is not so much that requiring that the quality of product meet the specifications as it is that requiring that the quality be found in the end to be satisfactory by those making use of the product. In this case, however, it must not be overlooked that there is a growing tendency on the part of the majority of users of most kinds of goods to place reliance upon the judgment of men or groups of men whom they accept as being technical authorities, such, for example, as national or international standardizing committees.

In the third place, as previously noted, a producer is sometimes bound because of his own future interests to adhere to a specification even when such adherence would not be demanded at the time by those whose wants the quality is supposed to satisfy. For example, the appreciation of high quality often comes through experiencing high quality. One who has never heard what a technician would consider to be good music, good quality of radio transmission, good quality of telephone transmission, or good quality of some musical instrument, might never have the desire to experience such. Progress, therefore, often comes by living up to a specification of quality even beyond the limits wanted by the majority of those concerned at a given time. In other words, the producer's personal interest is often more binding than either or both the bindingness of a specification made a part of a contractural relation and the immediate interests of the consuming group, if he is to lead the way in evolving standards that will later be wanted by the majority.

4.4 *Custom*

All of us are more or less creatures of habit; all of us are more or less influenced throughout life by the habits and the common methods of

acting of those around us. We early learn that society always takes a revenge of one form or another for a breach of any of its common ways of acting and hence as members of any group we feel more or less bound to follow the conventions of that group. For example, in our methods and means of communicating one with another, we are bound to a large extent to the customary use of symbols, either written or spoken. Even the meaning of a written specification of quality so far as the majority of a group or society is concerned inherently depends to a large extent upon the customary interpretation of words and other symbols used therein. It is to be expected that custom should play a part in the production of standards. Thus a long while before the development of written specifications of standards of quality there existed unwritten standards, as it were, fixed by the customs of certain groups. At least the meaning of certain words was sufficiently common to members of a group to enable the interchange of goods.

With the development of mass production practices first introduced in the eighteenth century, there has grown up an ever-increasing appreciation of the economic advantages to be attained by securing a high degree of uniformity in the quality characteristics of a given kind of thing. It is significant for what follows that there exist at least three ways in which customary quality may differ from specified quality in such a way as to constitute a part of the standard which is inherently binding upon the group.

In the first place, a given kind of product produced over an extended period of time in considerable quantities may exhibit a uniformity in quality characteristics not specifically expressed in the specifications of the form $S_{\bar{x}_i}$ and S_{r_i}. In the second place, one or more quality characteristics may be specified to have magnitudes lying within a definite range although experience has shown that over a certain period in the past in which many pieces of this kind have been produced the magnitudes of the particular quality characteristics thus specified have differed from their specified values but in a way which has been acceptable from the viewpoint of use. For example, take the case where the production of a new kind of product is started in which the specification of one of the important quality characteristics, such as length of life, is that it shall not be less than some specified value. Let us assume that N pieces of this kind of product have been made and put into service and that the experience thus obtained shows that the lengths of life of these N pieces of product have been distributed uniformly about an average length $\bar{L}$ considerably above the specified length S. Particularly if the number N of pieces of this kind of product that have gone into service is large and

if those making use of this kind of thing come to expect an average life of approximately $\bar{L}$, even though the specification simply calls for a life not less than S, most producers would feel bound in certain ways to maintain a quality not assignably less than $\bar{L}$. It is quite likely, to say the least, that some consumers of this kind of thing might feel justified in registering a complaint if they should find in the future that the length of life of this kind of thing was significantly lower than $\bar{L}$ even though it did not fall below S. In the third place, even though no specific mention is made of the fact that in the specification, users of a given kind of product may reasonably expect that observed variability in the quality characteristics specified should be no larger than that which for economic reasons should be left to chance. For example, consider the class of users of a given kind of thing such as an automobile. If we find upon comparing notes with our neighbors or others using the same make of car that ours differs from theirs in a way which we consider undesirable, we are likely to feel like registering a complaint.

In rendering quality judgments the quality judge must take into account at least these three ways in which custom may effectively constitute a part of the standard of quality binding in a given case. In fact, he not only must take into account custom in certain instances but in fact, as we have seen in the previous section, he must also in certain ways help in establishing custom, as, for example, in the analysis of results of inspection and the determination of economic limits of variability.

The ultimate source of binding force in maintaining uniformity is quite naturally the consumer's desire for uniformity. Such a common want, however, is in a certain sense potentially of legal binding in the sense that many statutes as well as common law have their origin in custom. In any case, the degree of binding depends among other things upon: the available evidence of the existence of a custom; how long and how continuously it has existed; whether or not the custom has been peaceably enjoyed; to what extent those affected have regarded it a duty to follow the custom; and whether or not the custom in question is consistent with all other accepted customs.

4.5 *Precedent*

To begin with, it is desirable to clarify the distinction here made between custom and precedent. Custom, as we have seen, is of the nature of an established practice that has more or less gradually come into existence. Precedent, on the other hand, arises in the judgment in re-

spect to the quality of product that has already been produced as to whether or not it is or was of standard quality. Precedent arises, in other words, in the finding of the quality judge. If it were feasible to write specifications of quality that were ideally necessary and sufficient for satisfaction, adequacy, and dependability at an economic cost, and if it were feasible to determine with certainty whether or not the quality of a given article met such specifications, there would be little, if any, occasion to consider the rôle of precedent. Since, however, this is not feasible, there are three types of judicial findings which are important in quality control.

Cases of non-conformance with specified requirements are bound to arise where the information available at the time justifies the judge of quality in concluding that, under the specific conditions existing in the case, the quality, even though non-conforming, is acceptable. Likewise, conditions are bound to arise where, even though the quality of a given thing does conform to specifications, it may not be acceptable. This follows at once from the fact that we are not able to state the necessary and sufficient quality requirements. This class of precedent arises as a natural consequence of looking at a standard as a means to an end, rather than as an end in itself.

Just as common law arises for the most part in the judicial recognition, interpretation, and formulation of custom, so also does the effective control of custom in standardization come about through the recognition, interrelation, and formulation of custom on the part of the judge of quality. Thus judicial declarations or recognitions of the existence of a custom constitute another source of precedent. In quality control one of the very important examples is the judicial decision as to whether or not a custom has been established with regard to the degree of variability which should be left to chance.

A third source of precedent is interpretation: first, interpretation of the operational meaning of a standard even in so far as it is specified; second, interpretation of the sampling technique required in order to give adequate information upon which to render a judgment; and third, interpretation of the rules of judging and interpreting evidence as to the quality of product.

V. CONCLUSION

The practical meaning and significance of a standard of quality is largely determined by the end which it is supposed to serve in use and by

the nature and degree of the binding force or sanction accorded it by the group interested in or affected by the standard. The standard itself may originate in one or more of the five sources: natural law, authority, specification, custom and precedent. In any case the judge of quality is not handed a standard ready made with which to compare the quality of any given kind of manufactured goods — instead he of necessity plays an important rôle in shaping and determining the standard as derived from these sources.

Contribution of Statistics to the Development Program of a Transformer for the L3 Carrier System

By G. J. LEVENBACH

(Manuscript received August 20, 1957)

Statistical methods played a significant part in the development program of the L3 system. Experiments were designed to assist in improving the manufacture of the input and output transformers of the amplifiers. Detailed analysis of a few of these experiments is presented.

I. INTRODUCTION

In previous issues of THE BELL SYSTEM TECHNICAL JOURNAL the problems in design, development and manufacture that were encountered in building the L3 coaxial carrier system are described. This system provides 1,860 one-way telephone channels or 600 one-way telephone channels plus one TV channel over each coaxial tube. The L3 system is capable of transmitting a television signal over a distance of approximately 1,000 miles and telephone signals, approximately 4,000 miles.

From the start of the development program, statistical methods have played a significant part. Special acceptance procedures have been set up to assure that the shipped product would meet certain distribution requirements.[1] Control chart techniques were generously applied both in the manufacture of component parts and for subassemblies.[2] This paper gives in part a case history of one of the difficult components. The viewpoint is that of the experiments designed to overcome difficulties in the initiation of the manufacturing process and to explore possibilities of improvement of the component.

A detailed discussion of the present manufacturing techniques of this component, the input and output transformer of the amplifier, has already been presented by Earle.[3] That paper will be used freely to provide the technical details and pictures necessary for an understanding of

the experiments. No basically new statistical designs were employed in this development. The main interest lies in the fact that these experiments together with the engineering design and the manufacturing operations, including the appropriate process controls and inspection techniques, were integrated in the development program.

An endeavor is made in this paper to point out the logical link between the statistical analysis and the engineering consequences. Advantages of the use of statistical methods in experimental work are as follows:

1. In designing an experiment (the adjective "statistical" will be implied from now on), the type of analysis to be performed on the data is a major consideration from the start. In some experiments one might wish to determine one or several of a larger number of factors which have an important effect. In this case the analysis should yield a statement about the significance of the effects of the operating factors, with a predetermined small risk of being wrong. In other cases one looks for quantitative measures of one or more properties and then the statistician will estimate intervals within which, on the basis of the experimental results, one can expect with a high probability, the true (unknown) value of these measures to lie.

2. Under the limits set by the requirements in the preceding paragraph the design will be such that the experimental effort is minimized.

3. The design will take into account the adverse effects on the precision of the experiment caused by known ambient conditions which are not completely under control of the experimenter.

4. In so far as possible, safeguards against effects from unknown factors will be incorporated in the designs.

The preceding points require that quantitative notions be introduced as much as possible, not only for the things measured but also for the operating factors and disturbances. The experimenter and the statistician try to agree on a statistical model, describing the expected behavior of the physical items in the experiment. Given the model, the statistician can suggest experimental arrangements, in an efficient way with respect to the experimental effort, which should yield reliable information about the problem at hand.

In many cases it turns out, when the observations become available, that the model has to be modified or that the experiment has not been performed according to the design. This usually increases the burden on the analysis. It happens occasionally that the data do not show definite results, and further experimentation is needed. In that case the careful statistical analysis might yield clues in what direction to proceed as well as useful quantitative information about disturbing factors, experimental errors, etc.

It has been pointed out that a difference between agricultural and industrial experiments lies in the time factor involved.[4] Extension or repetition of agricultural experiments is in most cases only possible at yearly intervals. In industry the time schedule is much less restricted. Therefore it pays to use involved designs in agriculture even at the cost of complex analyses. Where it is comparatively easy to start a new or partly new experiment, complexity may be too high a price to pay. Moreover when experimentation goes on parallel to a production process, speed in obtaining the results of an experiment is of prime importance. Simplicity of design is also valuable when the underlying model is not yet well understood, as in the early stages of exploratory development.

In the early stages of the manufacture of a complex component, the actual specification has to be written on the basis of the results on a comparatively small number of samples. It can hardly be expected that these samples are fully representative of the production items which will be manufactured. Nevertheless the design engineer will have to determine workable limits to give the manufacturer the opportunity to get his production rolling without producing too many items not acceptable for use. In the L3 system, studies of the over-all requirements of the system had indicated in which way they had to be broken down into the requirements for the components and subassemblies in order to assure satisfactory operation. In the case of the transformer under discussion the electrical transmission requirements were more or less fixed. It was the task of the design engineer to translate these requirements into mechanical tolerances which could be controlled during manufacture. On the basis of the equivalent diagram (Fig. 1) for the transformer, extensive calculations had been made to determine the relation between the variations of the electrical parameters and the over-all transmission response.[5, 6, 10] Each of the electrical parameters as shown in Fig. 1, a simplified picture of the equivalent diagram, does not necessarily correspond to a discrete part of the physical transformer, but the diagram can be considered to represent a model, which lends itself to mathematical treatment. Mathematical considerations, statistical or otherwise, on the basis of the model, help to establish the mechanical requirements for the manufacture, as will be shown later.

A few of the experiments performed to quantify the underlying relationships will be presented in a logical order. Although, through the pressure of circumstances, the actual experiments did not proceed in a strictly orderly fashion, the general line of experimentation was that described in this article. Production was progressing in parallel with this experimental program and, as described elsewhere,[2] control charts

showed several assignable causes of variation in the parameters, which were removed by improvements in manufacturing techniques.

The experiments selected to illustrate the development program will be discussed in some detail. In terms of their most important results these experiments can be described as follows:

1. Pinpointing the input and output network (Fig. 2) as the major source of variation. The transformer (Fig. 3) is the main component in

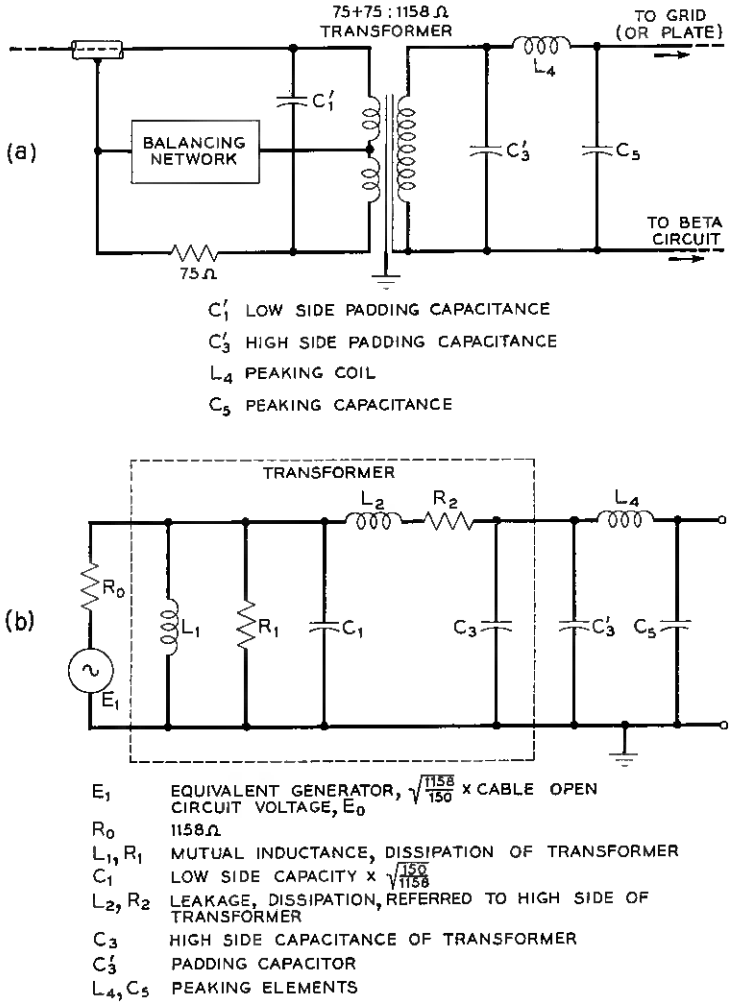

Fig. 1 — Coupling networks circuits. (a) Physical elements. (b) On ground equivalent circuit, adequate for gain and feedback computations in an amplifier configuration employing ground coupling networks.

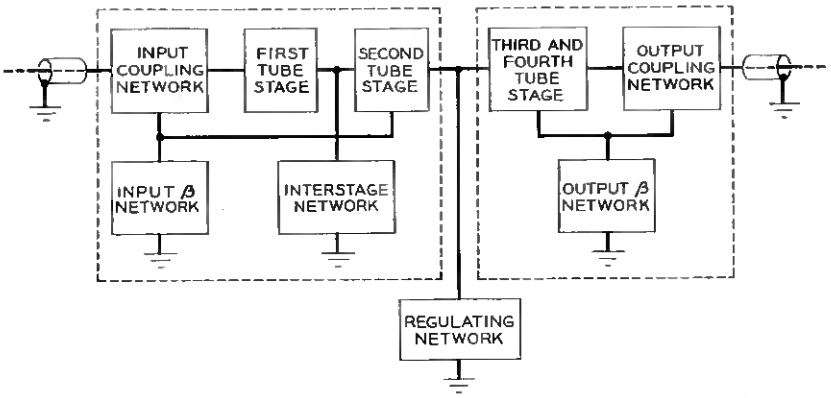

Fig. 2 — Block diagram of amplifier.

Fig. 3 — Transformer and separate inner and outer winding forms with windings.

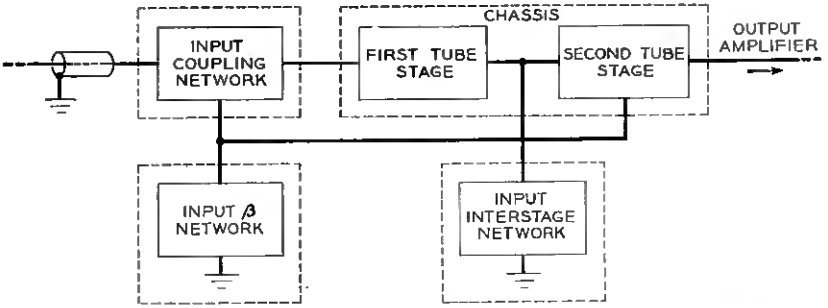

Fig. 4 — Input (sub) amplifier block diagram subdivision for hyper-graeco-latin square experiment.

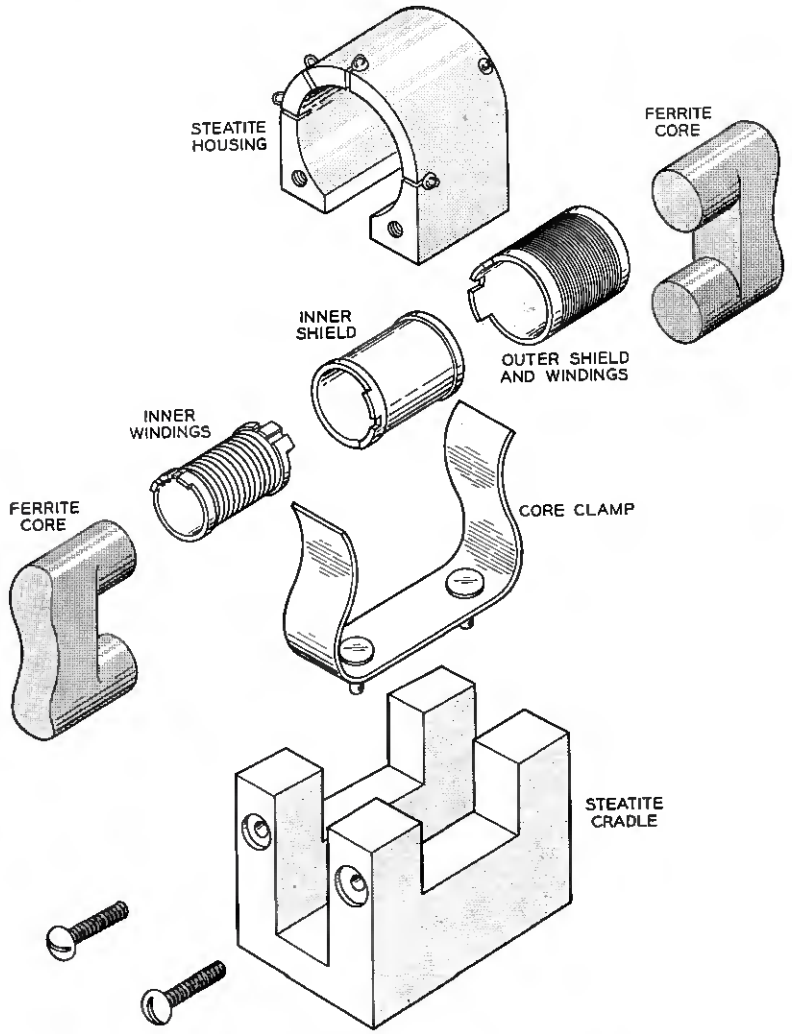

Fig. 5 — Exploded schematic of the 2504A transformer.

these networks so that subsequent experimentation was concentrated on the transformer.

2. Determining the required manufacturing limits for the wall thickness of the outer winding form of the transformer, (Fig. 5, 6).

3. Determining the required manufacturing limits for the "cutback" of the shield under the outer winding of the transformer. (The term "cutback" will be explained later.)

4. Comparing the over-all measured response of the complete amplifier with its predicted performance as based on a detailed knowledge of the components obtained from the designed experiments.

II. FINDING THE NETWORK CAUSING MOST OF THE UNWANTED VARIATIONS

From the first series of amplifiers manufactured, it appeared that the differences between the measured transmission gain curves for the various amplifiers were larger than could be tolerated.

For this discussion it is sufficient to represent the amplifier as in Fig. 2. The blocks represent subassemblies which are mechanically designed so that a high degree of reproducibility in the location of the components and the connected wiring is achieved. It is therefore feasible to inquire if one or two of the subassemblies are responsible for the bulk of the variability in measured gain. It is worth noting that the "large" variations are not large when compared to the capabilities of the measuring equipment. The over-all admissible amplifier gain variations are in the order of 0.2 to 0.3 db corresponding to voltage variations of less than 3 per cent.

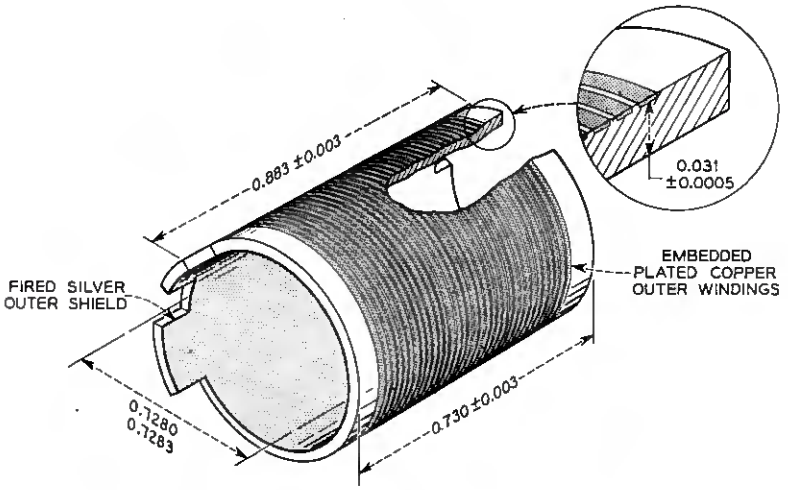

Fig. 6 — Outer winding form and detail to show "wall thickness."

Consequently, to be able to discriminate between the contributions of the individual components one must be able to measure reliably to as close as, say, 0.01 db, i.e., to detect voltage variations in the order of 0.1 per cent. This approaches the presently attainable precision of these types of measurements. Finally, these subassemblies are fairly expensive and were not in plentiful supply at the time these experiments had to be run.

Practically, it was reasonable to treat the input and output amplifiers, as indicated in Fig. 2, as separate entities. Each of these two subamplifiers can be measured accurately for its transmission gain in the same way as can be done with the completed amplifier. In this fashion a direct relationship exists between the results of sub- and complete amplifiers. This favorable condition does not exist with respect to the relationship between sub-amplifiers and its subassemblies which are also indicated in Fig. 2. To determine if the subassemblies meet the over-all requirements, it is necessary to combine them into sub-amplifiers and measure those.

Input and output amplifiers consist basically of the same subassemblies. The type of designed experiment used for both sub-amplifiers was identical so that a detailed example for the input-amplifier tells the main story. It was felt from engineering considerations that interactions between the various subassemblies in an input or output amplifier would be of a considerably smaller magnitude than the variations of interest and therefore could be neglected.

Four types of subassemblies make up a sub-amplifier, so these four should enter as factors in our experiment. As was pointed out above, a set of subassemblies has to be assembled into an amplifier to make transmission measurements possible. To evaluate this procedure, every time the set of available subassemblies was combined into sub-amplifiers it was considered a run. This gives the following factors to be used in the experiment:

Runs
Coupling Networks
Interstage Networks
Beta Networks
Chassis

The number of levels for each of the factors is determined below.

The experimental design should incorporate five factors and minimize the number of required subassembly units; however it does not have to measure interactions. An experimental design that lends itself to this type of situation is a hyper graeco-latin square.[7]

Assigning, as is shown in Table I, the rows to the different runs and

TABLE I. — HYPER GRAECO-LATIN SQUARE LAYOUT

Run No.	Chassis						
	1	2	3	4	5	6	7
1	A1α	B2β	C3γ	D4ϑ	E5ϵ	F6ζ	G7η
2	D3β	E4γ	F5ϑ	G6ϵ	A7ζ	B1η	C2α
3	G5γ	A6ϑ	B7ϵ	C1ζ	D2η	E3α	F4β
4	C7ϑ	D1ϵ	E2ζ	F3η	G4α	A5β	B6γ
5	F2ϵ	G3ζ	A4η	B5α	C6β	D7γ	E1ϑ
6	B4ζ	C5η	D6α	E7β	F1γ	G2ϑ	A3ϵ
7	E6η	F7α	G1β	A2γ	B3ϑ	C4ϵ	D5ζ

Latin letters—Coupling Networks
Greek letters—Beta Networks
Numerals—Interstage Networks

the columns to the different chassis, we can allocate the coupling networks, identified by latin letters, so that each occurs exactly once in each column and row. This results in a latin square. If we add to this structure two more arrays, one composed of greek letters, identifying the beta networks and one composed of numbers identifying the interstage networks, such that each letter or number occurs only once with each other symbol we have an (incomplete) system of "orthogonal squares". Data from such a pattern will allow us to obtain unbiased estimates of the main effects of the five factors incorporated, in the absence of interactions. Moreover, the estimates for one factor will be statistically uncorrelated with those for other factors.

The square in Table I is of size 7 × 7. This is the smallest practical size that could be applied. For 5 factors a square of size 5 × 5 could in theory be used as four different orthogonal squares of this size exist,[8] but we would have only four degrees of freedom to estimate our error.

No orthogonal squares of size 6 × 6 exist. In a 7 × 7 we have 49 observations and 18 degrees of freedom for error. For this experiment 7 units of each type had to be assembled 7 times into a set of 7 amplifiers each. The first set of 7 amplifiers was numbered 1 to 7 in random order, thus at the same time identifying the subassemblies. The complete layout of the experiment is given in Table I.

Measurements on the completed input amplifiers were made at the highest frequency of interest in the transmission band, 8.3 mc, and are listed in Table II. The analysis of variance computed in the usual manner from these data is presented in Table III. Apparent measurement standard deviation $\hat{\sigma} = \sqrt{0.000254} = 0.016$ db.

It is evident from the sums of squares column in the latter table that the coupling networks contribute a very sizeable part of the total variation. The experimental error as estimated from the residual mean

TABLE II. — TRANSMISSION MEASUREMENTS AT 8.3 MC IN DB

Run No.	Chassis No.						
	1	2	3	4	5	6	7
1	4.739	4.799	4.935	4.713	4.824	4.998	4.870
2	4.759	4.841	5.044	4.820	4.870	4.852	4.896
3	4.819	4.749	4.878	4.933	4.719	4.873	4.986
4	5.003	4.749	4.866	5.001	4.797	4.761	4.836
5	4.978	4.824	4.722	4.820	4.945	4.797	4.898
6	4.804	4.910	4.774	4.916	5.013	4.819	4.714
7	4.897	5.056	4.861	4.701	4.827	4.913	4.743

TABLE III. — ANALYSIS OF VARIANCE

Source	D/F	Sums of Squares	Mean Square	Significance Level
Coupling Networks	6	0.376359	0.062726	≦1%
Interstage Networks	6	0.037422	0.006237	≦1%
Beta Networks	6	0.003410	0.000568	not significant at 5% level
Chassis	6	0.003075	0.000512	
Runs	6	0.003381	0.000564	
Residual	18	0.004634	0.000254	
Total	48	0.428281		

squares amounts to 0.016 db. This disregards the effect of reassembling, as indicated by runs, which, however, is not significant at the 5 per cent level. It would be possible to pool the run, sum of squares, with that for error as estimated from the residual mean square to get more degrees of freedom for error but no new insight would be gained by this procedure. In the type of investigations described a level of significance of 5 per cent or smaller is generally applied. This implies that the chances are 5 per cent or less that, on the basis of the analysis, effects would be singled out for further engineering consideration when actually these effects are nonexistent.

To further illustrate the engineering implications, the results of Table III can be written in terms of the projected model for this experiment. It was assumed that the effects of the members of each of the subassemblies on the amplifier gain were normally distributed. The average value of the amplifier gain can be interpreted as the performance of an amplifier consisting of subassemblies of exact nominal values. The interesting part, however, is the gain variation from amplifier to amplifier, caused by the deviations from nominal of the subassemblies. These deviations

TABLE IV. — STANDARD DEVIATION ESTIMATES FOR THE VARIATIONS DUE TO THE DIFFERENT NETWORKS

Coupling Networks	0.094 db
Interstage Networks	0.029 db
Beta Networks	0.007 db
Chassis	0.006 db
Runs	0.007 db

TABLE V. — APPROXIMATE 90 PER CENT CONFIDENCE LIMITS FOR THE VARIATIONS DUE TO THE DIFFERENT NETWORKS

	Lower Limit (db)	Upper Limit (db)
Coupling Networks	0.065	0.181
Interstage Networks	0.019	0.056
Beta Networks	0.0	0.016
Chassis	0.0	0.015
Runs	0.0	0.016

can be measured by the standard deviation of their respective distributions. These standard deviations as derived from Table III are listed in Table IV and their approximate 90 per cent confidence limits in Table V.[9]

It appears again that the coupling networks contribute most to the variations in the transmission of the subamplifier. The interstage networks are of secondary importance, whereas the other three factors can be neglected. A similar picture emerged from the companion experiments on the output amplifier. It was therefore logical to concentrate first on trying to decrease the variability of the coupling network of which the transformer was the main part.

III. WALL-THICKNESS STUDIES ON THE OUTER COIL FORM OF THE TRANSFORMER

The transformer, even in its simplified form as in the equivalent circuit of Fig. 1, involves many parameters. By numerical evaluation the changes in transmission gain due to specified changes in these parameters were calculated on the basis of this circuit.[5, 6] As has already been pointed out, not all of the parameters in the equivalent diagram are directly represented in the physical transformer; therefore a relationship between the parameters and physical dimensions is not easy to establish.

From evaluation of the electrical circuit it was felt that the capacitance at the high inductance side of the transformer, C_3 in Fig. 1, would be a major contributor to the gain variation. Direct correlation between the behavior of this capacitance and various mechanical properties on

the basis of control charts did not yield sufficiently strong clues, partly due to the fact that the measurement accuracy in the production process was marginal in view of the small variations concerned. On the basis of engineering experience one of the strongly suspected mechanical variables was the wall thickness of the outer coil form of the transformer. The exploded views in Fig. 5 and Fig. 6 show that the outer form carries the winding with the highest number of turns. These turns are ground into the vycor glass body and they are subsequently copper plated. A silver shield is sprayed on the inside of the vycor glass form and fired subsequently. The "thickness" of the wall as measured between the bottom of the groove and the inner face is about 0.031″ and the geometry of the situation leads us to expect a strong dependence of the high side capacity on the wall thickness. (Fig. 6.)

The experiment to estimate the quantitative influence of wall thickness variations on electrical properties was set up as follows:

Two batches of 9 transformers each were produced in accordance with current production specifications except that batch "A" contained outer coil forms with "thick" walls and batch "B" with "thin" walls. On a nominal thickness of about 0.031″ batch A was on the average about 6 ten thousandths thicker than batch B. Due to the difficult grinding process it was impossible to make all coil forms of the same batch exactly alike to the limit of measurement, i.e., to within half a ten thousandth. The resulting variation in this thickness within a batch is indicated by the standard deviation of 1.5×10^{-4}.

All these transformers were measured in the same standard amplifier and the gain was observed at a number of frequencies. In addition, various short-circuit and open-circuit impedances were determined on the isolated transformers. Since these impedances bear a direct relation to the magnitude of the parameters in the equivalent diagram, one obtains information about the variations in the parameter values from the observed variations in the impedances. Allowing for these variations in predicting the performance of the circuit on the basis of the equivalent diagram, it is possible to compare the observed gain with that predicted. An example of such a comparison will be discussed later.

After a complete first run of measurements had been made on the transformers as manufactured, a second run was performed after the thick walled and thin walled coil forms had been interchanged between the transformers of batch "A" and "B".

Identifying the transformers without a coil form by capital letters and the forms by lower case ones in accordance with the batch to which they originally belonged, the actual set-up is given in Table VI. This table

TABLE VI. — BASIC DESIGN FOR WALL THICKNESS DEPENDENCY DETERMINATION

Coil Form	Transformer Batch	
	A	B
a	Run 1	Run 2
b	Run 2	Run 1

represents the experiment only "batchwise". It is important to note with respect to the model given below, that the interchange of one pair of coil forms (one thick and one thin) did not in general take place within one pair of transformers (one from batch A and one from batch B). If this had been done, a different analysis could have been performed on the same amount of data.

The mathematical model underlying this design takes into account the following effects:

μ = average level — $j = 1, 2.$

β_j = batch — $i = 1, 2, \cdots, 9$

$\varphi_{i,j}$ = transformer i in batch j — $j = 1, 2.$

ω_k = wall thickness — $k = 1, 2.$

ρ_l = runs — $l = 1, 2.$

$\epsilon_{i,j,k,l}$ = residual, being the difference between the measurements of the i^{th} transformer in the j^{th} batch and its prediction from wall thickness, batch and run effect.

With these definitions the observations $y_{i,j,k,l}$ can be expressed as follows:

$$y_{i,j,k,l} = \mu + \beta_j + \varphi_{i,j} + \omega_k + \rho_l + \epsilon_{i,j,k,l}.$$

From Table VI it is apparent that the wall thickness is measured by the row differences, the batch effect by the column differences and the run effect by the diagonal differences. The latter is indistinguishable from the row by column interaction, but there were reasons to believe that the interactions were of a smaller order of magnitude than the run effect.

The results of the gain measurements at one of the frequencies employed, 8.3 mc, are presented in Table VII, which gives only the fractional db, expressed in thousandths of db. A constant whole number of db is omitted throughout. This incorporates the fixed gains and attenuations of the measuring set up.

TABLE VII. — GAIN MEASUREMENTS AT 8.3 MC. EFFECT OF DIFFERENT WALL THICKNESS OF OUTER FORM

	Batch A			Batch B	
	Transformer	× 0.001 db		Transformer	× 0.001 db
Run 1 "Thick" Wall	1	744	"Thin" Wall	1	531
	2	778		2	510
	3	723		3	437
	4	698		4	487
	5	738		5	447
	6	644		6	608
	7	711		7	562
	8	670		8	476
	9	604		9	470
Run 2 "Thin" Wall	1	645	"Thick" Wall	1	674
	2	582		2	700
	3	556		3	634
	4	577		4	711
	5	582		5	512
	6	524		6	725
	7	550		7	658
	8	483		8	680
	9	547		9	676

TABLE VIII. — ANALYSIS OF VARIANCE OF WALL THICKNESS EXPERIMENT

Source	Sum of Squares	Degrees of Freedom	Mean Square	Significance Level
Between batches	20 449	1	20 449	5%+
Between transformers, within batches	75 126	16	4 695	1%
Between runs	880	1	880	not significant at 5% level
Between wall thickness	203 401	1	203 401	<1%
Within transformers corrected for runs and wall thickness (error)	20 888	16	1 305	
Total	320 744	35		

The analysis of variance of these data is presented in Table VIII.

It is readily seen from Table VIII that the wall thickness accounts for most of the variations, and that the effect of runs is indistinguishable from the error. It is possible just as was done in Table V to calculate the variance components for these effects and its limits. Both however are only based on one degree of freedom which makes this procedure hardly

profitable. The batch effect is tested against the "transformer within batches" variation, and the level of significance is a little over 5 per cent. This indicates that there was a systematic difference between the two batches.

An estimate of residual variation can be obtained from the two observations on the same transformer corrected for the estimated differences due to wall thickness and run effects. The standard deviation for error is $\hat{\sigma} = \sqrt{1305} = 36$ or 0.036 db in actual units. This can be compared to the stated goal of 0.01 db and the result of the preceding experiment 0.016 db. The two averages computed for the different wall-thickness groups, $y_{..k}$, provides us with an estimate of the effect of the average change in wall thickness on the gain:

For the "thick" wall the estimated gain is 0.682 db.

For the "thin" wall the estimated gain is 0.532 db.

Average increase of 0.006″ in wall thickness results in an increase of 0.150 db at 8.3 mc. In order to find out if the experiment was sensitive enough to find the dependence on wall thickness of the transmission measurements of the individual transformers, the residuals, as calculated from the equation on page 35, are plotted against the measured wall thickness, Fig. 7. The measurements of the wall thickness could be read to the nearest 0.00005″, but as seen in Fig. 7, the variations are too great to show any significant correlation with the fine structure of the wall thickness.

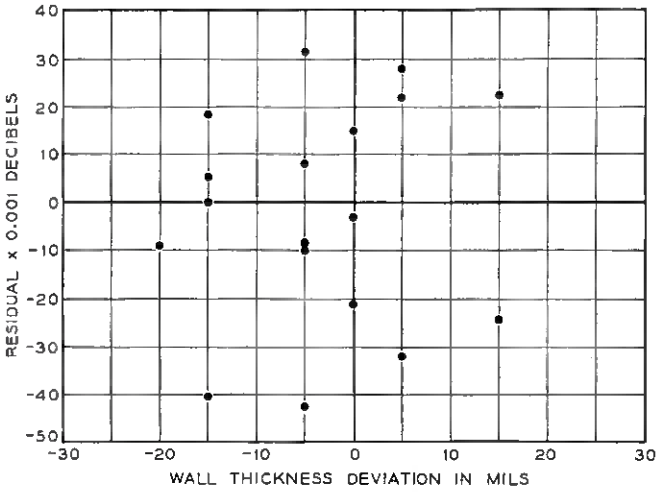

Fig. 7 — Residual variations, after the systematic effects have been removed, as a function of the wall thickness variation.

This experiment showed that it was necessary to control the wall thickness as closely as would be economical. The practical limit was known and the resulting transmission variations as estimated from the findings in this experiment, would be satisfactory from the over-all systems point of view.

IV. STUDY OF SHIELDING AND WINDING TERMINATION

Another mechanical variable to be considered is related to the termination of the winding on the outer form. One side of the winding (terminal No. 4) is connected to the shield that covers the inside of the coil form (Fig. 8). The other end has to be connected to one of the terminals (No. 5) on the body of the transformer. Electrically this latter point is sensitive and should be shielded as much as possible. On the other hand, in order to be able to connect the terminal lead to the winding a tab is inserted on the form. The shield must be cut back sufficiently to avoid short circuiting the winding via the tab. Originally a 0.150″ cutback was employed. Mechanical limitations make variations around the nominal cutback value unavoidable. The following experiment was set up to find out which nominal cutback value would result in the smallest variations in the transmission gain of the transformer.

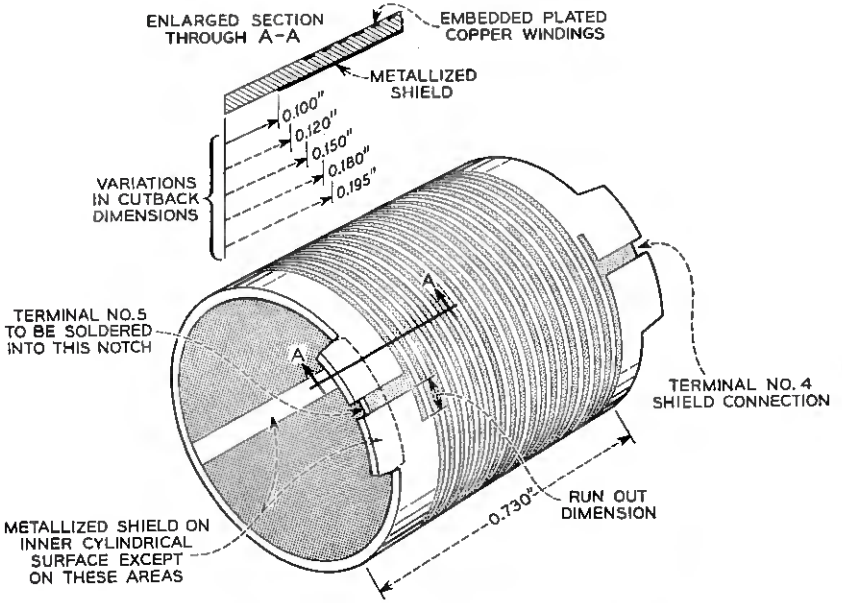

Fig. 8 — Side view of outer cylindrical spool, as per Fig. 6.

Another variable had been introduced into the problem inadvertently in the manufacturing process. This variable was related to the same sensitive point of the winding, and consisted of the amount of run-out or extra winding cut by the grinder beyond the point where the terminal tab No. 5 was connected to the winding. The run-out is measured in degrees of arc. Originally the run-out was kept close to 28°. After some manufacturing changes required for other reasons, the run-out variations became much larger. It was thought important to examine cutback and run-out at the same time to find any interaction effects if present.

An experiment to determine effects of cutback and run-out faces a difficulty similar to the previous one. The only hope to detect these effects is to try out the same transformer with different cutback and run-out values. This implies disassembling and re-assembling the transformers as many times as changes in the variables are made. In addition the change in variables can only go in one direction: the cutback can be increased by taking away a little bit of the shield and the run-out can be decreased by removing part of the run-out winding.

In accordance with these conditions an experiment was designed as indicated in the flow chart of Fig. 9, covering the possible combinations of applied changes in cutback and run-out in a systematic manner.

The cutback value of 0.150″ and the 28° run-out were the standard values in the manufacture at the time of the experiment. The stages of

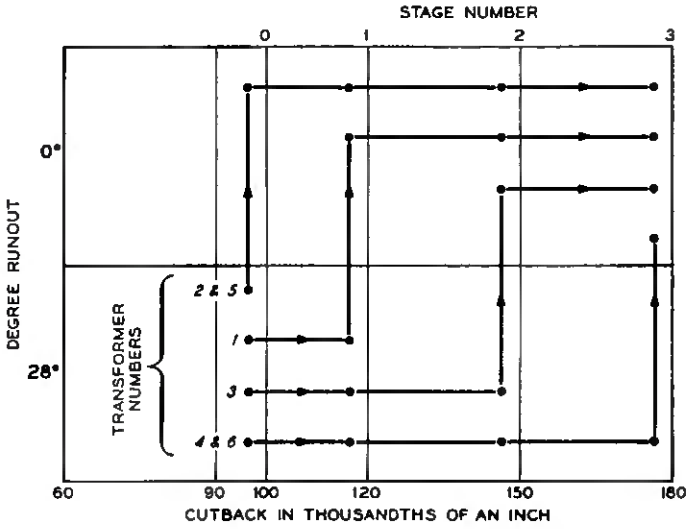

Fig. 9 — Flow chart of applied changes.

reassembly are indicated in order. The starting point for each transformer was 0.100″ cutback and 28° run-out.

This is an example of an experiment where several mishaps distorted the original design — a not unusual occurrence. Due to the time and costs involved the experiment was not repeated but a special effort was made to recover the information sought.

As in the previous experiment the transformers were measured in an amplifier to determine the gain characteristic as a function of frequency. In addition a few characteristic parameters were measured on the transformer itself.

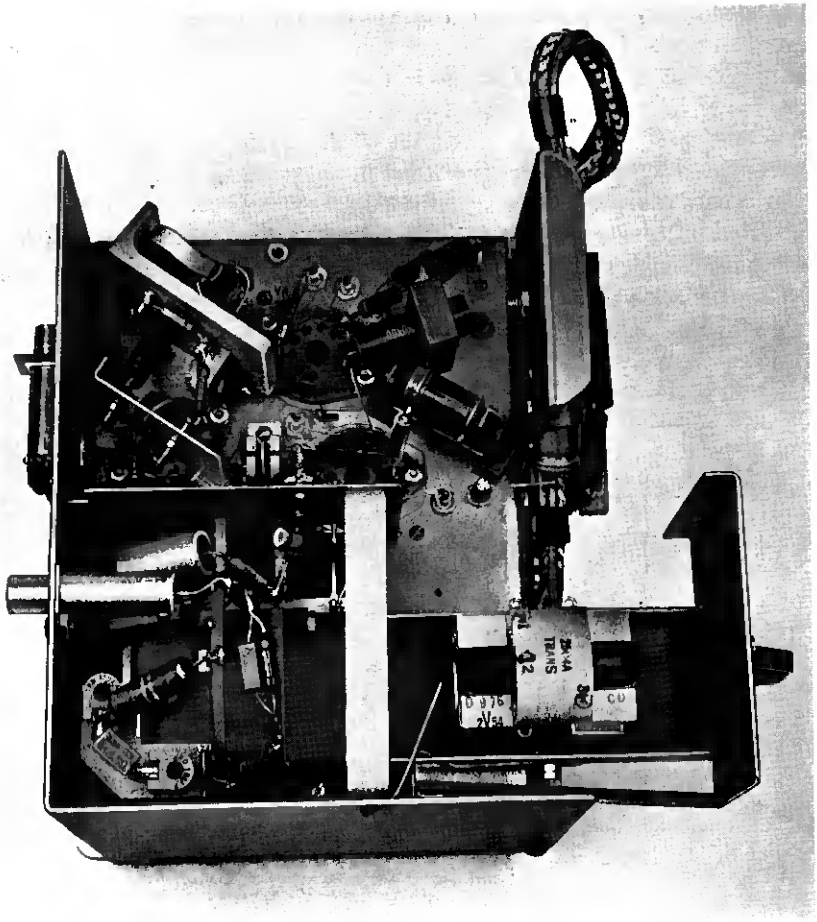

Fig. 10 — Jig for transformer measurement.

TABLE IX. — GAIN MEASUREMENTS AT 7.3 MC IN THOUSANDTHS OF DB

Transformer Number	Run-Out	Cut-Back × 0.001″				
		100	120	150	180	195
1	0	—	203	226	—	377
	2S	—	—	—	—	—
2	0	154	166	216	—	360
	2S	—	—	—	—	—
3	0	—	—	242	—	344
	2S	—	216	240	—	—
4	0	—	—	—	—	351
	2S	—	193	264	340	—
5	0	243	184	227	—	333
	2S	—	—	—	—	—
6	0	—	—	—	—	377
	2S	—	184	242	324	—

At the second stage of the experiment, Fig. 9, it appeared that the precision of measurement was rather poor due to the differences occurring when the transformer was disconnected from the amplifier and after the change in cutback and/or run-out reconnected by means of soldering. It was therefore decided to construct a contact fixture allowing the transformer to be plugged in and out of the amplifier.

For the first time after the fixture shown in Fig. 10 became available the transformers were measured twice — once soldered into the amplifier and once plugged in. This was done after the second reassembly and the previous measurements were adjusted to the fixture readings on the basis of this comparison. Almost all of the initial measurements (State 0) had to be discarded.

An additional deviation from the design occurred in the final stage when some of the transformers were cut back too far, to 0.195″ instead of 0.180″.

As an example the gain measurements at 7.3 mc are listed in Table IX.

When considering results such as in Table IX for further analysis the question arises what type of model should be fitted to the data. It goes without saying that apart from fitting the data the choice of the model must primarily make sense from an engineering standpoint. For designs like the hyper graeco-latin square of Section II and balanced designs in general the computational part of the analysis is small, measured in man-hours on a desk calculator. Changing the model in those designs by incorporating more factors or discarding alleged superfluous ones is simple, as the estimates of the effects of these factors in balanced situations are independent of the others.

In a case like in Table IX where no reasonable balance is left but where

the operating factors (cutback, etc.) are measurable or quasi measurable, regression models are indicated. The computational effort on a desk calculator to estimate the parameters in the regression model is considerable for three operating factors, as in our case. To explore a sufficient set of modifications of a model for four or more factors is only practical if an automatic computer is available.

As a first step in the analysis a linear multiple regression equation on three variables was calculated, the independent variables being:

x_1 : number of resolderings

x_2 : run-out

x_3 : cut-back.

The model fitted was:

$$Y - \bar{y} = \beta_1(x_1 - \bar{x}_1) + \beta_2(x_2 - \bar{x}_2) + \beta_3(x_3 - \bar{x}_3).$$

Estimates b of the β's resulted in

$$b_1 = -0.023 \text{ db/step}$$

$$b_2 = -0.0028 \text{ db/degree}$$

$$b_3 = 0.0052 \text{ db/mil.}$$

The corresponding analysis of variance table is Table X. Having a set of numbers it is always possible to go through the calculations and obtain estimates of the β's. The important part, however, is to determine how well the model fits. Looking at the analysis of variance Table X it appears in this case that a substantial part of the total observed variation as measured by the total sum of squares is explained by the model. The variations taken care of by the model are accounted for by the sum of squares for regression. The remainder measures our error. The estimated $\hat{\sigma}$ from the residual is $\sqrt{0.000630} = 0.025$ db.

TABLE X. — ANALYSIS OF VARIANCE FOR LINEAR REGRESSION ON 3 VARIABLES

Source	*SS*	*D/F*	*MS*
Regression	0.102845	3	0.034282
Residual	0.012596	20	0.000630
Total	0.115441	23	

TABLE XI. — ANALYSIS OF VARIANCE FOR LINEAR REGRESSION ON x_1 AND x_3

Source	*SS*	*D/F*	*MS*
Regression x_3 alone	0.096059	1	0.096059
Improvement due to x_1 added	0.002085	1	0.002085
	0.098144	2	
Residual	0.017297	21	0.000824
	0.115441		

Estimated $\hat{\sigma} = 0.029$.

It is of importance to find out the magnitude of the contribution by the individual independent variables x_i to our model. The general way of doing this is to drop one or more of the independent variables, recompute the estimates for the regression coefficients for the remaining variables and study the result in a new analysis of variance table.

As an example consider the simplified model

$$Y - \bar{y} = \beta_3(x_3 - \bar{x}_3)$$

and ask for the importance of incorporating the reassembly variate x_1 into this model. We can list the results as in Table XI. The improvement due to the addition of x_1 is not significant at the 5 per cent level.

Fig. 11 illustrates this procedure for a number of possibilities. Whatever model for fitting is chosen the total sum of squares is the same. The horizontal line at the top of the picture corresponds to this value of 0.115441 $(\text{db})^2$ (Table X). The length of the bars shows the part that is explained by incorporating in the model the variables listed at the bottom of each bar.

The run-out x_2 by itself does not appear to contribute anything appreciable, although in combination with resoldering x_1 it shows up a little. Cutback x_3 alone accounts for the bulk of the variation. Resoldering x_1 also shows up alone, but once x_3 is incorporated, addition of x_1 is not too important. This behaviour corresponds to the very strong correlation (correlation coefficient = 0.93) between the independent variates x_1 and x_3. This correlation stems from the fact that an increase in cutback necessarily corresponds to a later resoldering.

Engineering considerations suggested that the amount of non-linearity due to the cutback variable x_3 should also be examined. Cutbacks smaller than about 0.150″ do not reach under the first turn of the winding (Fig. 8) so they do not influence the shielding operation as strongly as when the cutback exceeds 0.150″.

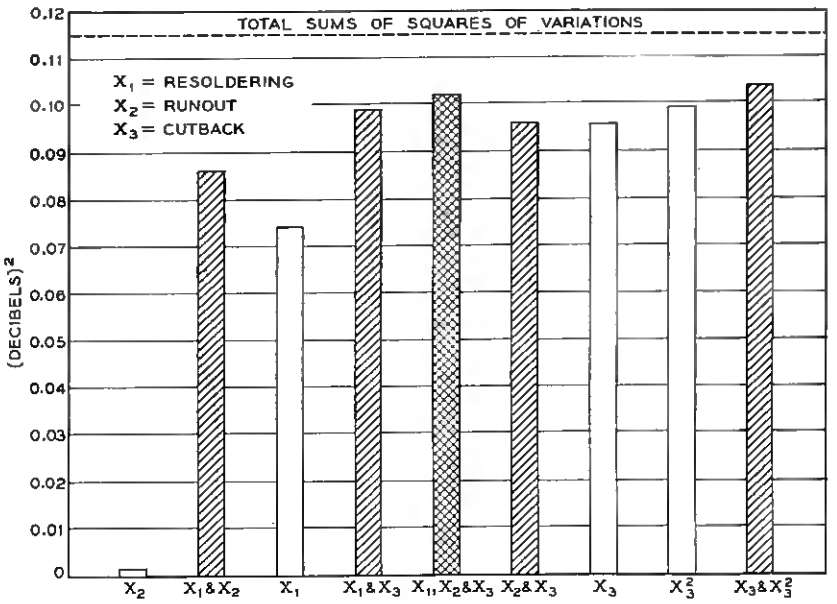

Fig. 11 — Contributions of various factors to the sums of squares of regression.

Introducing a quadratic term in the model

$$(Y - \bar{y}) = \beta_1(x_3 - \bar{x}_3) + \beta_{11}(x_3^2 - \overline{x_3^2})$$

gives the best fit to date as shown in Fig. 11. Run-out and resoldering are now left out, the former making no significant contribution and the latter being sufficiently taken care of by its correlation with the cutback. After all the resoldering was only of interest in the experimental situation, and did not occur in actual production.

Estimating the parameters yields

$$Y - 0.470 = 0.005\ x_3 + 0.000023\ x_3^2 \text{ db}$$

when x_3 is the cutback in 0.001″. The residual error standard deviation $\hat{\sigma} = 0.023$ db. Predicting some values

Cutback	*Gain*
0.120″	0.201 db
0.150″	0.238 db
0.180″	0.315 db

shows that 0.030″ less cutback with respect to 0.150″ makes a difference of about 0.04 db, whereas 0.030″ increase changes the gain by almost

0.08 db. Since gain should be insensitive to the variations in cutback which occur in manufacture, it was decided to keep the nominal cutback value at 0.120″.

In the analysis of each of the above experiments only one set of measurement results has been discussed. With the particular type of measuring set used, the gain of the amplifier is obtained as a continuous curve over the whole frequency range of interest. At about ten different frequencies ranging from 0.3 to 8.5 mc the results have been analyzed in the way described. In addition several discrete impedances in the transformer closely related to the elements in the equivalent diagram, Fig. 1, were measured directly.

In such a situation a very important check can be made about the assumptions underlying the experimentation and the analytical approach. On the one hand, we have the measurements of the performance of the transformer in the circuit and the measurements of various impedances connected with leakage, stray capacitances, etc. of the transformer. On the other hand, we have the analytical study of the model in the form of the equivalent diagram, Fig. 1, which provides us with a prediction of the over-all performance from the values of these impedances. If this prediction is sufficiently close to the measured over-all performance we can use control of the impedances to control the performance. In addition we can use the model for studying the consequences of contemplated major changes in the design.

From the point of view of guaranteeing reliability of complex systems it seems to be essential that a model as close to reality as possible be employed for prediction.

Comparisons between prediction from the equivalent diagram, Fig. 1, and measured curves have been made for the different experiments in the development program. Fig. 12 presents such a comparison for the previously described "wall-thickness" experiment. The changes in impedances observed corresponding to a change in wall thickness of 0.0006″ were fed into the formulas derived[5, 6] for the equivalent diagram. The resulting predicted gain values, together with the measured gain values, are plotted as a function of frequency in Fig. 12. Remembering the order of magnitude of the estimates for the error standard deviation, a few hundredths of a db in this type of transmission measurements, the agreement is satisfactory.

V. FINAL EVALUATION OF THE TRANSFORMER

The results of experiments like the ones described contributed to the tying down of specifications and controls in the manufacture of the

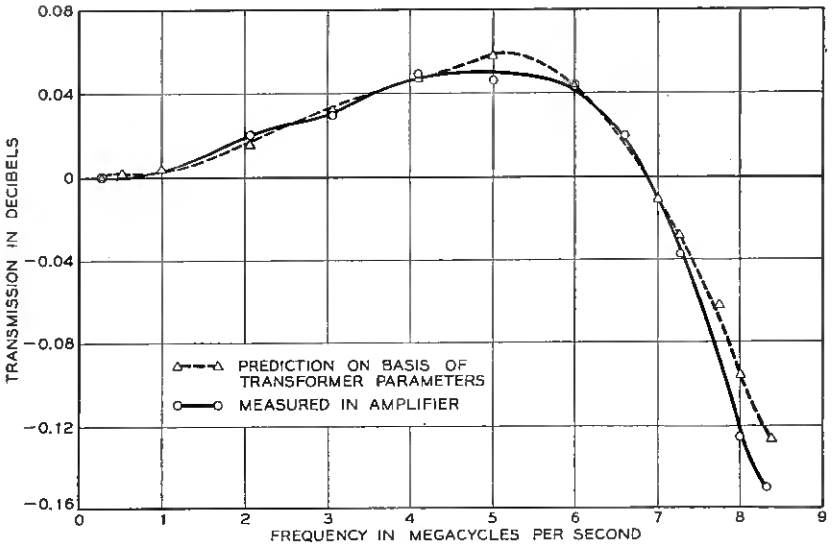

Fig. 12 — Comparison between measured and predicted transmission for a 0.6 mil increase in wall thickness.

transformer. As the measures derived from each of the experiments related only to a detail of the transformer, it was considered necessary to set up an experiment incorporating the results of the various tests, in order to examine the over-all performance of the transformer, in a complete amplifier.

In other words, it would be useful to confirm that the gain variations in the amplifier dependent on the (uncontrollable statistical) variations in the electrical parameters of the transformer are small enough to satisfy the systems designer. The experimental scheme adopted for this purpose called for a fair sized number of transformers basically belonging to two groups:

a. One group of transformers conforming to the current specifications and of recent manufacture at the time of this experiment.

b. One group of transformers consisting of recent rejects and all other old transformers that could be found, all having one or more parameters outside the specifications.

These transformers would be very carefully measured in the Laboratories, taking special care and using the best measuring equipment available. (The previous experiments described in this paper had been conducted in Western Electric factories.)

From the measured values of the parameters such as leakage induct-

ance, stray capacitances, etc., the predicted gain would be computed, again using the formulas derived on the basis of the equivalent diagram. The computed gains would finally be compared to the measured ones. It was hoped that this experiment would show two things:

1. That recently produced transformers which showed satisfactory parameter-measurement results would yield good amplifiers.

2. That the parameters chosen for control measurements in the transformer manufacturing process were adequate to reasonably predict the over-all transmission performance in the amplifier.

The experiment was preceded by a pilot experiment to test the gain-measuring equipment. In both steps of experimentation two jigs for gain measurements were to be used, consisting of almost identical sub-amplifiers, and measurements at 15 frequencies between 0.3 and 8.5 mc were to be made. The pilot experiment was designed such that an estimate of the jig differences and of the influence of time could be made. In addition the magnitude of residual error could be determined.

Eight transformers were measured twice in each of the two jigs in the following sequence. (Table XII.)

As an example let us again choose the results at a high frequency, as the sensitivity of the transformer and amplifiers for small deviations from the ideal increases with frequency.

The time effect will be judged by the difference between the first and second half of the experiments, called H_1 and H_2 respectively.

Disregarding the time sequence in each half, which can always be recovered if so desired by examining the residuals, the results coded as before in thousandths of db are given in Table XIII. The analysis of variance is given in Table XIV. Using the three-way interaction as a measure of residual variation Table XIV shows that the transformer by time and the jig by time interactions are unimportant. The transformer by jig interaction although not significant at the 5 per cent level is disturbing in an experiment of this kind. This might indicate that contact trouble exists between the transformer and the jig. The transformers were not soldered in the jigs but contact was made by means of springs.

TABLE XII. — TRANSFORMER NUMBERS IN TIME SEQUENCE OF MEASUREMENT FROM LEFT TO RIGHT

	H_1				H_2			
Jig 1	1, 2, 3, 4	5, 6, 7, 8	6, 5, 8, 7	2, 1, 3, 4	7, 8, 5, 6	3, 4, 1, 2	4, 3, 2, 1	8, 7, 6, 5
Jig 2								

TABLE XIII. — PILOT EXPERIMENTS 8.3-MC GAIN MEASUREMENTS ON "MICROBEL" TEST SET. UNITS 0.001 DB

	Jig 1		Jig 2	
	H_1	H_2	H_1	H_2
Tr. 1	765	777	890	888
2	652	672	797	777
3	812	814	920	910
4	760	747	915	927
5	832	840	961	938
6	775	743	909	887
7	756	757	889	878
8	698	705	832	820
Average for Jig 1	756		Average for Jig 2	884

TABLE XIV. — ANALYSIS OF VARIANCE OF PILOT EXPERIMENT

Source	*SS*	*D/F*	*MS*	Significance Level
Between Jigs	129159	1	129159	$\ll 1\%$
Between Transformers	79930	7	11419	$\ll 1\%$
Between Time	256	1	256	$> 10\%$
Transf × Jigs	2684	7	383	$\approx 7\%$
Jig × Time	229	1	229	$> 10\%$
Transf × Time	602	7	84	$> 25\%$
Transf × Jigs × Time	803	7	115	
Total	213663	31		

In the main experiment following this pilot one, contact trouble arose again. Moving up in the table the time effect appears negligible. The significant differences between transformers do not have to be considered as this reflects only the differences in their nominal gain, but the jig effect is highly significant even with respect to the transformer by jig interaction.

It would have been unrealistic to expect the jigs to be equal because of their complexity. What was hoped was that the difference between the two would be substantially constant. From the averages listed in Table XIII, we estimate the difference between Jig 1 and Jig 2 as 0.128 db, with 90 per cent confidence limits of 0.114 to 0.142 db based on standard deviation for the average difference of 0.008 db with 15 degrees of freedom. For this latter estimate the jig interactions were pooled with the "error" variance.

If the variations between jigs would remain within the above limits in

the main experiment yet to be made, this would be reasonable. However, the jig by transformer interaction tells us to be on guard.

The main experimental design following this pilot study is presented in Table XV. The intent was to obtain units with as wide a spread of properties as possible. Then, as explained in the beginning of this section, we could see if the formulas which predict the over-all gain from the detailed impedances of the transformer would hold over a wide enough range. In each period all the transformers listed were measured in one jig and then in the other. The jig sequence was varied from period to period. Transformers meeting specifications and rejects were collectively randomized over serial numbers. Therefore 50 good transformers of recent production were combined with 33 rejected ones. The latter were rejected for a variety of reasons and over a considerable period of time. In principle, no special design is necessary to obtain observations for comparing detailed measurements of a transformer to the

TABLE XV. — MEASURING SCHEDULE FOR TRANSFORMERS IN TERMS OF THEIR SERIAL NUMBERS

Runs = Days	1		2		3		4	
	Jigs							
	1	2	1	2	1	2	1	2
Morning	1	10	22	25*	43	50	64	72
	2	3	23	26	44	46	65*	73
	3	2	24*	27	45	44	66*	71
	4	1	25*	28	46	45	67	64
	5	6	25	22	47	49*	68	66*
	6	5*	27	29	48*	47	69	67
	7	4	28	30	49*	48*	70	65*
	8	8	29	31	50	51	71	69
	9*	7	30	23	51	52	72	70
	10	9	31	24*	52	43	73	68
	24*	40*	49*	57*	65*	80*	9*	20*
	35*	25*	59*	48*	77*	66*	14*	5*
Afternoon	11	20*	32	35*	53	59*	74	79
	12	13	33	36	54	57*	75	80*
	13	11	34	34	55	58	76	78
	14*	12	35*	38	56	62	77*	82
	15	16	36	32	57*	60	78	83
	16	14	37	39	58	61	79	74
	17	15	38	37	59*	63	80*	81
	18	19	39	42	60	53	81	76
	19	21	40*	33	61	56	82	77*
	20	17	41	40*	62	54	83	84
	21	18	42	41	63	55	84	75
	40*	24*	57*	49*	80*	65*	20	9
	25*	35*	48*	59*	66*	77*	5	14

performance of an amplifier containing the same transformer. But the time involved in measuring more than 80 transformers in each of two jigs is several days, so the possibility of time effects had to be watched. First, the numbers in the design were assigned at random to the pool of good and rejected transformers. Second, to keep a running check on the precision of the measurements a number of observations were repeated on different days (runs). In each pair of adjacent runs, and in the last and the first, a set of four transformers was replicated both in Jig 1 and Jig 2. From Table XV it can be seen that these linking sets are the following:

Run I and II	Transformers	24, 25, 35, 40
Run II and III	Transformers	48, 49, 57, 59
Run III and IV	Transformers	65, 66, 77, 80
Run IV and I	Transformers	5, 9, 14, 20

As a further precaution, which it was found not necessary to use in the analysis, half of the transformers in the sets above were replicated in the same period of the day, the other half in different periods. For Runs I and II we find from Table XV, in Jig 1, transformers 24 and 40 in the same periods, transformers 25 and 35 in different periods, in Jig 2, transformers 25 and 35 in the same periods, transformers 24 and 40 in different periods. A typical analysis for one linking set disregarding the period allocation, is shown in Table XVII for the observations taken at 8.3 mc and listed in Table XVI.

Both the interactions of jigs and runs and jigs and transformers are significant at the 5 per cent level. The run main effects mean square is not significant but the interactions with the jigs are disturbing. These interactions showed up to a greater or lesser extent in all the comparisons, both in those similar to this one and in the pilot experiment. The importance of the jig by run interaction can be illustrated if we list the

TABLE XVI. — TYPICAL SET OF LINKING MEASUREMENTS INCLUDED IN MAIN EXPERIMENT. UNITS IN 0.001 DB

Transformer	Run III		Run IV	
	Jig 1	Jig 2	Jig 1	Jig 2
65	4	230	15	195
66	−4	191	10	195
77	−65	92	−47	75
80	−45	152	−18	148

TABLE XVII. — ANALYSIS OF VARIANCE. TYPICAL LINKING SET IN MAIN EXPERIMENT

Source	*SS*	*D/F*	*MS*	Significance Level
Between Jigs	127449	1	127449	$\ll 1\%$
Between Runs	20	1	20	$> 25\%$
Between Transformers	22975	3	7658	$\ll 1\%$
Jigs × Runs	931	1	931	$< 5\%$
Jigs × Transformers	2262	3	754	$< 5\%$
Transformers × Runs	337	3	112	20%
Jigs × Runs × Transf.	170	3	57	

TABLE XVIII. — JIG COMPARISON

	Jig 2 − Jig 1 (in db)	90% Confidence Limits in db
Pilot	0.128	0.114 to 0.142
Run I & II	0.114	0.054 to 0.174
Run II & III	0.149	0.037 to 0.201
Run III & IV	0.170	0.120 to 0.220
Run IV & I	0.121	0.040 to 0.201

average differences between the jigs as observed in the various pairs of runs and in the pilot experiment. In Table XVIII are also calculated 90 per cent confidence limits for the jig difference based on a variance estimate incorporating the variances for the jig interactions. It was originally hoped to use an estimate of difference between the jigs to eliminate the jig effect from all the individual observation. The wide confidence limits of the jig difference estimates compared to the 0.01 db order of magnitude we are interested in, do not allow us to do this. Therefore the subsequent analysis was made separately for both jigs.

In addition to the gain measurements the following impedances were observed on all transformers: Resistive and Reactive component of leakage (R_R and R_L); Capacitance over the high winding (C_H); Stray Capacitances (C_{S_1} and C_{S_2}). These impedance results introduced in the formulas for the equivalent diagram of the amplifier yield a predicted gain, which should represent, if everything is all right, the measured gain values.

Using the coefficients m_i, $i = 1, 2, \cdots 5$, as computed from the equivalent diagram, we predict the transmission gain to be:

$$Y = m_0 + m_1 R_R + m_2 R_L + m_3 C_H + m_4 C_{S_1} + m_5 C_{S_2} .$$

Here, m_0 is an arbitrary constant, not important in these considerations, as in measuring amplifiers of this type, frequency-independent loss networks are often introduced, which add an additional constant in m_0.

Calling the measured transmission gain y, we will try to fit the model

$$y = \alpha + \beta Y.$$

The Y is taken as the independent variable as the transformer parameter measurements are more precise than the transmission measurements. In general for this type of regression line fitting the independent variable should be known without error.

If the equivalent diagram is adequate β should be equal to 1; our estimates b of β therefore should not differ significantly from that value. Table XIX lists for 8.3 mc the estimates of the slopes, their standard deviations, and the estimated standard deviations of the residual variations not accounted for by the regression. The intercept α like the parameter m_0 in the prediction equation, is of no interest as explained above.

It is seen that the agreement of the slopes with the theoretical value 1.00 is reasonably good, especially for Jig 2.

The rejects selected for this experiment fall into two classes, those in one set of recent manufacture not meeting the manufacturing specifications, but not too far removed from them, and the others left-over from the development program. Even for such groups with wide variations in their parameters not meeting the end requirements the agreement between prediction and measurement is reasonable. The Jig 1 results gen-

TABLE XIX. — COMPARISON BETWEEN THE REGRESSION PARAMETERS ESTIMATED FROM THE MEASUREMENTS IN BOTH JIGS. FREQUENCY 8.3 MC

	Slope b db/db		Standard error of slope s_b db/db		Standard error residual (db)	
	Jig 1	Jig 2	Jig 1	Jig 2	Jig 1	Jig 2
Standard production 50 units	1.38	0.97	0.18	0.11	0.04	0.02
Rejects from production 18 units	0.90	1.11	0.11	0.07	0.18	0.08
Rejects from development 15 units	0.78	0.82	0.18	0.08	0.07	0.03
All 83 units pooled	0.84	1.05	0.06	0.04	0.05	0.02

erally show a bigger deviation from the ideal value of 1 for the slope and, also, larger residual variations as indicated by the estimates of the variance. It will be remembered that from the pilot experiment and the "built-in" control in the main experiment it appeared that the difference between Jig 1 and Jig 2 was not constant. Subsequently a poor contact in Jig 1 was identified. However, the general result of the experiment was satisfactory, in that the feasibility of maintaining the overall performance of the amplifier within the required limits by controlling the parameters of the transformer was demonstrated.

VI. CONCLUSION

The foregoing describes some highlights in the statistical aspects of the development program of one of the critical components in the L3 system. It will be clear that statistics can be a very powerful help, when integrated in the engineering efforts.

VII. ACKNOWLEDGEMENTS

Many individuals throughout the Bell Laboratories and the Western Electric departments concerned contributed to the success of the L3 system. Most intimately concerned with the development of the transformer were C. W. Thulin and W. L. Brune. The systems aspects, both in development and measurements, were the responsibility of B. J. Kinsburg and G. R. Leopold. J. H. Bash performed most of the measurements discussed. F. E. Stehlik and S. A. Levin performed the equivalent-network studies. N. E. Earle and his co-workers represent the Western Electric effort, and J. W. Tukey and M. E. Terry of the Mathematical Research Department contributed substantially to the design and analysis of the experiments.

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Runs Determined in a Sample by an Arbitrary Cut

By PAUL S. OLMSTEAD

(Manuscript received August 9, 1957)

This paper, after making a critical review of the literature pertaining to runs above and below in a fixed sample, provides the following extensions:

1. *Sample arrangement distributions for runs of length at least s on one, each, and either side of* any selected *cut for samples of 10 and 20,*
2. *Sample arrangement distributions for runs of length at least s on one, each, and either side of the* median *for samples of 10, 20, 40, 60, 100, and 200,*
3. *Sample arrangement distributions for runs of length at least s on each side of* all possible *cuts for samples of 10, 20, 40, and 100,*
4. *Asymptotic values of the probabilities of such arrangements when the sample size and length of run are large,*
5. *Convenient charts and tables for probabilities of 0.01, 0.10, 0.50, 0.90, and 0.99 to facilitate use by engineers and scientists, and*
6. *Discussion of a simple application.*

The inclusion of the case for runs of length at least s on each side of all possible *cuts should prove very useful because it provides a quantitative measure for a common operational procedure for which the exact probabilities were heretofore unknown.*

I. SUMMARY

This paper discusses certain nonparametic measures for use in detecting the presence of assignable causes in experimental data. Specifically, it assumes that a sample of n observations of a characteristic, X, has been obtained and that a particular arrangement, X_1, X_2, $\cdots\cdots$ X_n, e.g., by the time order of determination or other considerations, increases the value of the sample as evidence. Assuming a cut at a particular value of X, such as A, such a series may be divided into groups of consecutive observations that lie, alternately, above and below the cut. The length of such a group is called a run.

The paper also presents charts, tables, and formulas relating to such sample arrangement distributions for runs above and below *any selected* and *all possible* cuts or demarcation values. Specifically, it contains:

a. A review of the literature relating to runs above and below (Section II).

b. Appropriate charts and tables for the convenience of the engineer or other user (Section II and III).

c. An example (Section III) and reference to others (Section II).

d. A procedure for obtaining the probability that a randomly selected arrangement of a sample of size n will contain one or more runs of length at least s *on each side* of at least one of *all possible* cuts or demarcation values that do not coincide with one of the numerical values in the sample (Section VI).

e. Relationships between n and s for constant probability (Section VIII).

f. The probability that a randomly selected arrangement of a sample of size n will contain one or more runs of length at least s on each side of a selected cut or demarcation value such that n_1 numerical values are above and n_2 numerical values are below ($n = n_1 + n_2$). Similar probabilities are given for arrangements with runs *above*, with runs *below* and with runs *on either side* of such a cut or demarcation value (Section IV).

g. Simplified formulas for runs above and below the median that are equivalent to those given by Mosteller[4] (Section V).

h. Asymptotic values of these probabilities for both n and s large (Section VII).

II. HISTORICAL BACKGROUND AND DISCUSSION

Runs above and below the average, the median, or some other selected value have been used by a number of engineers to assist in detecting and identifying assignable causes of variation in connection with research and development work. In order to have a clear picture of the problems of such work, it may be worthwhile to set down some statements which characterize it:

a. A repetitive process that has not been examined for control by statistical methods and that has not subsequently been brought into control is very unlikely to be in statistical control,

b. Causes of lack of control often occur sporadically, being present for relatively short intervals of time,

c. Such causes of lack of control may often be detected by taking account of order either in manufacture or in taking observations, and

d. A basis for determining what fractions or portions of the observations may have been affected by an undesired cause is the application of statistical tests to the pattern of the individual values of the measurements in the order in which they were obtained.

Runs above and below have been particularly useful in assisting in the identification of such assignable causes. Their use in engineering has progressed through the following steps:

1. Using a procedure based on the work of Cochran,[2] Shewhart[5] showed the distribution with respect to length of the runs above and below the average. It was his observation that a run of length 7 was often associated with a cause that could be found. Cochran had derived the distribution of runs of lengths s (our notation) of two complementary events E_1 and E_2 of known probability, p, and $q = 1 - p$, respectively. In applying Cochran's formula, Shewhart chose two statistics, $\bar{X}$ and p, from his observed data. Recognizing that this might invalidate the use of Cochran's formula, he suggested to the writer that this loophole could be avoided by working out the distribution for run lengths relative to the median. This distribution was worked out and recorded in a memorandum dated October 14, 1940.

2. About the same time, Mood[3] was working on his "Distribution Theory of Runs" for which the distribution relative to the median is a special case. He included in his results expressions for the variances and covariances. Campbell[1] made use of the distribution of lengths of run relative to the median.

3. The next step was to obtain the distribution of possible arrangements with runs of at least a given length relative to the median. Mood[3] gave a general analysis of the problem, which was supplemented in a form more easily comprehensible to the engineer by Mosteller.[4] Mosteller gave criteria based on sample size at given probability levels for length of run on one side and on either side of the median. While this paper was in preparation, Olmstead had been examining the problem of the probability of arrangements with runs of at least a given length on each side of the median. When this was brought to Mosteller's attention, his paper was revised to include this case which had its inception in the engineering idea that if two cause systems were operating in separate periods they would be likely to produce separate groups of high and low values.

4. Following this, attention was given to the distribution of arrangements, as indicated in Section V of this paper, where division for runs above and below was made at some location other than the median. Validity in use of the probabilities calculated on this basis was dependent on the choice of division location prior to the test and often left the en-

gineer and the statistician uncertain concerning the risks that were being taken when the division location was chosen after looking at the data. Because of assumption (a) above, this did not worry the engineer as much as it did the statistician, particularly when the engineer could find a cause associated with long runs identified in this way. The fact that he usually found such a cause indicated that some other way of considering the problem from the viewpoint of mathematical statistics would be fruitful.

5. The obvious next step was to find a procedure for counting all of the possible arrangements of n numbers, no two alike, that would have one or more runs of length at least s on each side of at least one of all of the possible division points that do not coincide with one of the numerical values in the sample. One way of doing this is first to write down or plot all $(n!)$ possible arrangements of the n numbers. Assume that the numerical values of the numbers are the y-coordinates and the order in which they occur in an arrangement is indicated by the x-coordinates of such a plot. All such plots could then be examined to see what y-division not at one of the y-values would give the longest run of consecutive y-values on each side of the division. In this way, each arrangement would be assigned to a category where a particular length of run was equalled or exceeded on each side for at least one of the possible y-divi-

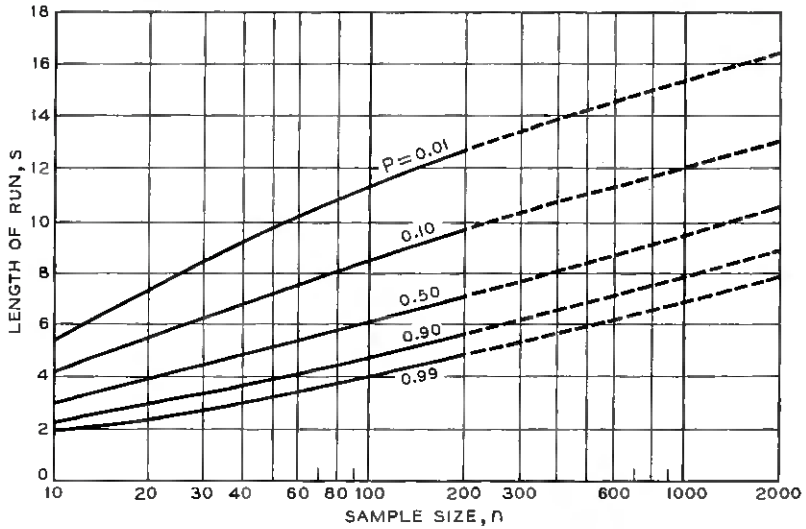

FIG. 1 — Length, s, of run on *one* side of *median* versus sample size, n, for selected values of probability, P.

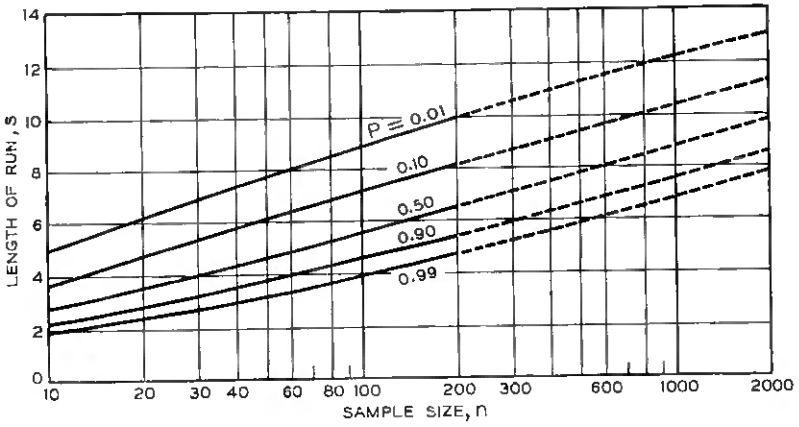

FIG. 2 — Length, s, of run on *each* side of *median* versus sample size, n, for selected values of probability, P.

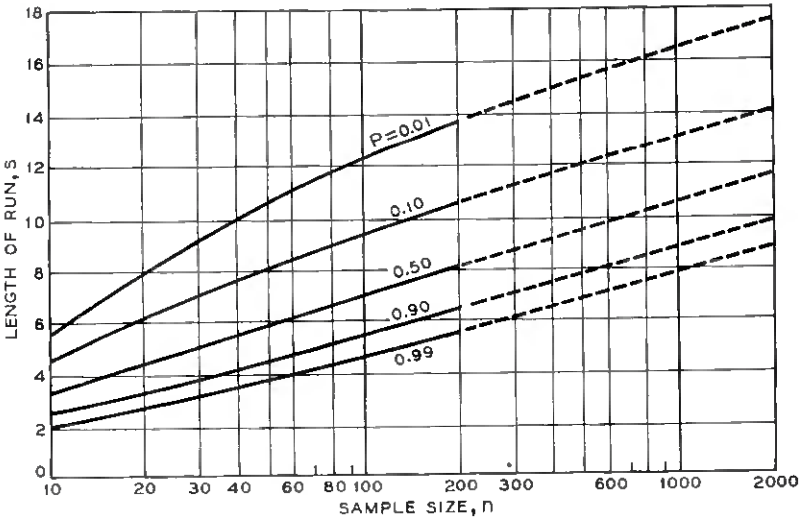

FIG. 3 — Length, s, of run on *either* side of *median* versus sample size, n, for selected values of probability, P.

sions. The process presented in Section VI is the mathematical equivalent of carrying out such a count. This process is gratifying to the engineer and the statistician alike because of the freedom permitted in setting the division location after examining the data so as to obtain the longest lengths of run on each side of the selected value. Use of this information

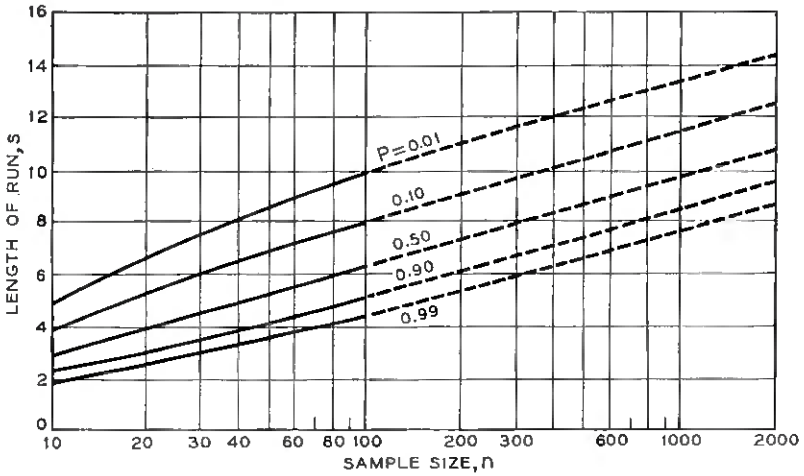

FIG. 4 — Length, s, of run on *each* side for *any cut* versus sample size, n, for selected values of probability, P.

was made first in an article by Walker and Olmstead.[6] Its part in detecting the type of an assignable cause appeared first in an article by Olmstead.[7]

6. In connection with the investigation undertaken for this paper, the asymptotic relationships for determining probabilities when n and s are large have been obtained (Section VII) and the results compared with those given by the exact relationships. The exact relationships applying to the median have been calculated for sample sizes of 60, 100, and 200 extending this information beyond the range usually covered by research workers. For the convenience of such workers, four charts (Figs. 1, 2, 3, and 4) have been prepared to show the relationships between s and n for $P = 0.01, 0.10, 0.50, 0.90$, and 0.99 for the primary types of runs.

III. WORKING TECHNIQUES

As just mentioned, Figs. 1, 2, 3, and 4 present graphically five percentage points of each of the four "above" and (or) "below" run distributions for all sample sizes from 10 to 2,000. The same information is furnished in tabular form in Tables I, II, III, and IV. How these are derived and calculated is discussed later (Sections V, VI, and VIII). Specifically, the four types of distribution thus made available are:

a. The probability, P, of the event that the length of the longest run on one pre-chosen side of median equals or exceeds s; if above, the prob-

ability is designated $P(s/-,$ medium); if below, $P(-/s,$ median). The notation, $P(s/-,$ median) may be read — the probability that an arrangement will contain a run of length at least s above the median.

b. The probability of the event that the length of the shorter of the longest run above and the longest run below the median equals or exceeds s: designated $P(s/s,$ median), where s/s means that there is a run of length at least s, on each side of the median.

c. The probability, P, of the event that the length of the longer of the longest run above and the longest run below the median equals or exceeds s: designated $P(s,$ median), where s means the longer of $(s/-$ median) and $(-/s,$ median).

d. The probability, P, of the event that the length of the shorter of

TABLE I

Minimum sample sizes, n, that exceed selected probabilities, P, for a given length, s, of run on *one* side of median calculated from Table XVI and equations (23) and (27) to three significant figures.

Run Length s	Probability, P				
	0.01	0.10	0.50	0.90	0.99
1	2	2	2	2	2
2	4	4	6	8	12
3	6	6	12	22	38
4	8	10	22	54	100
5	10	16	46	116	230
6	14	26	92	260	490
7	18	44	182	530	1044
8	26	78	360	1104	2140
9	38	142	714	2240	4370
10	56	256	1424	4530	8980
11	86	480	2850	9190	18240
12	140	930	5680	18540	37200
13	234	1838	11330	37600	75500
14	410	3630	22700	75700	151700
15	748	7160	45300	151700	303000
16	1446	14190	90600	303000	607000
17	2830	28100	181200	607000	1214000
18	5530	56100	362000	1214000	2430000
19	10860	117300	725000	2430000	4850000
20	21500	235000	1450000	4850000	9710000

Examples of use:

	Observed Data		*Probability, P*
Case 1	$n = 96$	$s = 4$	$0.90 < P < 0.99$
2	54	10	$P < 0.01$
3	56	10	$0.01 < P < 0.10$

Table II

Minimum sample sizes, n, that exceed selected probabilities, P, for a given length, s, of run on each side of median calculated from Table XVI and equations (24) and (27) to three significant figures.

Run Length s	Probability, P				
	0.01	0.10	0.50	0.90	0.99
1	2	2	2	2	2
2	4	4	6	10	14
3	6	8	14	26	44
4	8	14	30	68	116
5	12	26	68	152	252
6	20	50	140	322	552
7	34	98	290	676	1164
8	62	194	596	1390	2390
9	116	390	1208	2830	4930
10	216	782	2440	5650	10140
11	446	1182	4910	11750	20700
12	884	2360	9840	23800	42500
13	1762	4720	19890	48600	86700
14	3510	9450	39900	98600	174200
15	6990	18900	80500	197300	348000
16	13930	37800	161300	395000	697000
17	27900	75600	323000	789000	1394000
18	55500	151200	645000	1578000	2790000
19	111000	302000	1290000	3160000	5570000
20	222000	605000	2580000	6310000	11150000

Table III

Minimum sample sizes, n, that exceed selected probabilities, P, for a given length, s, of run on either side of median calculated from Table XVI and equations (25) and (27) to three significant figures.

Run Length s	Probability, P				
	0.01	0.10	0.50	0.90	0.99
1	2	2	2	2	2
2	4	4	4	8	10
3	6	6	8	16	28
4	8	8	16	36	64
5	10	14	30	76	136
6	12	20	58	152	282
7	16	32	106	296	568
8	22	52	200	580	1150
9	32	86	388	1174	2310
10	42	150	758	2350	4640
11	62	262	1488	4720	9330
12	94	500	2920	9460	18730
13	156	962	5860	10660	37700
14	256	1876	11250	21300	75700
15	418	3670	22600	42600	151600
16	766	7330	45200	85300	303000
17	1472	14090	90100	170500	606000
18	2860	27900	180300	341000	1213000
19	5570	55500	361000	682000	2430000
20	10860	111100	721000	1364000	4870000

TABLE IV

Minimum sample sizes, n, that exceed selected probabilities, P, for a given length, s, of run on each side of any cut calculated from Table XVI and equations (26) and (27) to three significant figures.

Run Length s	Probability, P				
	0.01	0.10	0.50	0.90	0.99
1	2	2	2	2	2
2	4	4	6	8	12
3	6	8	12	22	34
4	8	12	22	48	76
5	12	18	46	96	162
6	16	34	86	192	380
7	24	58	166	382	668
8	38	108	324	760	1342
9	66	204	638	1518	2690
10	118	400	1266	3030	5410
11	228	790	2530	6070	10870
12	444	1568	5050	12130	21500
13	878	3130	10070	24300	43100
14	1750	6220	20100	48500	86200
15	3480	12490	40300	97000	172300
16	6790	25000	80600	194100	345000
17	13860	49900	161100	388000	689000
18	27700	99900	322000	776000	1379000
19	55400	199800	644000	1553000	2760000
20	110800	400000	1289000	3110000	5510000

TABLE V

Speedometer readings at one minute intervals.

Time	MPH	Time	MPH	Time	MPH	Time	MPH
1	48	15	55	29	52	43	60
2	50	16	53	30	58	44	58
3	48	17	48	31	55	45	55
4	50	18	50	32	57	46	57
5	52	19	50	33	58	47	57
6	49	20	55	34	58	48	53
7	50	21	55	35	58	49	57
8	47	22	55	36	58	50	58
9	51	23	55	37	58	51	58
10	50	24	55	38	58	52	56
11	49	25	51	39	55	53	58
12	52	26	53	40	56	54	63
13	53	27	52	41	57	55	60
14	53	28	51	42	56	56	50

the longest run above and the longest run below a cut chosen to maximize this length equals or exceeds s: $P(s/s,$ any cut) with meaning similar to that for $P(s/s,$ median) but for the case where the cut has been chosen to maximize the shorter of the longest runs on each side.

The use of these distributions can be illustrated by the calculation of the various run length statistics for a specific example. The 56 speedometer readings presented in Table V and Fig. 5 were observed at one minute intervals during a driver's first trip on a toll highway with

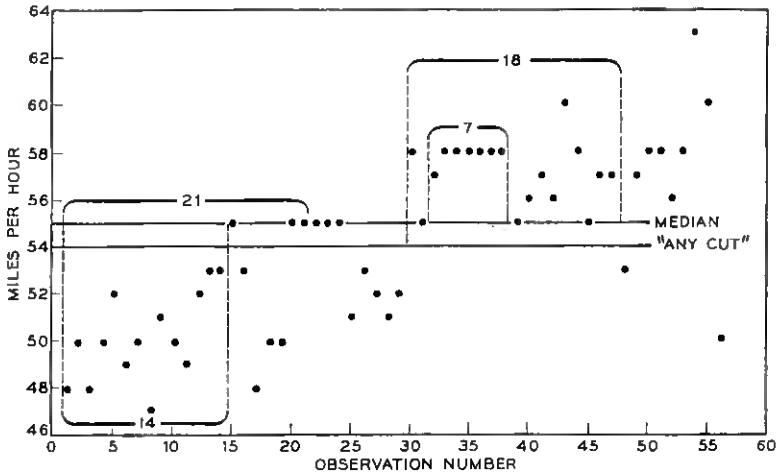

FIG. 5 — Readings at one-minute intervals.

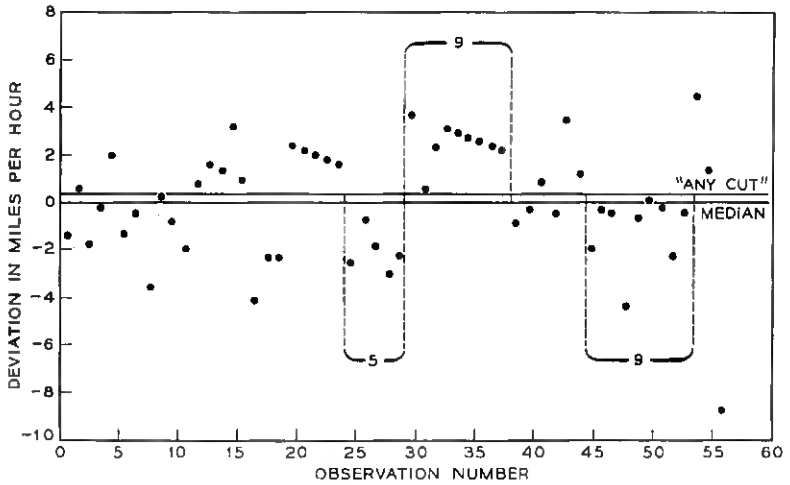

FIG. 6 — Chart for deviations from trend line.

separate traffic lanes. In this instance, nine observations occur at the median (55) with 22 above and 25 below. This is not unusual in experimental work where ties are likely at or near the median. (It should be pointed out that the occurrence of ties makes this a difficult example. Later, this example will be modified by removing a trend and then it will be simpler. Consideration will first be given to runs with respect to the median and then to "any cut.") Various methods of resolving such ties are possible. The most conservative is to use a tied median to terminate a run. The least conservative is to use the tied median or medians for inclusion in the run. Intermediate between these is to consider all possible allocations and their effects on run length. Here, in order to obtain 28 above and 28 below the median, it is necessary to allocate the nine tied at the median so that 6 will be above and 3 below. The run length associated with each such combination would then be obtained and, if desired, the average computed. In this case, the lengths of the various runs obtained by these three methods are as follows:

Type of Run	Run Lengths, s			Limit for $P \leqq 0.01$	Per Cent Below Limit
	Most Conservative	Average	Least Conservative		
Above	7	13.7	18	11 (Table I)	33
Below	14	15.8	21	11 (Table I)	0
Each Side	7	12.8	18	8 (Table II)	1
Either Side	14	16.6	21	11 (Table III)	0
Each Side, Any Cut	14	—	—	9 (Table IV)	—

It will be observed that only one answer results for the "each side of any cut." Also, three of the five tests on the most conservative basis are above their respective limits for a P of 0.01 and all on the other bases. This happens quite frequently in engineering problems.

It is apparent, however, in this case, that there is a consistent trend throughout the set of data. In Fig. 6, this has been removed and the median lies between the 28 points above and the 28 points below. The following statistics are obtained:

Type of Run	Run Lengths, s		P for Observed Run
	Observed Run	Limit for $P \leqq 0.01$	
Above	9	11 (Table I)	0.03
Below	5	11 (Table I)	0.60
Each Side	5	8 (Table II)	0.42
Either Side	9	11 (Table III)	0.05
Each Side, Any Cut	9	9 (Table IV)	0.008

In this case, only the statistic for the longest run on each side of any cut has a P as low as 0.01. Two others were of the order of 0.05 and the remaining two near 0.50. The explanation of the indicated nonrandomness was identified with human behavior under conditions of learning.

This example raises a question about the treatment of odd sized samples, where the median is a single observation. These may all be reduced to even sized samples by omitting the median. This is unnecessary in the case of the longest run on "each side of any cut" where the P values for a given s for the odd sized sample lie between those for the adjacent even sized samples.

IV. SOME SPECIFIC SAMPLES

Table VI presents the values of probabilities $P(s/-,\ n_1/n_2)$ and $P(-/s,\ n_1/n_2)$, for every possible separation of $10 = n_1 + n_2$ observations into n_1 on one side of a cut and n_2 on the other. Table VII does the same for $20 = n_1 + n_2$ observations. Similarly, the values of $P(s/s,\ n_1/n_2)$ and $P(s/-$ or $-/s,\ n_1/n_2)$ are given in Tables VIII and IX, and Tables X, and XI, respectively.

In Tables XII, XIII, and XIV, the table presented by Mosteller[4] for the three kinds of runs with respect to the median, that is, where $n_1 = n_2$, has been extended to include samples of 60, 100, and 200.

The values of $P(s/s$, any cut) for $n = 10$, 20, 40, and 100 are given in Table XV. It will be noted that the values of the probabilities in this table differ only slightly from those in Table XIII for $P(s/s$, median). For large sample sizes, other considerations suggest that the s-values

Table VI

Probability of an arrangment with a run of length at least s on "one side" of a demarcation value for $n_1 + n_2 = 10$ calculated from equation (1) or (2).

Length of Run s	Total on the "one side", i.e., n_1 or n_2								
	9	8	7	6	5	4	3	2	1
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.000	1.000	1.000	1.000	0.976	0.833	0.533	0.200	
3	1.000	1.000	0.967	0.786	0.500	0.233	0.067		
4	1.000	0.933	0.667	0.357	0.143	0.033			
5	1.000	0.667	0.333	0.119	0.024				
6	0.800	0.400	0.133	0.024					
7	0.600	0.200	0.033						
8	0.400	0.067							
9	0.200								

TABLE VII

Probability of an arrangement with a run of length at least s on "one side" of a demarcation value for $n_1 + n_2 = 20$ calculated from equation (1) or (2).

Length of Run s	Total on the "one side", i.e., n_1 or n_2																		
	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.990	0.956	0.871	0.718	0.509	0.284	0.100	
3	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.988	0.950	0.870	0.742	0.582	0.415	0.260	0.140	0.060	0.016		
4	1.000	1.000	1.000	1.000	0.996	0.971	0.900	0.779	0.622	0.457	0.307	0.187	0.101	0.046	0.017	0.004			
5	1.000	1.000	1.000	0.986	0.920	0.790	0.621	0.447	0.295	0.178	0.098	0.047	0.019	0.006	0.001				
6	1.000	1.000	0.982	0.889	0.721	0.527	0.351	0.214	0.119	0.060	0.026	0.009	0.005	0.000					
7	1.000	0.995	0.898	0.707	0.492	0.309	0.177	0.092	0.043	0.017	0.006	0.001	0.000						
8	1.000	0.947	0.751	0.509	0.307	0.167	0.082	0.035	0.013	0.004	0.001	0.000							
9	1.000	0.853	0.579	0.341	0.179	0.083	0.034	0.012	0.003	0.001	0.000								
10	1.000	0.711	0.421	0.217	0.098	0.038	0.012	0.003	0.001	0.000									
11	0.900	0.568	0.295	0.130	0.049	0.015	0.004	0.001	0.000										
12	0.800	0.442	0.196	0.072	0.022	0.005	0.001	0.000											
13	0.700	0.332	0.123	0.036	0.008	0.001	0.000												
14	0.600	0.237	0.070	0.015	0.002	0.000													
15	0.500	0.152	0.035	0.005	0.000														
16	0.400	0.095	0.014	0.001															
17	0.300	0.047	0.004																
18	0.200	0.016																	
19	0.100																		

TABLE VIII

Probability of an arrangement with a run of length at least s on "each side" of a demarcation value for $n_1 + n_2 = 10$ calculated from equation (4).

Length of Run s	n_1 or n_2 / n_2 or n_1	1 / 9	2 / 8	3 / 7	4 / 6	5 / 5
1		1.000	1.000	1.000	1.000	1.000
2			0.200	0.533	0.833	0.960
3				0.067	0.224	0.333
4					0.029	0.056
5						0.008

TABLE IX

Probability of an arrangement with a run of length at least s on "each side" of a demarcation value for $n_1 + n_2 = 20$ calculated from equation (4).

Length of Run s	n_1 or n_2 / n_2 or n_1	1 / 19	2 / 18	3 / 17	4 / 16	5 / 15	6 / 14	7 / 13	8 / 12	9 / 11	10 / 10
1		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2			0.100	0.284	0.509	0.718	0.871	0.958	0.990	0.999	1.000
3				0.016	0.060	0.140	0.260	0.413	0.581	0.727	0.784
4					0.004	0.017	0.046	0.100	0.179	0.245	0.274
5						0.001	0.006	0.012	0.042	0.056	0.064
6							0.000	0.002	0.007	0.011	0.013
7								0.000	0.001	0.002	0.002
8									0.000	0.000	0.000
9										0.000	0.000
10											0.000

TABLE X

Probability of an arrangement with a run of length at least s on "either side" of a demarcation value for $n_1 + n_2 = 10$ calculated from equation (5).

Length of Run s	n_1 or n_2 / n_2 or n_1	1 / 9	2 / 8	3 / 7	4 / 6	5 / 5
1		1.000	1.000	1.000	1.000	1.000
2		1.000	1.000	1.000	1.000	0.992
3		1.000	1.000	0.967	0.795	0.667
4		1.000	0.933	0.667	0.362	0.230
5		1.000	0.667	0.333	0.119	0.040
6		0.800	0.400	0.133	0.024	
7		0.600	0.200	0.033		
8		0.400	0.067			
9		0.200				

Table XI

Probability of an arrangement with a run of length at least s on "either side" of a demarcation value for $n_1 + n_2 = 20$ calculated from equation (5).

Length of Run s	n_1 or n_2 / n_2 or n_1	1 / 19	2 / 18	3 / 17	4 / 16	5 / 15	6 / 14	7 / 13	8 / 12	9 / 11	10 / 10
1		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3		1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.989	0.966	0.956
4		1.000	1.000	1.000	1.000	0.996	0.971	0.901	0.787	0.684	0.640
5		1.000	1.000	1.000	0.986	0.920	0.790	0.622	0.452	0.337	0.293
6		1.000	1.000	0.982	0.889	0.721	0.527	0.351	0.217	0.134	0.106
7		1.000	0.995	0.898	0.707	0.492	0.309	0.177	0.092	0.046	0.032
8		1.000	0.947	0.751	0.509	0.307	0.167	0.082	0.035	0.014	0.007
9		1.000	0.853	0.579	0.341	0.179	0.083	0.034	0.012	0.003	0.001
10		1.000	0.711	0.421	0.217	0.098	0.038	0.012	0.005	0.001	0.000
11		0.900	0.568	0.295	0.130	0.049	0.015	0.004	0.001	0.000	
12		0.800	0.442	0.196	0.072	0.022	0.005	0.001	0.000		
13		0.700	0.332	0.125	0.036	0.008	0.001	0.000			
14		0.600	0.237	0.070	0.015	0.002	0.000				
15		0.500	0.158	0.035	0.005	0.000					
16		0.400	0.095	0.014	0.001						
17		0.300	0.047	0.004							
18		0.200	0.016								
19		0.100									

Table XII

Probability of an arrangement with a run of length at least s on "one side" of median calculated from equation (1) or (2).

Length of Run s	Sample size, n					
	10	20	40	60	100	200
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.97619	0.99994	1.00000	1.00000	1.00000	1.00000
3	0.50000	0.86973	0.99225	0.99956	1.00000	1.00000
4	0.14286	0.45713	0.79885	0.92695	0.99049	0.99994
5	0.02381	0.17849	0.44954	0.63645	0.84289	0.98093
6		0.05960	0.20733	0.33935	0.54439	0.82160
7		0.01703	0.08697	0.15952	0.29185	0.54174
8		0.00395	0.03438	0.07046	0.14251	0.30295
9		0.00065	0.01290	0.02996	0.06642	0.15529
10		0.00006	0.00458	0.01235	0.03015	0.07621
11			0.00153	0.00494	0.01344	0.03656
12			0.00047	0.00192	0.00589	0.01731
13			0.00014	0.00072	0.00255	0.00813
14			0.00004	0.00026	0.00108	0.00378
15			0.00001	0.00009	0.00045	0.00175
16			0.00000	0.00003	0.00019	0.00080
17			0.00000	0.00001	0.00008	0.00037
18			0.00000	0.00000	0.00003	0.00017
19			0.00000	0.00000	0.00001	0.00007
20			0.00000	0.00000	0.00000	0.00003
21				0.00000	0.00000	0.00001
22 or over				0.00000	0.00000	0.00000

TABLE XIII

Probability of an arrangement with a run of length at least s on "each side" of median calculated from equation (6) or (4).

Length of Run s	Sample Size, n					
	10	20	40	60	100	200
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.96032	0.99989	1.00000	1.00000	1.00000	1.00000
3	0.33333	0.78582	0.98519	0.99912	1.00000	1.00000
4	0.05556	0.27412	0.66809	0.88729	0.98159	0.99987
5	0.00794	0.06356	0.24933	0.44250	0.72496	0.96284
6		0.01288	0.06820	0.14723	0.33308	0.68619
7		0.00249	0.01647	0.03992	0.10591	0.31377
8		0.00045	0.00379	0.00992	0.02919	0.10573
9		0.00008	0.00085	0.00238	0.00747	0.03027
10		0.00001	0.00019	0.00056	0.00185	0.00800
11			0.00004	0.00013	0.00045	0.00203
12			0.00001	0.00003	0.00011	0.00051
13			0.00000	0.00000	0.00002	0.00013
14			0.00000	0.00000	0.00000	0.00003
15			0.00000	0.00000	0.00000	0.00001
16 or over			0.00000	0.00000	0.00000	0.00000

TABLE XIV

Probability of an arrangement with a run of length at least s on "either side" of median calculated from equation (5).

Length of Run s	Sample Size, n					
	10	20	40	60	100	200
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.99206	0.99999	1.00000	1.00000	1.00000	1.00000
3	0.66667	0.95564	0.99931	1.00000	1.00000	1.00000
4	0.23016	0.64014	0.92961	0.98660	0.99938	1.00000
5	0.03968	0.29342	0.64975	0.83041	0.96082	0.99901
6		0.10632	0.34646	0.53147	0.75569	0.95701
7		0.03157	0.15747	0.27911	0.47779	0.76970
8		0.00741	0.06497	0.13100	0.25582	0.50017
9		0.00122	0.02495	0.05754	0.12538	0.28031
10		0.00011	0.00897	0.02414	0.05846	0.14443
11			0.00302	0.00975	0.02642	0.07108
12			0.00093	0.00380	0.01168	0.03411
13			0.00028	0.00144	0.00506	0.01613
14			0.00008	0.00052	0.00216	0.00753
15			0.00002	0.00018	0.00090	0.00349
16			0.00000	0.00006	0.00038	0.00160
17			0.00000	0.00002	0.00016	0.00074
18			0.00000	0.00000	0.00006	0.00034
19			0.00000	0.00000	0.00002	0.00014
20			0.00000	0.00000	0.00000	0.00006
21				0.00000	0.00000	0.00002
22 or over				0.00000	0.00000	0.00000

TABLE XV

Probability of an arrangement with a run of length at least s on "each side" of at least one of all possible demarcation values calculated from equation (22).

Length of Run s	Sample Size, n			
	10	20	40	100
1	1.00000	1.00000	1.00000	1.0000
2	0.97937	0.99997	1.00000	1.0000
3	0.46190	0.89748	0.99713	1.0000
4	0.08413	0.44121	0.83760	0.9986
5	0.00794	0.12994	0.43401	0.9125
6		0.02943	0.15840	(0.5863)*
7		0.00559	0.04544	(0.2561)*
8		0.00093	0.01179	0.0876
9		0.00013	0.00277	0.0263
10		0.00001	0.00066	0.0073
11			0.00015	(0.0020)*
12			0.00003	(0.0005)*
13			0.00001	(0.0001)*
14 or over			0.00000	(0.0000)*

* Values in parentheses were interpolated or extrapolated.

will increase by unity. All this is in accord with the experience of the engineer who did not hesitate to use available information for $P(s/s$ median) as being a good first approximation to $P(s/s$, any cut).

V. SAMPLE ARRANGEMENT DISTRIBUTIONS WITH RUNS OF LENGTH AT LEAST s ABOVE AND BELOW ANY SELECTED CUT

Assume a finite sample of $n = n_1 + n_2$ numbers, of which n_1 have the common property of being above the selected cut and, similarly, n_2 are below. Clearly, the n_2 numbers may be considered as providing $(n_2 + 1)$ cells or partitions of the n_1 numbers above. Some of these cells or partitions will, of course, be empty, particularly when n_1 is less than $(n_2 + 1)$. If at least s of the n_1 numbers are to be in one partition, it would first appear that the number of ways would be proportional to the number of possible partitions, $n_2 + 1$, and also to the number of ways in which the partition boundary points, n_2, may be selected from the remaining numbers, $n - s$, i.e., the combination of $(n - s)$ things taken n_2 at a time. This, however, gives an over-estimate because it counts twice each arrangement that has two partitions of s each, three times for each arrangement that has three partitions of s each, etc. Taking these factors into account, it is found that the number of ways of partitioning the n_1 numbers by means of the n_2 numbers so as to obtain one

or more partitions that contain s or more elements is:

$$\sum_{j=1}^{\left[\frac{n_1}{s}\right]} (-1)^{j+1} \binom{n_2+1}{j} \binom{n-js}{n_2}.$$

Having this, we may write down the probability of an arrangement of n numbers that will contain at least one run of length s or more among the n_1 numbers that are *above* our demarcation value by dividing by $\binom{n}{n_2}$:

$$P(s/-, n_1/n_2) = \frac{1}{\binom{n}{n_2}} \sum_{j=1}^{\left[\frac{n_1}{s}\right]} (-1)^{j+1} \binom{n_2+1}{j} \binom{n-js}{n_2}. \quad (1)^*$$

In a similar manner, we may, by interchanging n_1 and n_2, write down the probability of an arrangement of n numbers that will contain at least one run of length s or more among the n_2 numbers that are *below* our demarcation value:

$$P(-/s, n_1/n_2) = \frac{1}{\binom{n}{n_1}} \sum_{j=1}^{\left[\frac{n_2}{s}\right]} (-1)^{j+1} \binom{n_1+1}{j} \binom{n-js}{n_1}. \quad (2)$$

To assist in determining the probability that an arrangement will contain at least one run of length s or more *on each side* of the demarcation value, let us assume that we have partitioned the n_1 numbers above into r runs of which at least one is of length at least s. These r runs may be associated with $(r-1)$ runs or partitions of the n_2 in only one way, with $(r+1)$ runs of the n_2 in only one way, but with r runs of the n_2 in two ways. Each of these sets of possible runs must contain at least one run of length s or more. The resulting partitioning count for s, n_1, and r is:

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* Some readers may wish to note that

$$\binom{n}{n_1} P(s/-, n_1/n_2)$$

is the coefficient of x^{n_1} in the expansion of $(1 + x + x^2 + \cdots)^{n_2+1} - (1 + x + x^2 + \cdots + x^{s-1})^{n_2+1}$

$$B(n_i, r) = \sum_{j=1}^{\left[\frac{n_i - r}{s-1}\right]} (-1)^{j+1} \binom{r}{j} \binom{(n_1 - 1) - j(s-1)}{r-1} \text{ for } i = 1,2 \quad (3)^*$$

and $B(n_2, r-1)$ and $B(n_2, r+1)$ are obtained by substituting $(r-1)$ and $(r+1)$ respectively for r in (3). All that is needed to secure the desired probability is to find the count of the possible arrangements in both n_1 and n_2 corresponding to each r, sum with respect to r and divide by the total possible arrangements:

$$P(s/s, n_1/n_2) = \frac{1}{\binom{n}{n_2}} \sum_{r=1}^{n_1 - s + 1} B(n_1, r)[B(n_2, r-1) + 2B(n_2, r) + B(n_2, r+1)]. \quad (4)$$

To find the probability that an arrangement will contain at least one run of length s or more *on either side* of the demarcation value, it should be noted that (4) is counted in both (2) and (1). Thus, this probability is simply:

$$P(s/- \text{ or } -/s, n_1/n_2) = P(s/-, n_1/n_2) + P(-/s, n_1/n_2) - P(s/s, n_1/n_2) \quad (5)$$

where the probabilities on the right hand side of (5) are given by (1), (2), and (4) respectively.

When the median is used as the demarcation value, $n_1 = n_2$, so that $P(s/-, \text{median}) = P(-/s, \text{median})$. In addition, by rearranging terms, $P(s/s, \text{median})$ may be written in the simplified form:

$$P(s/s, \text{median}) = \frac{1}{\binom{2n_1}{n_1}} \sum_{r=0}^{n_1 - s + 1} [B(n_1, r) + B(n_1, r+1)]^2 \quad (6)$$

where $B(n_1, r)$ and $B(n_1, r+1)$ are defined by (3) as before. Equation (6) has been used for the new calculations reported here. (See Section IV)

VI. SAMPLE ARRANGEMENT DISTRIBUTIONS FOR RUNS OF LENGTH s OR MORE ON EACH SIDE OF AT LEAST ONE OF ALL POSSIBLE DEMARCATION VALUES

When this derivation was first discussed with a mathematical statistician, he questioned whether anyone would want a criterion based on

* Here, $B(n_i, r) \equiv B(n_i, r, s)$ is the coefficient of x^{n_i} in

$$(x + x^2 + \cdots)^{n_2+1} - (x + x^2 + \cdots + x^{s-1})^{n_2+1}$$

such a distribution. To make it clear that the engineer does want it, assume that we have a set of data and Tables VI to XI, inclusive. The engineer might look for the longest run on either side of the median. Having found it, he might pick a demarcation value that would just include this run. This would give him, for instance,

$$n_2 < \frac{n_1 + n_2}{2}$$

on one side of his demarcation value and

$$n_1 > \frac{n_1 + n_2}{2}$$

on the other side. He might then look for the longest run on the n_1 side. This would give him two long runs that might be equal in length or one shorter than the other. In either case, he could obtain a value of s for the length of run that is equalled or exceeded on each side of his demarcation value. If his total sample happened to be 20, he could obtain $P(s/s,\ n_1/n_2)$ from (4) or Table IX for n_1 , n_2 , and s. This probability, however, is based on his having chosen n_1 and n_2 before the experiment and therefore does not indicate what the true probability associated with this process is. At the same time, it is reasonably certain that this is a procedure that many engineers would be inclined to follow if they did not have prior knowledge concerning where to set the demarcation value.

To facilitate the solution, it will be assumed that no two of the n values in a sample of size n are identical. For the analysis given here, n is taken to be even. Study of small samples shows that when n is odd, $P(s,\ n-1) \leqq P(s,\ n) \leqq P(s,\ n+1)$. Taking (6) (with the median as initial cut) as a starting point, assume that the demarcation value is moved so that $(n_1 + 1)$ values are on one side and $(n_1 - 1)$ values on the other. This adds a fraction of the total arrangements with runs of length s or more on each side of the new demarcation value equal to:

$$\Delta_1 P(s/s,\ n_1 + 1/n_1 - 1)$$

$$= \frac{1}{(n_1 + 1)\binom{2n_1}{n_1 - 1}} \sum_{r=1}^{n_1 - s} B(n_1 - 1, r)[A(n_1 + 1, r - 1) + 2A(n_1 + 1, r) + A(n_1 + 1, r + 1)] \tag{7}$$

where $B(n_1 - 1, r)$ is given by (3) above and

$$A(n_1+1,r) = r\sum_{j=0}^{r-1}(-1)^j[r+s-(n_1+1) + (j+1)(s-1)]\binom{r-1}{j} \cdot\left[\binom{n_1+1-s-j(s-1)}{r-1} - \binom{n_1+1-2s-j(s-1)}{r-1}\right] \quad (8)$$
$$+ r(r-1)\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j} \cdot\left[\binom{n_1+1-s-j(s-1)}{r} - \binom{n_1+1-2s-j(s-1)}{r}\right].$$

The essential points in the derivation of (7) and (8) may be perceived most easily by considering some typical computations. Suppose that we wish to derive $\Delta_1 P(4/4, 6/4)$, having previously derived all of the values of $P(s/s, 5/5)$ from (6). The possible combinations with a run of at least 4 on each side of a cut with 6 above and 4 below have the following orders:

1. 6 above and 4 below, or 4 below and 6 above,
2. 5 above, 4 below, and 1 above, or 1 above, 4 below, and 5 above, and
3. 4 above, 4 below, and 2 above, or 2 above, 4 below and 4 above.

The simplest of these is the first. Starting with the value of $P(4/4, 5/5)$ as given by (6), we now wish to determine how much additional probability is associated with moving the cut from the median to a point where 6 are above and 4 are below. Since there are 6 possible locations in the new arrangement for the value that was moved from below to above the cut and $\binom{10}{6}$ ways for arranging 6 above and 4 below, the total possible combinations of these provides the factor given in the denominator of (7), in this case $6\binom{10}{6}$. Since there is only one combination possible for 4 items taken 4 at a time, $B(4, 1)$ as given by (3) is as might be expected unity. Then, since we must have at least one run above the cut, $A(6, 0)$ must be 0. The first important question relates to the value of $A(6, 1)$. Since there is only one run of 6, it is easy to see that a run of length 4 or more must have occurred above the median if the value

moved from below is now in position 1, 2, 5, or 6 in the new run. Hence, there are only two possible locations for the value moved that give new combinations that have not been counted with respect to the median. At this point, it will be observed that for this case, $r = 1$, this value is given by $2s - (n_1 + 1)$. Since there is exactly one run on each side of the new cut, the coefficient 2 appears before the $A(n_1 + 1, r)$ in (7) to take account of the two ways that these runs may be arranged, namely, 6 above followed by 4 below and 4 below followed by 6 above.

Now consider the ways in which we may have two runs with the restriction that one must be of length 4 or more. This is to be given by $A(6, 2)$. In this case, there are two such run combinations, one with runs of lengths 5 and 1, and one with runs of lengths 4 and 2. Obviously, the value that was moved could not have been in the short run in either case because these arrangements would have had long runs of length 4 or more that would have been counted with respect to the median. In the case of the run of length 5, it could not be on either end but in the run of length 4, it could be at any one of the positions in the run. We also observe that with two runs of dissimilar lengths, the positions of the runs may be interchanged. This gives in this case a factor 2. Hence, we find that $A(6, 2)$ is $2 \cdot 3 + 2 \cdot 4$, or 14. To conform with (8), this sum would have to be written as $2 \cdot 3 \cdot 2 + 2 \cdot 1$, although, at this point, it may not be clear that this is a reasonable thing to do. However, by extending the investigation step by step, it is found that the various terms in (8) are required. Specifically, the j becomes necessary when $n_1 + 1$ becomes greater than $2s - 1$ and the binomial coefficients with terms in $2s$ are introduced so that any combination that already has a run of length s on the basis of the median will not be counted again.

Obviously, this process may be continued by moving the cut to include $(n_1 + 2)$ values on one side and leave $(n_1 - 2)$ values on the other. Proceeding in this way, the fraction added in going from $(n_1 + i - 1)$ values above and $(n_1 - i + 1)$ values below to $(n_1 + i)$ above and $(n_1 - i)$ below is given by:

$$\Delta_i P(s/s, n_1 + i/n_1 - i) = \frac{1}{(n_1 + i)\binom{2n_1}{n_1 - i}} \sum_{r=1}^{(n_1-i)-(s-1)} B(n_1 - i, r) \cdot [A(n_1 + i, r - 1) + 2A(n_1 + i, r) + A(n_1 + i, r + 1)] \tag{9}$$

where $B(n_1 - i, r)$ is defined by (3) above and

$$A(n_1 + i, r) = r \sum_{j=0}^{r-1} (-1)^j [r + s - (n_1 + i) + (j + 1)(s - 1)] \cdot \binom{r-1}{j} \left[\binom{n_1 + i - s - j(s-1)}{r-1} - \binom{n_1 + i - 2s - j(s-1)}{r-1} \right] + r(r-1) \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \cdot \left[\binom{n_1 + i - s - j(s-1)}{r} - \binom{n_1 - i - 2s - j(s-1)}{r} \right]. \quad (10)$$

One of each of these Δ's is added in going from the median to each side. Therefore, the desired probability of an arrangement with runs of length s or more on each side of at least one of all possible demarcation values is:

$$P(s/s, \text{ any cut}) = P(s/s, n_1/n_1) + 2 \sum_{i=1}^{n_1 - s} \Delta_1 P(s/s, n_1 + i/n_1 - i). \quad (11)$$

VII. ASYMPTOTIC DISTRIBUTIONS

Intuitively, the asymptotic distribution of arrangements with 0, 1, 2 etc., runs of length s or more for $n_1/n = e_1$, a constant, would be expected to become Poisson Exponential as n becomes large. Referring to Mood,[3] the expected number of runs of length s or more on one side of a demarcation value is his expression (3.13), which may be written:

$$E(r_{1s}) = (n_2 + 1) \frac{n_1^{(s)}}{n^{(s)}} \sim n e_1^{\,s} e_2 \qquad \text{for } n_1 \text{ and } n_2 \text{ large} \quad (12)$$

where $E(r_{1s})$ is the expected number of runs of length s or more on the side of the cut designated 1; superscript (s) designates a factorial moment, e.g.,

$$n^{(s)} = n(n-1)(n-2) \cdots (n - s + 1) \quad (13)$$

and e_1 and e_2 are written for n_1/n and n_2/n, respectively.

The variance is his expression (3.15), or

$$\tilde{\sigma}_{r_{1s} r_{1s}} = \tilde{\sigma}_{ss} = \frac{(n_2 + 1)^{(2)} n_1^{(2s)}}{n^{(2s)}} + (n_2 + 1) \frac{n_1^{(s)}}{n^{(s)}} \cdot \left(1 - (n_2 + 1) \frac{n_1^{(s)}}{n^{(s)}}\right) \quad (14)$$

$$\sim n e_1^{\,s} e_2 (1 - s^2 e_1^{\,s-1} e_2^{\,2} - e_1^{\,s})$$

$$\sim n e_1^{\,s} e_2 = E(r_{1s}) \quad \text{for } s, n_1, \text{ and } n_2 \text{ large.}$$

Corresponding expressions for the side designated 2 may be obtained by interchanging the subscripts, 1 and 2, in equations (12) and (14).

Mood[3] also derives an expression (3.18) for the covariance of numbers of runs equal to or greater than specified lengths on the two sides of the demarcation value. For runs of length s or more on each side, this becomes:

$$\begin{aligned}\tilde{\sigma}_{r_{1s}r_{2s}} &= \frac{n_1^{(s+1)}n_2^{(s+1)}}{n^{(2s)}} + \frac{2n_1^{(s)}n_2^{(s)}}{n^{(2s-1)}} - \frac{(n_1+1)(n_2+1)n_1^{(s)}n_2^{(s)}}{n^{(s)}n^{(s)}} \\ &\sim n\, e_1^{s}e_2^{s}(s^2e_1e_2 - s + 1) \\ &\sim n\, s^2e_1^{s+1}e_2^{s+1} \text{ for } s,\ n_1,\text{ and } n_2 \text{ large.}\end{aligned} \tag{15}$$

From (14) and (15), it is clear that the covariance between long runs on the two sides becomes negligible for s, n_1, and n_2 large and the occurrence of long runs on each side may be treated as independent.

Since Mood[3] has shown (his Theorem I) that the distribution of the number of runs of length s or more on one side is asymptotically normal and by (12) and (14) above, the first two moments are those of a Poisson Exponential, the asymptotic probabilities of arrangements with runs of length s or more may be approximated by:

On side 1:

$$P(s/-,\ n_1/n_2) \doteq 1 - e^{-ne_2e_1^{s}}; \tag{16}$$

On side 2:

$$P(-/s,\ n_1/n_2) \doteq 1 - e^{-ne_1e_2^{s}}; \tag{17}$$

On each side:

$$P(s/s,\ n_1/n_2) \doteq (1 - e^{-ne_2e_1^{s}})(1 - e^{-ne_1e_2^{s}}); \tag{18}$$

On either side:

$$P(s/-\text{ or }-/s,\ n_1/n_2) \doteq 1 - e^{-ne_1e_2(e_1^{s-1}+e_2^{s-1})}. \tag{19}$$

When the median is being used as the demarcation value, that is, when $e_1 = e_2$, these become:

On side 1 or on side 2 alone:

$$P(s/-,\ \text{median}) = P(-/s,\ \text{median}) \doteq 1 - e^{-n\cdot 2^{-(s+1)}}; \tag{20}$$

On each side:

$$P(s/s,\ \text{median}) \doteq (1 - e^{-n\cdot 2^{-(s+1)}})^2; \tag{21}$$

On either side:

$$P(s/-\text{ or }-/s,\ \text{median}) \doteq 1 - e^{-n\cdot 2^{-s}}. \tag{22}$$

Asymptotic relationships of this type do not add much to the solution of the practical problem of calculating probabilities associated with samples of 100 or less. They do, however, suggest that doubling sample sizes for a given probability should increase s by unity. This is in close agreement with the calculations for finite sample sizes. This observation suggested the treatment in the next section.

VIII. RELATIONSHIPS BETWEEN s AND n FOR CONSTANT PROBABILITY

From (20), (21), and (22), it is clear that, for constant probability, s is asymptotically a simple function of n for each of the arrangement distributions considered for runs relative to the median. Specifically, we obtain:

On side 1 or on side 2:

$$s = \frac{\log n - \log(-\log_e(1-P))}{\log 2} - 1 \qquad \text{where} \quad P = P(s/-, \text{ median}) = P(-/s, \text{ median}); \tag{23}$$

On each side:

$$s = \frac{\log n - \log(-\log_e(1-\sqrt{P}))}{\log 2} - 1 \qquad \text{where} \quad P = P(s/s, \text{ median}); \tag{24}$$

On either side:

$$s = \frac{\log n - \log(-\log_e(1-P))}{\log 2} \qquad \text{where} \quad P = P(s/- \text{ or } -/s, \text{ median}). \tag{25}$$

After considering equations (23), (24), and (25), it is quite obvious that an equation similar to (24) in the same way that (25) is similar to (23) could be written, i.e.;

$$s = \frac{\log n - \log(-\log_e(1-\sqrt{P}))}{\log 2} \qquad \text{where } P = P[(s/- \text{ or } -/s)/(s/- \text{ or } -/s), \text{ median}] \tag{26}$$

but what is the meaning of P? It is clear that the P in (26) is approximately the square of the P in (25). So far, however, no analytic justification for (26) has been obtained, although the P in (26) is obviously

TABLE XVI

Constants for equation (27) calculated from equations (23) to (26) and tables VII to X

Equation	Table	P	A	B	C	Differences at # equal to					
						10	20	40	60	100	200
23	VII	0.001	5.151	126.6	−266.5	0	0	0	0	−0.01	+0.01
		0.01	4.863	53.19	−105.1	0	+0.02	−0.02	−0.01	0	+0.02
		0.02	4.445	39.61	−79.16	0	+0.01	−0.01	−0.01	0	+0.02
		0.025	4.306	35.34	−71.08	0	+0.01	−0.01	−0.01	0	+0.02
		0.05	3.127	28.03	−57.71	0	+0.01	−0.01	−0.01	0	+0.02
		0.10	3.127	13.95	−32.83	0	+0.01	−0.01	−0.01	0	+0.01
		0.50	0.5297	−3.126	−0.1306	0	0	0	0	0	0
		0.90	−2.576	−6.757	10.96	0	0	0	+0.01	0	0
		0.95	−3.442	−7.244	13.84	0	0	0	0	0	0
		0.975	−4.227	−7.185	15.44	0	0	0	0	0	−0.01
		0.98	−4.441	−7.326	16.29	0	−0.01	0	0	0	−0.01
		0.99	−5.128	−7.164	17.78	0	−0.01	+0.01	0	0	−0.01
		0.999	−7.829	0.3596	8.278	0	0	−0.01	−0.01	+0.06	−0.04
24	VIII	0.001	−0.3002	23.36	−32.54	0	−0.01	+0.02	0	0	−0.01
		0.01	0.0467	10.93	−13.30	0	0	0	0	0	0
		0.02	0.0048	8.596	−11.14	0	0	0	0	0	0
		0.025	−0.0005	7.612	−9.868	0	0	0	0	0	0
		0.05	−0.0672	4.695	−6.288	0	0	0	0	0	0
		0.10	−0.2136	1.668	−2.139	0	0	0	0	0	0
		0.50	−1.573	−4.142	6.576	0	0	0	+0.01	0	−0.01
		0.90	−3.660	−6.518	13.04	0	0	0	0	0	−0.01
		0.95	−4.408	−6.591	14.75	0	0	0	0	0	−0.01
		0.975	−5.039	−6.665	16.21	0	0	0	0	0	−0.01
		0.98	−5.218	−6.728	16.72	0	0	0	+0.01	0	−0.01
		0.99	−5.822	−7.068	18.19	0	0	0	+0.01	0	−0.01
		0.999	−7.430	−6.861	23.26	0	−0.01	0	+0.01	0	−0.01
25	IX	0.001	4.879	154.6	−324.2	0	0	+0.01	0	−0.02	+0.01
		0.01	4.902	73.86	−145.9	0	+0.01	−0.01	−0.01	0	+0.02
		0.02	4.764	55.83	−109.5	0	+0.01	−0.01	−0.01	0	0
		0.025	4.611	51.88	−102.5	0	+0.01	−0.01	−0.01	0	+0.02
		0.05	4.432	35.72	−71.31	0	+0.02	−0.02	−0.01	0	+0.02
		0.10	3.847	25.64	−54.79	0	+0.01	−0.01	−0.01	0	+0.02
		0.50	2.524	1.680	−13.88	0	+0.01	−0.01	0	0	+0.01
		0.90	1.141	−9.694	7.601	0	0	0	0	0	0
		0.95	0.759	−12.16	12.83	0	0	0	0	0	0
		0.975	0.422	−14.08	17.14	0	0	0	0	0	0
		0.98	0.356	−14.74	18.61	0	0	0	0	0	0
		0.99	0.081	−16.61	22.91	0	0	0	+0.01	0	0
		0.999	−0.600	−21.68	34.97	0	−0.01	+0.02	0	−0.02	+0.01
26	X	0.001	−4.176	60.34	−69.24	+0.01	−0.01	0	—	+0.01	—
		0.01	−2.176	39.32	−49.98	+0.01	+0.01	−0.01	—	0	—
		0.02	−1.762	34.71	−47.65	0	+0.01	−0.02	—	−0.02	—
		0.025	−1.427	32.01	−45.06	0	+0.02	−0.01	—	−0.01	—
		0.05	−0.830	26.26	−41.04	0	+0.02	−0.01	—	−0.02	—
		0.10	−0.356	21.25	−38.10	0	+0.01	−0.01	—	−0.01	—
		0.50	1.069	1.932	−14.67	0	+0.01	−0.01	—	0	—
		0.90	1.136	−10.55	5.014	0	+0.02	−0.02	—	0	—
		0.95	1.369	−17.16	18.39	0	+0.01	−0.02	—	+0.02	—
		0.975	1.222	−19.89	24.12	−0.01	+0.01	−0.02	—	+0.02	—
		0.98	1.007	−19.78	24.69	−0.01	+0.01	−0.02	—	+0.01	—
		0.99	0.679	−21.66	30.15	−0.01	0	−0.02	—	+0.02	—
		0.999	0.408	−29.73	48.79	0	0	−0.02	—	+0.02	—

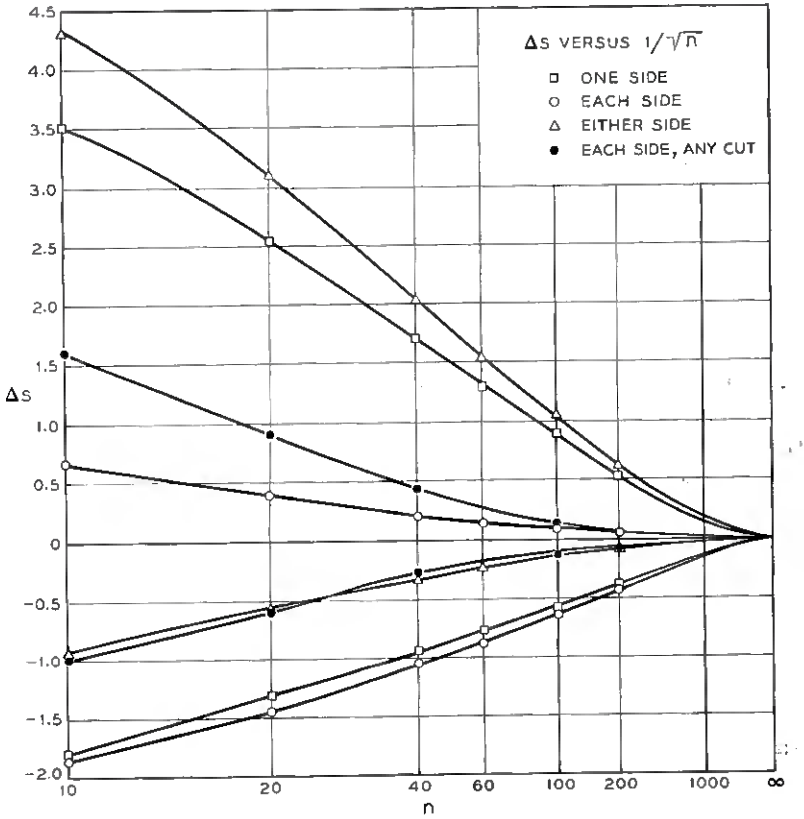

FIG. 7 — Differences between interpolated values of s computed from Tables XII to XV, inclusive, and appropriate equations (23) to (26), inclusive, for $P = 0.01$ and 0.99.

the maximum value possible for $P(s/s, \text{any cut})$. Nevertheless, as we shall see below, it appears to predict empirically the large sample behavior of runs above and below any cut even better than (23), (24), and (25) predict the large sample behavior of the other types of run.

For this comparison, values of s corresponding to particular values of P were interpolated (in a few cases, extrapolated) from the exact determinations of Tables XII to XV. Since the distributions for each sample size in these tables had been found to be mildly deviant from log-normal, the interpolation process first obtained a three point log-normal relationship in the P area of interest by changing the s-scale to an $(s + a)$-scale. Here, a is the constant that must be added to s to produce the log-normal relationship in the interval under consideration. Values of s for each P, n, and type of run were obtained to four decimal places.

In each case, the difference between the interpolated value and that given by the appropriate equation (23), (24), (25), or (26) was calculated. At this point, it was found that some of these differences for a particular P and type of run could be approximated by linear equations in $1/n$ or $1/\sqrt{n}$. In view of this, all have been fitted by the equation:

$$\Delta s = \frac{A}{\sqrt{n}} + \frac{B}{n} + \frac{C}{\sqrt{n^3}} \tag{27}$$

The constants, A, B, and C, have been recorded in Table XVI. The agreement between the values given by this equation and the differences on which they were based seldom exceed 0.02. Thus, it was assumed that (27) provided a reasonable approximation for extrapolation to the larger sample sizes for which values are shown in Tables I to IV and in Figs. 1 to 4.

To illustrate the agreement with (27), some typical results for P's of 0.01 and 0.99 are given in Fig. 7. All show that the differences converge in a reasonably uniform manner to zero at infinity.

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Properties of Control Chart Zone Tests

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This paper is concerned with the statistical properties of tests composed of the standard control chart test supplemented by one or more tests for runs of points in various zones into which the control chart is partitioned. The basic properties of the resultant tests, called zone tests, are illustrated graphically. A procedure for determining the properties of many zone tests of practical interest is described.

1. INTRODUCTION

1.1 *General*

In using an $\bar{X}$ control chart to maintain control of a process average, we periodically measure n units of the product and plot the average measurement $\bar{X}_n$ on the control chart in its chronological position. The control chart presents a pictorial summary of production history that is useful in: (a) detecting changes in the process average, and (b) providing clues to the causes of such changes. Various run tests have proved useful in application (b).[1] Most of the literature on run theory pertains to this application. There are tests for runs up and for runs up and down; there are tests for the number of runs and for the lengths of runs. The control chart is particularly suitable for run tests. We shall consider the use of a particular type of run test in application (a).

In application (a), as each point is plotted we decide whether or not to look for trouble (to take action to eliminate the cause of the change in the process average). Using the standard control chart test,[2, 3] we look for trouble if a point falls in a zone outside of two control limits symmetrically placed on either side of a line representing the nominal process average. The control limits, called the 3σ (3-sigma) limits, are placed at $\bar{X}_0' \pm 3(\sigma'/\sqrt{n})$, where $\bar{X}_0'$ and σ' are the nominal process average and standard deviation, respectively, and n is the sample size, or number of units of product measured for each point. We shall assume that $\bar{X}_0'$ and σ' are known, and that σ' remains fixed.

In using a statistical test to decide at each point whether or not to look for trouble, we are subject to two types of errors:

(1) We make Type 1 errors when we decide to look for trouble when in fact none is present.

(2) We make Type 2 errors when we decide not to look for trouble when trouble is actually present.

Few Type 1 errors are made when the standard control chart test is used — an average of about one point in 370 falls outside of the 3σ limits when the process average is at its nominal level. Type 2 errors occur at consecutive points following a change until the test used indicates that a change has occurred. Small changes may result in long sequences of Type 2 errors because the probability of a point falling outside of the 3σ limits may be small, though larger than it was when the process average was at its nominal level. This definition of the two types of errors makes a sharp distinction between the presence and absence of trouble — a distinction more theoretical than practical — in order to simplify the exposition of the subject.

Experience indicates that, in general, the standard control chart test maintains an economic balance between the two types of errors in a wide range of industrial applications (Reference 2, pp. 276–7; Reference 3, p. 11). However, other tests may be more attractive economically in applications where early detection of relatively small changes is important. It has been suggested (Reference 4, p. 128) that supplementary run tests may prove useful in such applications. Various run tests are used in practice to supplement the standard control chart test,* but little has been published on the properties of the resultant tests,† though it is quite apparent that each additional supplementary run test employed decreases the number of Type 2 errors made and increases the number of Type 1 errors.

There are several alternative ways to reduce the number of Type 2 errors made; we can:

(1) Set the limit lines closer to the nominal process average $\bar{X}_0'$.

(2) Increase the sample size.[5, 7]

(3) Replace the standard test with a single test for runs of points outside of appropriate limits.[6]

(4) Supplement the standard test with one or more run tests.

(5) Temporarily modify the sampling procedure — e.g., increase the sample size or frequency of sampling — whenever a point falls outside of "warning" limits but inside of the "action" limits (3σ limits).[4, 7]

* See footnote, page 89.

† After the page proofs of this paper had been received, the author was advised of Reference 9, which deals primarily with the test $T_{12}(L_1, L_2)$.

(6) Use a control chart for a statistic other than $\bar{X}_n$; for example, plot points representing the moving average of k consecutive $\bar{X}_n$'s.

The improvements generally require extra information or more complicated tests, or they result in an increased frequency of Type 1 errors.

In this paper we study the properties of various run tests that either replace or supplement the standard control chart test in application (a). We limit our study to a particular type of run tests which we call "zone tests" because they test for runs of points in various zones into which the control chart is partitioned. For example, we study such tests as $T_{12'}$ (3, 2),† which calls for action if a single point falls outside of the 3σ limits or if two of three consecutive points fall outside of a 2σ limit line. We limit our studies to tests used on charts of the statistic $\bar{X}_n$; zone tests can be useful on other charts, but their properties depend on the properties of the particular statistic plotted. Our results apply for any sample size and frequency of sampling.

We use $T_k(L_k)$ to denote a test for k consecutive points outside of one of the pair of limit lines at $\bar{X}_0' \pm L_k(\sigma'/\sqrt{n})$, and $T_{k'}(L_k)$ to denote the test for k out of $k + 1$ consecutive points outside of the limit lines. If we combine two tests, we let $T_{k_1k_2}(L_{k_1}, L_{k_2})$ denote the combined test that calls for action on the occurrence of *either* type of run; k_1 and k_2 are integers less than nine, either primed or unprimed.

For simplicity of notation we may eliminate the brackets on the test notation if the subscripts provide sufficient information. For this purpose, we adopt standard limits for certain runs. Thus we may use T_1 rather than $T_1(3)$ to denote the standard control chart test. Also, we use the 2σ limits, the 1σ limits, and $\bar{X}_0'$ itself as standard for runs of lengths 2, 4, and 8, respectively. Thus $T_{12'}$ means $T_{12'}(3, 2)$, and T_8 means $T_8(0)$. We use an asterisk to denote one-sided tests — those with limit lines on only one side of $\bar{X}_0'$. Test T_1* has a single limit line, at $\bar{X}_0' + 3(\sigma'/\sqrt{n})$.

1.2 *Process Model*

We use a process model in which the process average is $\bar{X}' = \bar{X}_0' + \Delta$, where Δ is subject to change. A picture showing how Δ changes with time would show a series of rectangular pulses (positive or negative) of various heights, separated by periods with $\Delta = 0$. The beginning of a pulse corresponds to the occurrence of an assignable cause of variation, and the height of the pulse is a function of the particular cause. The pulse ending corresponds to the elimination of the trouble. The distribution of

† Read subscript as 1, 2'.

the lengths of the pulses depends on the test we use to detect changes; the test should be designed to keep the lengths reasonably short.

The sample average, $\bar{X}_n$, is assumed to have a normal distribution defined by its expected value $\bar{X}'$ and standard deviation $\sigma'/\sqrt{n}$.

Whenever $\Delta = 0$, the process average is at its nominal level, and we say the process is in State 1. Whenever $\Delta \neq 0$, there is trouble present, and we say the process is in State 2. We assume that no additional changes occur while the process remains in State 2.

At each point we look for certain runs that rarely occur in State 1. In the absence of such runs there is no indication that the process is not in State 1, and accordingly we do not look for trouble. We do not attempt to define the probability that the process is in State 1 at any point. In this model, we stop the process to look for trouble on the *first* occurrence of a run for which we are testing. When the process starts again it is assumed to be in State 1; consequently, the testing procedure ignores previous points.

Relatively straight-forward mathematics can be used to describe the properties of certain tests acting within the framework of this process model. Alternative, and perhaps more realistic, assumptions can easily lead to much more complicated problems of description. In many cases the results obtained here can be used to describe qualitatively the properties of tests applied to more complex processes.

1.3 *Measuring the Two Types of Decision Errors*

As each point is plotted on the control chart we decide either that the process is in State 1 — in which case we leave it alone — or that it is in State 2 — in which case we look for trouble. We make a Type 1 error when we say that the process is in State 2 when actually it is in State 1; Type 1 errors initiate needless action. We make a Type 2 error when we say that the process is in State 1 when actually it is in State 2; Type 2 errors fail to initiate needed action. We generally make a series of consecutive errors of Type 2 before detecting the change in state.

Let the random variable y denote the number of points plotted while the process remains in State 2. Then $y - 1$ consecutive errors of Type 2 are made. Let $E(y)$ denote the expected, or average, value of y; then $E(y - 1)$ is the average length of a series of Type 2 errors.

$E(y)$ depends on Δ, the amount by which the process average changes; we sometimes note this dependence by writing $E(y; \Delta)$. $E(y; \Delta)$ is a monotonically decreasing function of the magnitude of Δ; that is, the larger the change, the smaller is $E(y)$. In other words, tests are more sensitive to large changes than to small changes.

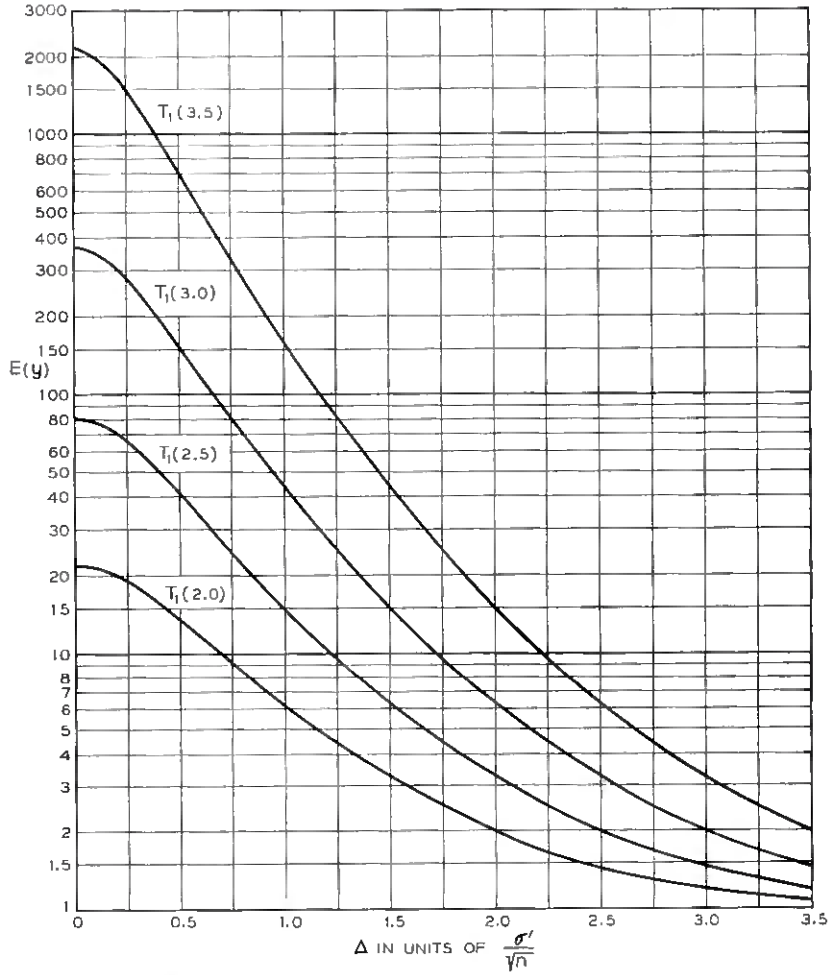

Fig. 1 — $E(y)$ versus Δ for $T_1(L_1)$ for various limits.

Fig. 1 shows curves of $E(y)$ versus Δ for $T_1(L_1)$ for $L_1 = 2, 2.5, 3$, and 3.5. Note on the curve for $T_1(3)$, for example, that $E(y) = 15$ at $\Delta = 1.5\ (\sigma'/\sqrt{n})$; this means that following a change of this magnitude, an average of 15 points are plotted before a point falls outside of a 3σ limit. Note that as Δ approaches zero, $E(y)$ approaches 370, which corresponds to the average number of points between consecutive Type 1 errors while the process remains in State 1.

In Fig. 1 and later figures the abscissa is Δ, and it is measured in units of $\sigma'/\sqrt{n}$, which is the standard deviation of $\bar{X}_n$. The particular ab-

scissa that applies to a change of a given physical magnitude is proportional to $\sqrt{n}$; for example, if n is doubled in the above example where $\Delta = 1.5(\sigma'/\sqrt{n})$, then the appropriate abscissa on Fig. 1 increases from 1.5 units, with ordinate $E(y) = 15$ on curve T_1, to $1.5\sqrt{2} = 2.121$ units, with $E(y) = 5.4$. The positions of the curves relative to one another are independent of n.

If the process were to remain in State 1 indefinitely, $E(y; 0)$ would represent the average number of points between consecutive Type 1 errors, and $1/[E(y; 0)]$ would be the asymptotic probability of a Type 1 error. In comparing tests with respect to Type 1 errors, we compare their values of $E(y; 0)$.

In comparing tests with respect to Type 2 errors, we compare their values of $E(y)$, or $E(y - 1)$, for various non-zero values of Δ.

1.4 *Comparing the Statistical Properties of Various Zone Tests*

We are primarily interested in the distribution of y. The distribution of y for all of the zone tests we consider can be adequately summarized by one parameter — its average value $E(y)$ (see Section 3.1). Therefore, in comparing the statistical properties of various tests, we compare their curves of $E(y)$ versus Δ. From such curves we can determine the asymptotic probability of Type 1 errors, $1/[E(y; 0)]$, and the average number of consecutive Type 2 errors, $E(y - 1; \Delta)$, for any Δ different from zero.

Figure 1 illustrates how the properties of zone tests can be changed by changing the limit lines. By changing the limit lines of $T_1(L_1)$ from $L_1 = 3$ to $L_1 = 2$, we reduce $E(y)$ for all values of Δ: when $\Delta > 0$, this means that the Type 2 errors are reduced; when $\Delta = 0$, this means that Type 1 errors are increased.

A choice between two tests should be based partially on the relative values of the two types of decision errors. We can fix the Type 1 errors at any desired level by an appropriate setting of the zone limits; then the Type 2 errors alone serve as a basis of comparison.

II. SUMMARY OF RESULTS

Section 4 shows how to determine the distribution of y, and in particular its average value $E(y)$, for one-sided tests $T_k{}^*(L_k)$ and $T_{k'}{}^*(L_k)$, for any k. Simple substitutions into equations for the above one-sided tests allow us to determine the properties of any test of type $T_{1k}{}^*(L_1, L_k)$ or $T_{1k'}{}^*(L_1, L_k)$. We then determine the properties of two-sided tests from the properties of the corresponding one-sided tests.

We show that the average values of y in any two separate tests provide upper and lower bounds to the average value of y in their combined test. Thus with subscript t_1 denoting test T_{t_1}, t_2 denoting T_{t_2}, and t_1t_2 de-

noting the test $T_{t_1 t_2}$ combining T_{t_1} and T_{t_2}, we have upper bounds

$$E_{t_1 t_2}(y; \Delta) \leqq E_{t_1}(y; \Delta), \qquad E_{t_1 t_2}(y; \Delta) \leqq E_{t_2}(y; \Delta), \tag{1}$$

and a lower bound

$$\frac{1}{E_{t_1 t_2}(y; \Delta)} \leqq \frac{1}{E_{t_1}(y; \Delta)} + \frac{1}{E_{t_2}(y; \Delta)}. \tag{2}$$

An application of (2) to the determination of the properties of two-sided tests in terms of the properties of their component one-sided tests yields

$$\frac{1}{E(y; \Delta)} \cong \frac{1}{E^*(y; \Delta)} + \frac{1}{E^*(y; -\Delta)}, \tag{3}$$

where the asterisks denote one-sided test results.

We can determine the properties of the following tests: $T_k(L_k)$, $T_{k'}(L_k)$, $T_{1k}(L_1, L_k)$ and $T_{1k'}(L_1, L_k)$, for any k. With $L_1 = 3$, the last two types of tests supplement the standard control chart test $T_1(3)$ with one other zone test.

Equations (1) support the logical conclusion that the more criteria we have to indicate the presence of trouble, the more quickly we will look for trouble when it is present as well as when it is not present. Thus, in supplementing the standard control chart test with other tests, we decrease the Type 2 errors at the expense of more frequent Type 1 errors. The question of how far to go in supplementing the standard control chart test must be answered in light of the relative importance of the two types of errors in the particular application considered.

Section III presents a series of charts to show the properties of several particular tests, including T_1, T_{12}, $T_{12'}$, T_{18}, and $T_{12'4'8}$. The last test* illustrates the effect of supplementing T_1 with more than one additional test; its properties were determined through the use of Monte Carlo techniques. We also show $E(y)$ versus Δ for several tests when their zone limits are translated away from the center line so that their Type 1 errors are comparable to those of T_1. It is through such translations of zone limits that we can offset the undesirable effect on Type 1 errors that occurs when we add new tests to our testing procedure.

III. CHARTS SHOWING PROPERTIES OF VARIOUS ZONE TESTS

3.1 *Distribution Function of y*

The cumulative distribution function of the random variable y, $Q_j = \text{Prob}\ (y > j)$, is shown in Figs. 2 and 3 for various zone tests.

* This test is similar to one that has been used by the Western Electric Company in its quality control training program; somewhat different criteria for taking action are used and therefore the statistical properties differ.

The curves are applicable only at integral values of j. If $y > j$, there have been no indications of a changed process average in the first j points following the change from $\bar{X}_0'$ to $\bar{X}_0' + \Delta$.

Fig. 2 shows curves for T_1, $T_{2'}$, $T_{4'}$, and T_8 for $\Delta = 0, \sigma'/\sqrt{n}$, and $2(\sigma'/\sqrt{n})$. Fig. 3 compares T_1, $T_{2'}$, $T_{12'}$, and $T_{12'4'8}$ for $\Delta = \sigma'/\sqrt{n}$; it illustrates the effect of additional tests on the distribution of y.

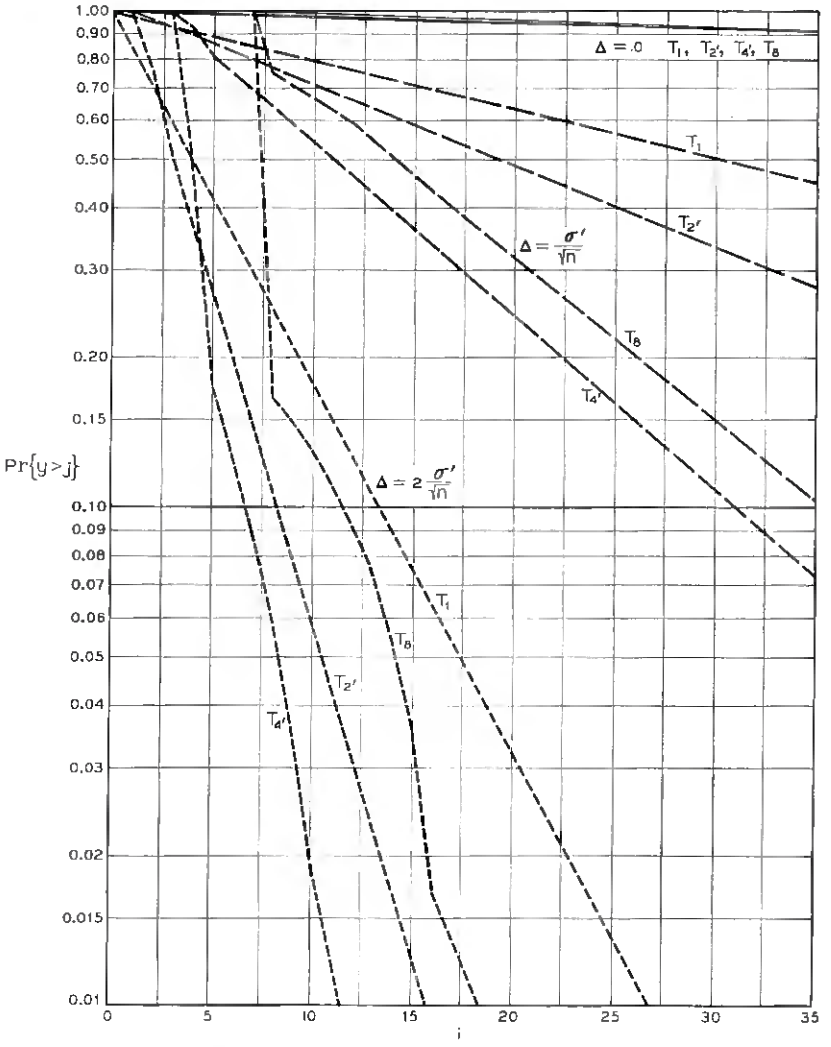

Fig. 2 — Cumulative distribution of y for T_1, $T_{2'}$, $T_{4'}$, and T_8 for $\Delta = 0, \sigma'/\sqrt{n}$, and $2(\sigma'/\sqrt{n})$.

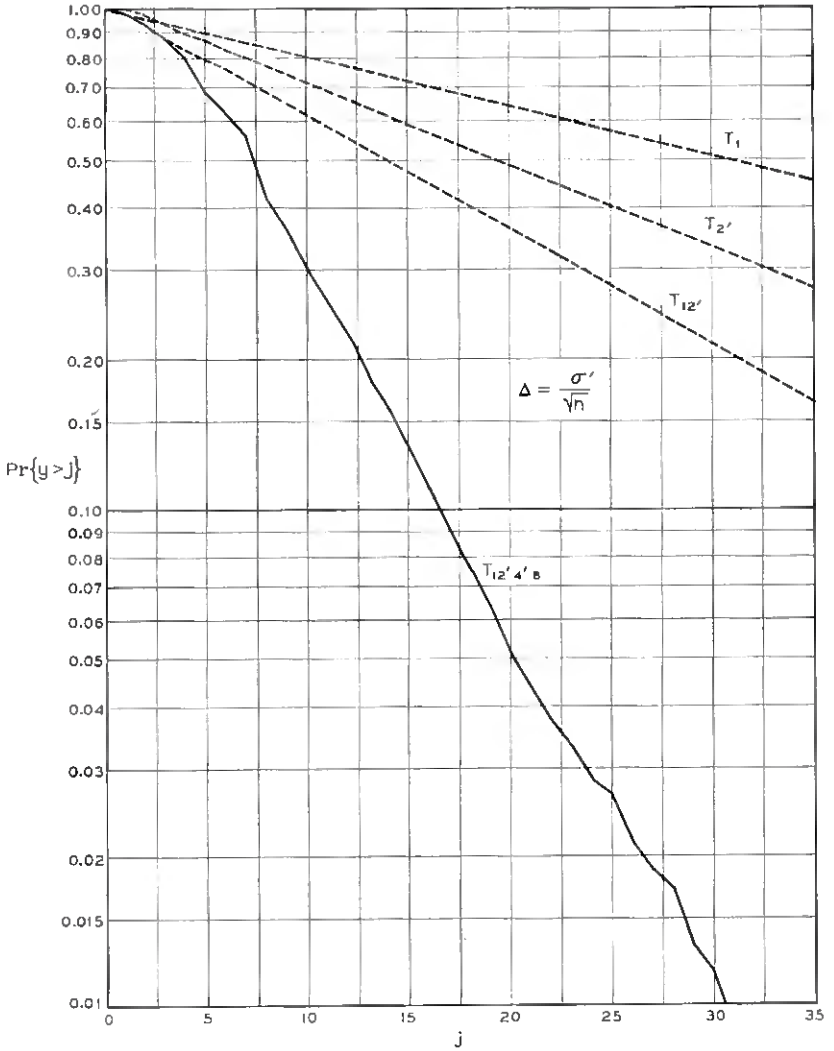

Fig. 3 — Cumulative distribution of y for T_1, $T_{2'}$, $T_{12'}$ and $T_{12'4'8}$ for $\Delta = \sigma'/\sqrt{n}$.

The curves of Figs. 2 and 3, plotted on semilogarithmic paper, can be approximated for practical purposes by straight lines. Thus, the distributions are approximately geometric, or discrete exponential, distributions that can be described by a single parameter $E(y)$ and an initial value. It is for this reason that $E(y)$ adequately summarizes their statistical properties.

Figs. 2 and 3 illustrate the fact that single tests for long runs, such as T_8, do not become fully effective immediately following a change.

3.2 *Curves of $E(y)$ Versus Δ for Tests with Standard Zone Limits*

Figs. 4, 5, and 6 illustrate typical curves of $E(y)$ versus Δ. Fig. 4 shows curves for T_1, $T_{2'}$, $T_{4'}$, and T_8. Fig. 5 shows the effect of broadening the criteria for looking for trouble — T_2 calls for action only if two consecutive points fall outside of a 2σ limit, whereas $T_{2'}$ calls for

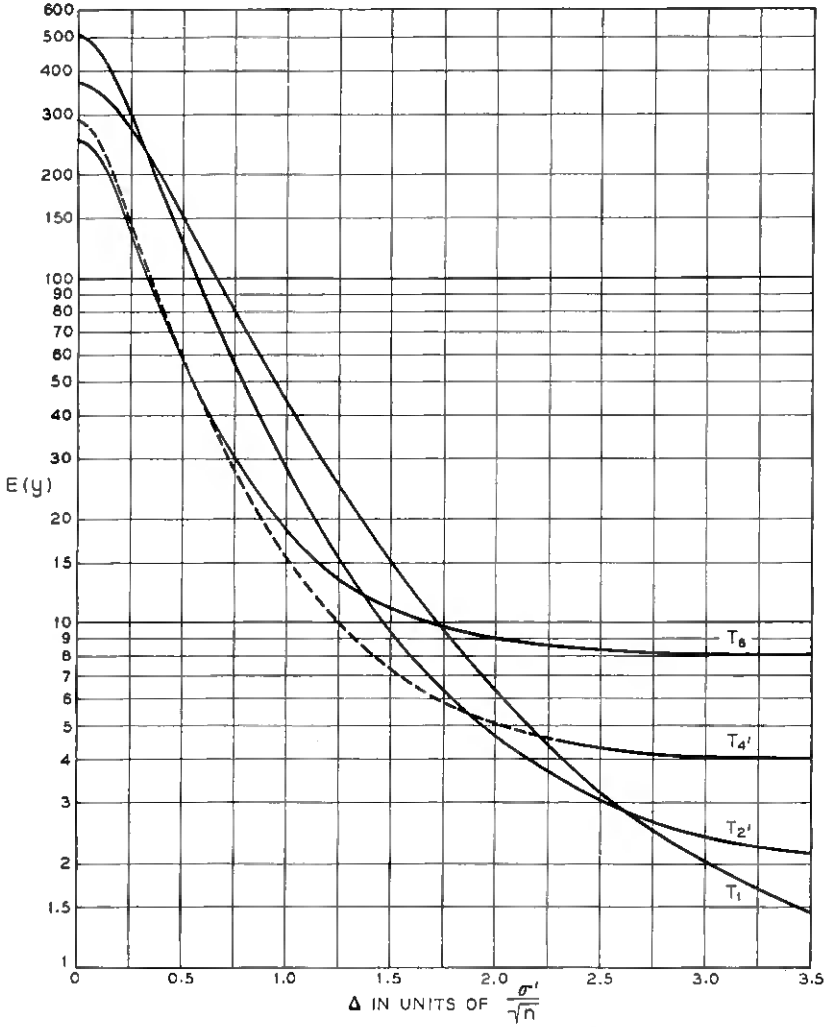

Fig. 4 — $E(y)$ versus Δ for T_1, $T_{2'}$, $T_{4'}$, and T_8.

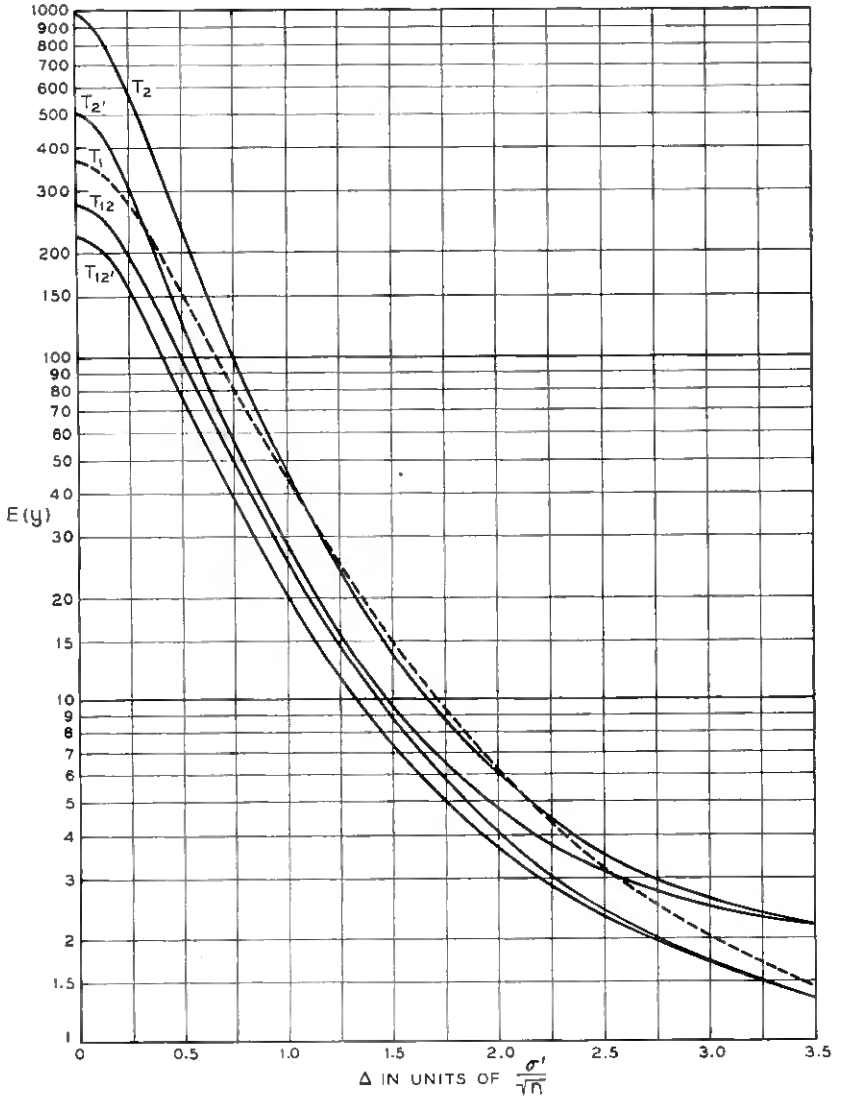

Fig. 5 — $E(y)$ versus Δ for T_1 , T_2 , $T_{2'}$, T_{12} , and $T_{12'}$.

action whenever T_2 does and also whenever two points falling outside a 2σ limit are separated by a single point not falling outside of the 2σ limit. $E(y)$ is less for $T_{2'}$ than for T_2 for all values of Δ; this difference is reflected in the curves for T_{12} and $T_{12'}$, which supplement T_1 with T_2 and $T_{2'}$, respectively.

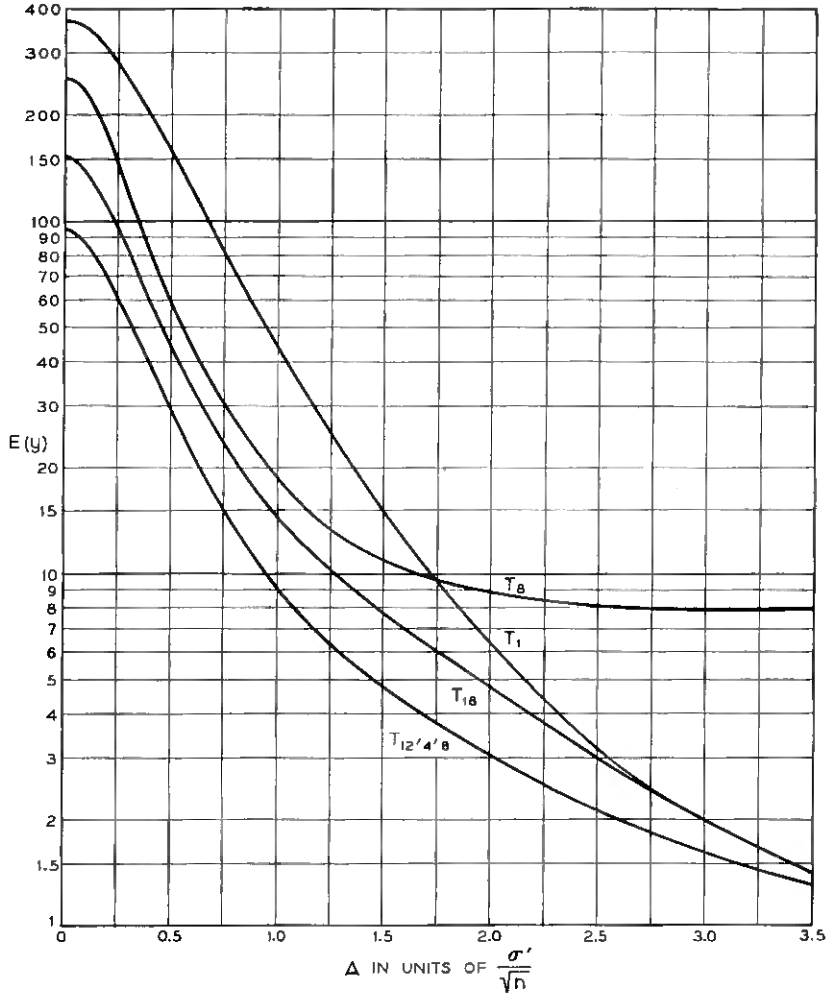

Fig. 6 — $E(y)$ versus Δ for T_1, T_8, T_{18}, and $T_{12'4'8}$.

Fig. 6 illustrates the effect of supplementing T_1 first with T_8 and then with T_8, $T_{2'}$, and $T_{4'}$. Notice how the Type 1 errors become more frequent as Type 2 errors decrease.

3.3 *Curves of $E(y)$ Versus Δ with Limits Set for a Selected Probability of Type 1 Errors*

Fig. 7 shows curves of $E(y)$ versus Δ for tests for k ($k = 1, 2, 3, 4, 6, 8$) consecutive points outside of limits that are set for each k so that the

probability of a Type 1 error is comparable to that of T_1. Tests for long runs clearly are most effective against small process changes, while T_1 itself is most effective against large process changes.

Fig. 8 shows curves for T_1, $T_8(0.065)$, and $T_{18}(3.19, 0.19)$. The last test is composed of the first two tests with all zone limits translated away from $\bar{X}_0'$. Notice that T_1 and $T_8(0.065)$ taken individually are more effective than $T_{18}(3.19, 0.19)$ in certain ranges of Δ. Fig. 9 illustrates

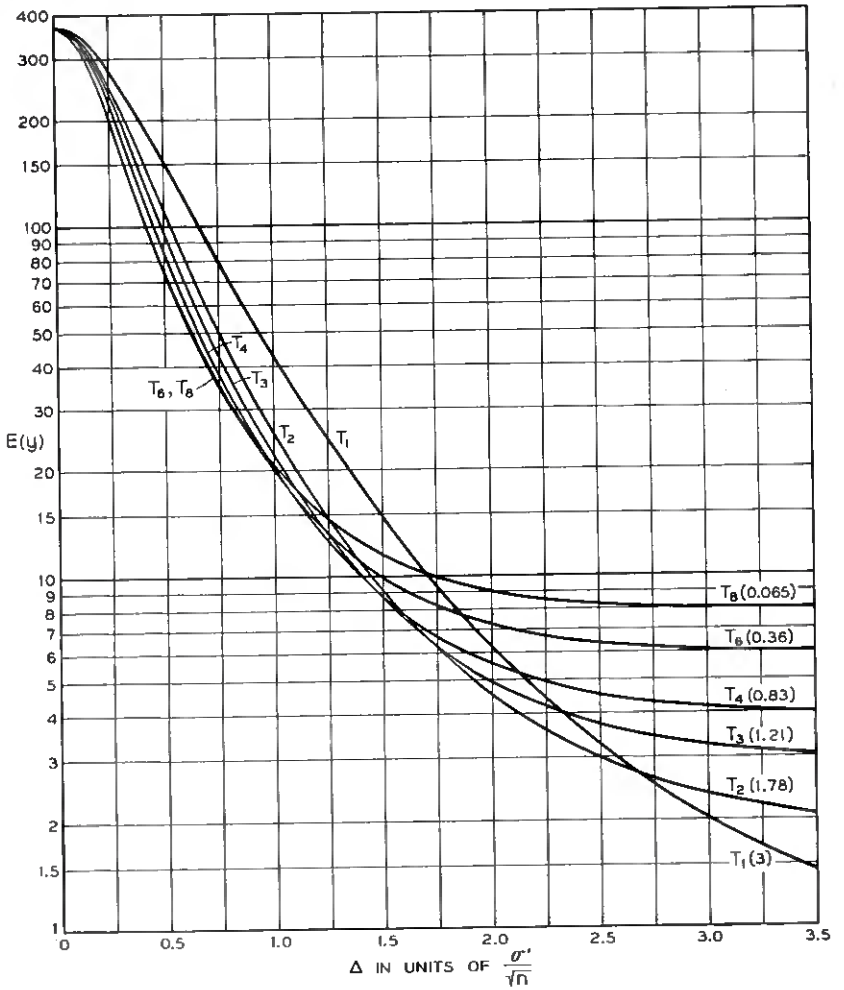

Fig. 7 — $E(y)$ versus Δ for $T_k(L_k)$ for $k = 1, 2, 3, 4, 6$, and 8 with limits set for the same probabilities of Type 1 errors.

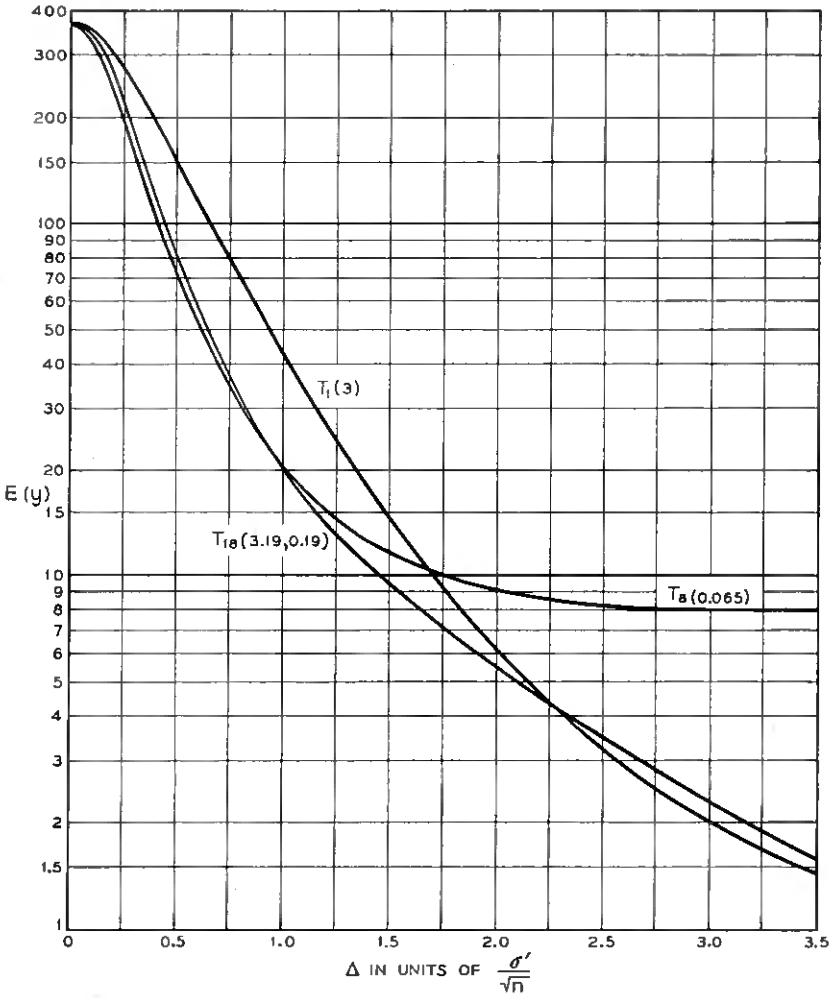

Fig. 8 — $E(y)$ versus Δ for T_1, $T_8(L_8)$, and $T_{18}(L_1, L_8)$ with limits set for the same probabilities of Type 1 errors.

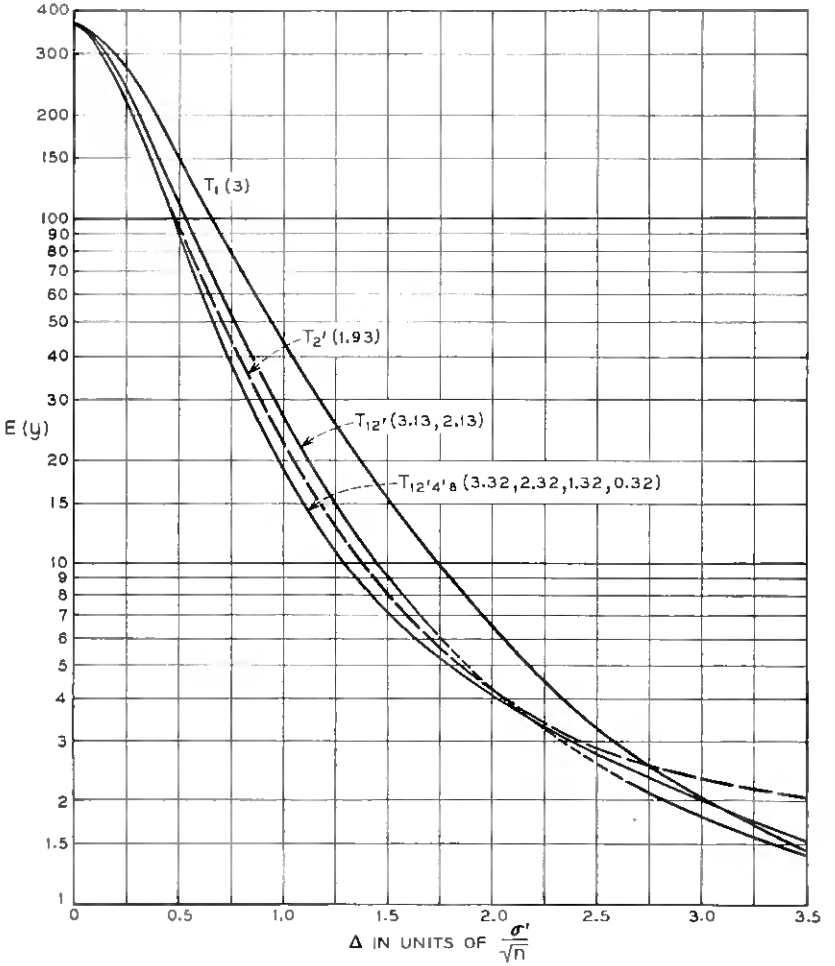

Fig. 9 — $E(y)$ versus Δ for T_1, $T_{2'}(L_{2'})$, $T_{12'}(L_1, L_{2'})$, and $T_{12'4'8}(L_1, L_{2'}, L_{4'}, L_8)$ with limits set for the same probabilities of Type 1 errors.

the same general ideas as Fig. 8, with the addition of $T_{12'4'8}$ with its zone limits translated away from $\bar{X}_0'$.

Because logarithmic scales are used for $E(y)$, the differences $E_1(y) - E_t(y)$ between curves for T_1 and other tests are distorted; Fig. 10 shows the difference on an arithmetic scale for two of the curves of Fig. 9.

Fig. 11 supports the theory that $T_{k'}(L_{k'})$ is slightly more sensitive to small changes than $T_k(L_k)$ when the limits are set so that the two tests have the same probabilities of Type 1 errors. Further graphical

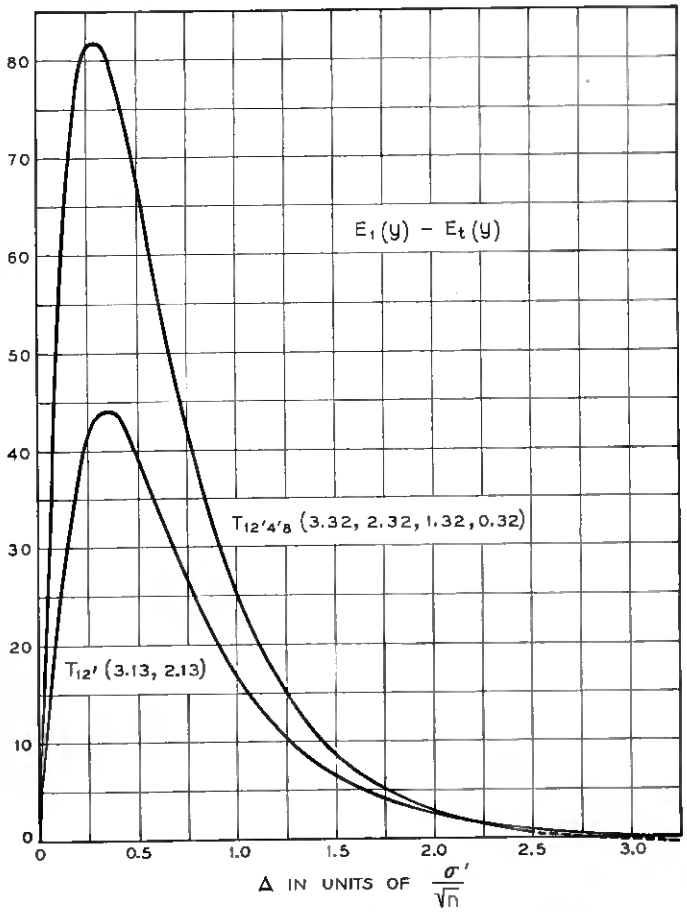

Fig. 10 — The difference between ordinates of curves of Fig. 9 shown on an arithmetic scale.

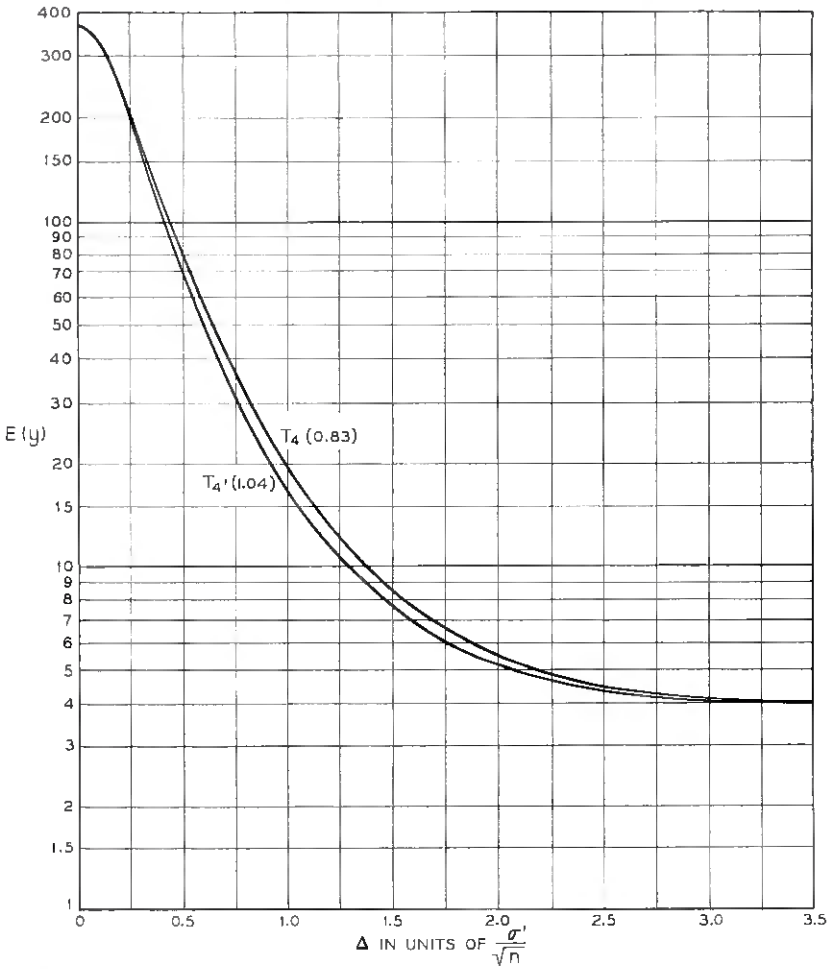

Fig. 11 — $E(y)$ versus Δ for $T_4(L_4)$ and $T_4'(L_4')$ with limits set for the same probabilities of Type 1 errors.

support is given by curves for $T_{2'}(1.93)$ of Fig. 9 and $T_2(1.78)$ of Fig. 7. No analytical proof has been developed.

IV. DETERMINING THE STATISTICAL PROPERTIES OF ZONE TESTS

4.1 *General Procedure*

With the control chart partitioned into mutually exclusive zones A, B, C, D, $\cdots$, R, we represent a sequence of points falling consecutively

into zones B, C, D, and B, for example, by the sequence *bcdb*. The lower case letters such as b serve a dual purpose — they denote the fact that a point falls into a particular zone, and they denote the probability of that particular event, or outcome. For example, the probability of a particular sequence *bcdbcdb* is $b^3c^2d^2$. Where there is danger of confusion, we may denote outcome b by ϵ_b and its probability by p_b. A sequence *bcdb* is considered to represent the outcome of a sequence of independent trials, each of which has fixed probabilities of outcomes $a,b,c, \cdots, r$.

Since the control chart points represent an average measurement $\bar{X}_n$ that has a normal distribution with average $\bar{X}_0' + \Delta$ and standard deviation $\sigma'/\sqrt{n}$, we use normal probability tables to determine the probability b, which remains constant from point to point as long as the process remains in a given state. If $\Phi(x)$ is the area under the normal curve above x, and if zone B is between limit lines at $\bar{X}_0' + L_2(\sigma'/\sqrt{n})$ and $\bar{X}_0' + L_1(\sigma'/\sqrt{n})$, where $L_2 \leqq L_1$, then probability

$$b = \Phi\left(L_2 - \frac{\Delta\sqrt{n}}{\sigma'}\right) - \Phi\left(L_1 - \frac{\Delta\sqrt{n}}{\sigma'}\right) .$$

When the process changes from State 1 to State 2, the probabilities of points falling into the various zones change. At the first point in State 2, zone tests see one point from State 2 preceded by a sequence of points from State 1; at each subsequent point in State 2 a single point from State 1 is dropped from consideration, until at last all points considered are from State 2. The zone tests are such that the probability of a point from State 2 falling into a critical zone is greater than the probability of a point from State 1 falling into the same zone. Consequently, the probability of the occurrence of a run of points in a critical zone is greatest if all of the points are from State 2. For simplicity and clarity we neglect points from State 1 while considering the results of testing points from State 2. This means that T_8, for example, does not become effective until the eighth point in State 2 appears. This simplifying assumption will affect the results little; its effect can be eliminated by calculating the probability of detecting the change in the first few points and adjusting our results. As an illustration, $T_8(0.065)$ of Fig. 7 should approach 7.1, rather than 8, as Δ approaches infinity.

If a control chart is partitioned into three mutually exclusive zones A, B, and C, outcomes a, b, and c are associated with the events that points fall in the respective zones, and probabilities a, b, and c ($a + b + c = 1$) are the corresponding probabilities of the events, or outcomes. The possible outcome of the first j trials, or points, can be enumerated by the ordered expansion of the multinomial $(a + b + c)^j$.

For example, with $j = 2$, we have:

$$(a + b + c)^2 = aa + ab + ac + ba + bb + bc + ca + cb + cc.$$

The probabilities of the various sequences occurring are obtained simply by multiplying the individual terms; for example, sequence aa has probability a^2. The probability of a particular event such as the event that either a or b occurs at least once in the first two trials is determined by selecting those sequences in which this event occurs and cumulating their probabilities; in this case it is $a^2 + b^2 + 2ab + 2ac + 2bc$.

If we wished to determine the probability Q_j of no occurrences in the first j trials of an event ϵ (a run of eight consecutive points in zone A, for example), we could enumerate all of the 3^j possible outcomes, pick out those we were interested in, and determine their probabilities. This procedure becomes very tedious as j increases, and we soon look for shortcuts. We attempt to find a recursion equation defining Q_j in terms of a limited number of terms Q_{j-1}, Q_{j-2}, etc. If we can find such an equation, we need to enumerate all pertinent outcomes only to the point where the equation becomes effective.

A recursion equation for Q_j, together with a set of initial conditions, leads to a generating function $Q(s)$ whose power series expansion exhibits Q_j as the coefficient of s^j:

$$Q(s) = 1 + Q_1 s + Q_2 s^2 + \cdots + Q_j s^j + \cdots = \sum_{j=0}^{\infty} Q_j s^j. \quad (4)$$

The generating function is useful in obtaining moments of the distribution of y. In particular, we obtain $E(y)$ by setting $s = 1$ in the equation for $Q(s)$: $E(y) = Q(1)$.

The simplest zone tests are those in which a point is classified in one of two categories; it represents either event ϵ_p with probability p or event ϵ_q with probability $q = 1 - p$. We arbitrarily call ϵ_p a success and ϵ_q a failure.* We call a test for success runs a simple run test. A compound run test is composed of more than one simple run test; for example, a test for a run of two consecutive points above the $+2\sigma$ limit is a simple run test, but a test for a run of two consecutive points above the $+2\sigma$ limit *or* below the -2σ limit is a compound run test composed of two simple run tests. A simple run test classifies points in two ways; a compound run test classifies points in more than two ways.

The test for a run of two consecutive points above the $+2\sigma$ limit is a one-sided zone test; the test for a run of two consecutive points above

* This terminology may seem incongruous, since we hope for events ϵ_q, which we term failures. Alternatively, we could change the definition, and say that we test for failure runs, but this conflicts with standard terminology.

the $+2\sigma$ limit or below the -2σ limit is a two-sided zone test. We derive the properties of two-sided tests from those of one-sided tests.

In Sections 4.2 and 4.3 we present recursion equations and generating functions for Q_j, the probability that $y > j$, for the following simple run tests:

(1) k consecutive successes $\quad k = 1, 2, 3, 4, \cdots,$

(2) k successes in $k + 1$ (or k) consecutive trials $\quad k = 2, 3,$ and 4.

In addition, we describe a procedure for extending k in (2) to any value. Equations for $E(y)$ are also presented. The results apply to one-sided zone tests.

Section 4.4 describes a procedure for determining the properties of two-sided zone tests from the properties of one-sided zone tests.

Section 4.5 presents a procedure for determining the properties of any run test combined with a test for a single point in a critical zone. Simple substitutions into the equations for a particular one-sided zone test lead to a description of the properties of that test in combination with the standard control chart test T_1*.

Section 4.6 develops upper and lower bounds to $E(y)$. Section 4.7 shows how to determine easily the properties of some tests whose zone limits are non-standard. Section 4.8 discusses the use of Monte Carlo techniques for determining the properties of tests more complex than those considered here.

4.2 *The First Occurrence of k Consecutive Successes*

We separate those sequences of outcomes having no occurrences of k consecutive successes in the first j trials (that is, $y > j$) into mutually exclusive categories according to whether the last failure occurred on trial $j, j-1, j-2, \cdots$ or $j-k+1$. With Q_j denoting the probability that $y > j$, we let $Q_{j,i}$ denote the probability that $y > j$ and that trial $j - i$ resulted in a failure and the succeeding i trials resulted in successes. Then, since i can be no greater than $k-1$, we have the equation:

$$Q_j = Q_{j,0} + Q_{j,1} + Q_{j,2} + \cdots + Q_{j,k-1}. \tag{5}$$

We enumerate the possible results:

Sequence Endings	Probabilities of Occurrence
$\cdots\cdots\cdots q$	$Q_{j,0} = q(Q_{j-1,0} + Q_{j-1,1} + \cdots + Q_{j-1,k-1})$
$\cdots\cdots qp$	$Q_{j,1} = pq(Q_{j-2,0} + Q_{j-2,1} + \cdots + Q_{j-2,k-1})$
$\cdots\cdots qpp$	$Q_{j,2} = p^2q(Q_{j-3,0} + Q_{j-3,1} + \cdots + Q_{j-3,k-1})$
$\cdots\cdots qppp$	$Q_{j,3} = p^3q(Q_{j-4,0} + Q_{j-4,1} + \cdots + Q_{j-4,k-1})$
$\vdots$	
$\underbrace{qpp\cdots p}_{k-1\ p\text{'s}}$	$Q_{j,k-1} = p^{k-1}q(Q_{j-k,0} + Q_{j-k,1} + \cdots + Q_{j-k,k-1})$

The equations on the right reduce to $Q_{j,i} = p^i q Q_{j-i-1}$. We obtain the desired recursion equation by summing over all values of i,

$$Q_j = qQ_{j-1} + pqQ_{j-2} + p^2qQ_{j-3} + \cdots + p^{k-1}qQ_{j-k}. \tag{6}$$

We can use (6) to calculate Q_j for $j \geqq k$, noting that $Q_j = 1$ for $j < k$.

We obtain the generating function of Q_j from (6),

$$Q(s) = \frac{1 - p^k s^k}{1 - s + qp^k s^{k+1}}. \tag{7}$$

Then $E(y)$ is obtained by setting $s = 1$ in (7),

$$E(y) = \frac{1 - p^k}{qp^k}. \tag{8}$$

These results are well known.[8]

4.3 *The First Occurrence of k Successes in $k + 1$ Consecutive Trials*

As in the preceding section, we separate those sequences having no occurrence of the event in question — in this case k successes in $k + 1$ consecutive trials — into mutually exclusive categories according to whether the last failure occurred on trial $j, j - 1, j - 2, \cdots,$ or $j - k + 1$. In the current problem, however, we are also interested in the location of the next-to-the-last failure since if the event in question has not occurred there must be at least two failures in the preceding $k + 1$ trials. If the last failure was on trial $j - (k - 2)$, for example, there must be at least one other failure in the preceding two trials. Here an enumeration of possible results yields:

Sequence Endings	Probabilities of Occurrence	
$\cdots\cdots\cdots q$	$Q_{j,0} = q(Q_{j-1,0} + Q_{j-1,1} + \cdots + Q_{j-1,k-2} + Q_{j-1,k-1})$	(9.0)
$\cdots\cdots\cdots qp$	$Q_{j,1} = pq(Q_{j-2,0} + Q_{j-2,1} + \cdots + Q_{j-2,k-2})$	(9.1)
$\cdots\cdots\cdots qpp$	$Q_{j,2} = p^2q(Q_{j-3,0} + \cdots + Q_{j-3,k-3})$	(9.2)
.	.	
.	.	
.	.	
$qp \cdots ppp$	$Q_{j,k-2} = p^{k-2}q(Q_{j-(k-1),0} + Q_{j-(k-1),1})$	(9.(k − 2))
$\underbrace{qpp \cdots pp}_{k-1\ p\text{'s}}$	$Q_{j,k-1} = p^{k-1}q(Q_{j-k,0})$.	(9.(k − 1))

Each equation in (9) has one term less than the equation immediately above it. We adopt a standard procedure for deriving a recursion equation for Q_j from equations (9). First we find from (9.0) that:

$$Q_{j,0} = qQ_{j-1}. \tag{10}$$

Then we substitute (10), with j reduced by k, into (9.(k − 1)):

$$Q_{j,k-1} = p^{k-1}q^2Q_{j-k-1}. \tag{11}$$

Next we translate the final term on the right-hand side of (9.0) to the left-hand side, and substitute (10) and (11), the latter with j reduced by one. Then, if we multiply through the new (9.0) by p and reduce j by one, its right-hand side is identical to that of (9.1). Then we have

$$Q_{j,1} = pqQ_{j-2} - p^k q^3 Q_{j-k-2} . \tag{12}$$

We substitute (10) and (12), with j reduced by $(k - 1)$, into (9.$(k - 2)$) to obtain

$$Q_{j,k-2} = p^{k-2}q^2 Q_{j-k} + p^{k-1}q^2 Q_{j-k-1} - p^{2k-2}q^4 Q_{j-2k-2} . \tag{13}$$

We proceed step by step, taking equations from the top and then from the bottom, to find equations for the $Q_{j,i}$'s in terms of Q_j's. Then we add all of the equations together to obtain the recursion equation for Q_j, which will depend on some of the $k(k + 1)/2$ immediately preceding Q_j's. The recursion equation is used with $k(k + 1)/2$ initial Q_j's to derive the generating function $Q(s)$.

4.31 *The First Occurrence of Two Successes in Three Consecutive Trials*

As in (9), we have:

Sequence Endings	Probabilities of Occurrence	
$\cdot q$	$Q_{j,0} = q(Q_{j-1,0} + Q_{j-1,1})$	(14.0)
qp	$Q_{j,1} = pq(Q_{j-2,0})$.	(14.1)

Then $Q_{j,0} = qQ_{j-1}$, $Q_{j,1} = pq^2 Q_{j-3}$, and the recursion equation is

$$Q_j = qQ_{j-1} + pq^2 Q_{j-3}, \qquad j > 2. \tag{15}$$

With (15) and the initial conditions $Q_0 = Q_1 = 1$ and $Q_2 = 1 - p^2$, we derive the generating function for Q_j:

$$Q(s) = \frac{1 + ps + pqs^2}{1 - qs - pq^2 s^3}. \tag{16}$$

$E(y)$ is obtained by setting $s = 1$ in (16);

$$E(y) = \frac{1 + p + pq}{p^2(1 + q)} . \tag{17}$$

4.32 *The First Occurrence of Three Successes in Four Consecutive Trials*

The initial conditions are:

$$Q_0 = Q_1 = Q_2 = 1,$$
$$Q_3 = 1 - p^3,$$
$$Q_4 = 1 - p^3 - 3p^3 q,$$
$$Q_5 = 1 - p^3 - 3p^3 q - 3p^3 q^2.$$

For $j > 5$ we follow the standard procedure to find a recursion equation for Q_j in terms of the $3 \cdot 4/2 = 6$ preceding Q_j's:

$$Q_j = qQ_{j-1} + pqQ_{j-2} + p^2q^2Q_{j-4} - p^3q^3Q_{j-6}, \qquad j > 5. \quad (18)$$

The generating function is

$$Q(s) = \frac{1 + ps + p^2s^2 + p^2qs^3 - p^3qs^4 - p^3q^2s^5}{1 - qs - pqs^2 - p^2q^2s^4 + p^3q^3s^3}, \quad (19)$$

and the expected value of y is

$$E(y) = \frac{1 + p + p^2 + p^2q^3}{p^3(1 + q + q^3)}. \quad (20)$$

4.33 *The First Occurrence of Four Successes in Five Consecutive Trials*

Here the $4 \cdot 5/2 = 10$ initial Q_j's are:

$$\begin{aligned}
Q_0 &= Q_1 = Q_2 = Q_3 = 1, \qquad Q_4 = 1 - p^4, \\
Q_5 &= Q_4 - 4p^4q, \\
Q_6 &= Q_5 - 4p^4q^2, \\
Q_7 &= Q_6 - 4p^4q^3 - 3p^5q^2, \\
Q_8 &= Q_7 - 4p^4q^4 - 7p^5q^3 - 2p^6q^2, \\
Q_9 &= Q_8 - 4p^4q^5 - 11p^5q^4 - 9p^6q^3 - p^7q^2.
\end{aligned}$$

For $j > 9$ the following recursion equation holds:

$$\begin{aligned}
Q_j = {} & qQ_{j-1} + pqQ_{j-2} + p^2q^2Q_{j-4} + 2p^3q^2Q_{j-5} \\
& - p^4q^3Q_{j-7} - p^6q^4Q_{j-10}.
\end{aligned} \quad (21)$$

The generating function of Q_j is

$$Q(s) = \frac{\begin{aligned}1 + ps + p^2s^2 + p^2s^3 + 2p^3qs^4 - p^4qs^5 \\ - p^4q^2s^6 - p^6q^2s^8 - p^6q^3s^9\end{aligned}}{\begin{aligned}1 - qs - pqs^2 - p^2q^2s^4 - 2p^3q^2s^5 \\ + p^4q^3s^7 + p^6q^4s^{10}\end{aligned}}. \quad (22)$$

Then

$$E(y) = \frac{1 + p + 2p^2 + 2p^3q - p^4q - p^4q^2 - p^6q^2 - p^6q^3}{p^3(1 + q - 2q^2 + pq^3 + p^3q^4)}. \quad (23)$$

4.4 *Properties of Two-Sided Zone Tests*

The results presented in Sections 4.2 and 4.3 are applicable to the study of the statistical properties of one-sided zone tests for runs of

points in the zone above an upper limit line at $\bar{X}_0' + L_k(\sigma'/\sqrt{n})$. Generally, we also test for the same types of runs below a lower limit line at $\bar{X}_0' - L_k(\sigma'/\sqrt{n})$, in which case the test is a two-sided zone test and each point falls into one of three mutually exclusive zones.

Let A denote the zone above the upper limit, B denote the zone between the two limits, and C denote the zone below the lower limit. Consider an infinite sequence of independent trials having possible outcomes a, b, and c with fixed probabilities a, b, and c. When the outcome of the jth trial completes a pattern of outcomes describing an event ϵ we say that ϵ occurs on the jth trial. Event ϵ is defined by a set of outcome patterns and a counting, or testing, rule. If when ϵ occurs on the jth trial we treat trial $j + 1$ as though it were the first trial, ignoring the results of the first j trials, then ϵ is a recurrent event.[8]

Let

u_j = Probability that ϵ occurs on the jth trial,
f_j = " " " " for the first time on the jth trial,
Q_j = Probability that ϵ does not occur in the first j trials.

Denote the generating functions of u_j, f_j, and Q_j by $U(s)$, $F(s)$, and $Q(s)$, respectively.

The following equation can be used to determine the Q's in terms of the f's:

$$Q(s) = \frac{1 - F(s)}{1 - s} \qquad -1 < s < 1. \quad (24)$$

If ϵ is a *recurrent* event the following equation holds [Reference 8, p. 243]:

$$u_j = f_j + f_{j-1}u_1 + f_{j-2}u_2 + \cdots + f_1 u_{j-1}. \quad (25)$$

Equation (25) leads to the following identity (setting $f_0 = 0$, $u_0 = 1$):

$$U(s) = \frac{1}{1 - F(s)}. \quad (26)$$

From (24) and (26) we have

$$(1 - s)\,U(s) = \frac{1}{Q(s)}, \qquad -1 < s < 1, \quad (27)$$

for recurrent event ϵ. We shall consider only recurrent events which have finite recurrence times; in these cases $F(1) = f_1 + f_2 + \cdots = 1$, and $U(1)$ is infinite. The limit of $(1 - s)U(s)$ as s approaches unity from below is (using L'Hospital's Rule):

$$\lim_{s \to 1} (1 - s)\,U(s) = \frac{1}{F'(1)} = \frac{1}{Q(1)} = \frac{1}{E(y)}, \quad (28)$$

where y denotes the number of the trial of the first occurrence of ϵ, and $E(y)$ denotes its expected value. $E(y)$ is also the average recurrence time (average number of trials between consecutive occurrences) of recurrent event ϵ.

Consider recurrent events ϵ_1, ϵ_2, and ϵ_{12}, defined, respectively, by the sets of outcome patterns α, β, and α or β and a counting rule that requires counting to start from scratch on trial $j (j > 1)$ if and only if the event under consideration occurs on trial $j - 1$. Assume that ϵ_1 and ϵ_2 are mutually exclusive — that is, they cannot both occur on the same trial.

For an example, let the single pattern $a\ c\ a$ define the set α and the pattern $c\ a\ c$ define the set β — then the set α or β has the two patterns $a\ c\ a$ and $c\ a\ c$. Consider an outcome sequence:

trial number:	1	2	3	4	5	6	7	8	9
trial outcome:	*a*	*c*	*a*	*c*	*a*	*c*	*a*	*b*	*a*

The event ϵ_1 occurs on trials 3 and 7; the event ϵ_2 occurs on trial 4; and the event ϵ_{12} occurs on trials 3, 6, and 9.

Let $E_1(y)$, $E_2(y)$, and $E_{12}(y)$ denote the average recurrence times of ϵ_1, ϵ_2, and ϵ_{12}, respectively. Under what conditions can we determine $E_{12}(y)$ from known values of $E_1(y)$ and $E_2(y)$?

Consider events ϵ_1'' and ϵ_2'' defined by outcome patterns α and β, respectively, and a counting rule that requires counting to start from scratch on trial j if and only if *either* ϵ_1'' *or* ϵ_2'' occurs on trial $j - 1$. Events ϵ_1'' and ϵ_2'' differ from ϵ_1 and ϵ_2 only in counting rules. In the example previously considered, we see that ϵ_1'' occurred on trials 3 and 9, and ϵ_2'' occurred on trial 6. Either ϵ_1'' or ϵ_2'' (but not both) occurs on every trial on which ϵ_{12} occurs; this leads to the equation

$$u_{12,j} = u_{1,j}'' + u_{2,j}'', \tag{29}$$

where $u_{12,j}$, $u_{1,j}''$, and $u_{2,j}''$ denote, respectively, the probabilities that ϵ_{12}, ϵ_1'', and ϵ_2'' occur on trial j.

Multiplying (29) through by s^j and summing over j from one to infinity, we obtain an equation relating the generating functions of the probabilities in (29):

$$U_{12}(s) = U_1''(s) + U_2''(s) - 1. \tag{30}$$

The constant appears because $u_0 = 1$ in all cases.

Events ϵ_1'' and ϵ_2'' are recurrent events, and equations (25) through (28) can be used to determine their mean recurrence times $E_1''(y)$ and $E_2''(y)$. (The fact that (25) applies allows us to call ϵ_1'' and ϵ_2'' recurrent events).

If we multiply through (30) by $(1 - s)$ and take the limit of each side as s approaches unity (see (28)), we obtain

$$\frac{1}{E_{12}(y)} = \frac{1}{E_1''(y)} + \frac{1}{E_2''(y)}. \tag{31}$$

In any sequence of trial outcomes, if a pattern in α occurs for the first time on trial j, then ϵ_1 occurs for the first time on trial j, and ϵ_1'' occurs for the first time either on trial j or on a *later* trial; ϵ_1'' will occur for the first time on a later trial if ϵ_2'' occurred while this first pattern in α was being formed. Thus we have

$$E_1(y) \leqq E_1''(y) \tag{32}$$

where the equality sign holds if and only if no pattern in β overlaps a pattern in α. A pattern in β overlaps a pattern in α if the terminating outcomes of the former correspond to the beginning outcomes of the latter. Thus outcome pattern $c\,a\,c$ overlaps $a\,c\,a$ because the terminating outcomes $a\,c$ of the former correspond to the beginning outcomes $a\,c$ of the latter. If no pattern in β overlaps a pattern in α then the occurrence ϵ_2'' does not "cancel out" the beginning of any patterns in α, and therefore ϵ_1 and ϵ_1'' always occur on the same trials.

From (31) and (32) we have

$$\frac{1}{E_{12}(y)} \leqq \frac{1}{E_1(y)} + \frac{1}{E_2(y)}, \tag{33}$$

where the equality sign holds if and only if ϵ_1 and ϵ_2 are defined by non-overlapping patterns, in which case we shall say that ϵ_1 and ϵ_2 are non-overlapping events. From our example it is clear that mutually exclusive events are not necessarily non-overlapping.

We can use (33) to find $E(y)$ for two-sided tests in terms of the $E^*(y)$'s of the component one-sided tests. Note that a given change Δ looks like a $-\Delta$ to one of the component tests. Then

$$\frac{1}{E(y; \Delta)} \leqq \frac{1}{E^*(y; \Delta)} + \frac{1}{E^*(y; -\Delta)}. \tag{34}$$

For $\Delta = 0$, $E(y; 0) \leqq (E^*(y; 0)/2)$. The equality sign holds in (34) for $T_k(L_k)$ and $T_{1k}(L_1, L_k)$. For $T_{k'}(L_k)$ and $T_{1k'}(L_1, L_k)$, (34) defines lower bounds which are very close approximations to $E(y)$. For $T_{2'}$ equation (34) leads to a lower bound of 510.6, which compares with the true value $E(y; 0) = 510.7$. The degree to which the approximation approaches the true value depends on the probability of overlap, which in cases we consider is very small; for this reason we can consider (34) to be an approximation rather than a lower bound.

4.5 *Properties of Tests Combining $T_1(L_1)$ with One Other Test Whose Properties Are Known*

Consider any test T for which we partition the control chart into mutually exclusive zones $A, B, C, \cdots, R$. The possible outcomes of the first j trials can be enumerated by an ordered expansion of

$$(a + b + c + \cdots + r)^j.$$

With the letters denoting the probabilities of points falling into the various zones $(a + b + c + \cdots + r = 1)$, we pick out all of those terms corresponding to outcomes in which the event ϵ does not occur, and denote their sum by

$$Q_{t,j} = g_j(a, b, c, \cdots r). \tag{35}$$

Clearly $Q_{t,j}$, or g_j, is the sum of a series of terms such as $a^2bc^3 \cdots$, representing the probabilities of particular outcomes.

If we wish to find the probability $Q_{1t,j}$ of no occurrences of event ϵ *and* no occurrence of a point falling in zones A or B, say, we simply eliminate from g_j those terms in which either a or b occurs. We can do this by substituting zeros for a and b wherever they occur in g_j : $Q_{1t,j} = g_j(0, 0, c, d, \cdots, r)$. By multiplying and dividing each remaining term in $g_j(0, 0, c, d, \cdots, r)$ by $(1 - a - b)^j$, we derive an alternative expression:

$$Q_{1t,j} = g_j\left(0, 0, \frac{c}{1 - a - b}, \frac{d}{1 - a - b}, \cdots, \frac{r}{1 - a - b}\right)(1 - a - b)^j, \tag{36}$$

showing that the conditional probability of no ϵ given no points in A or B uses the same function required for $Q_{t,j}$. This enables us to write the generating function of $Q_{1t,j}$ as

$$Q_{1t}(s) = h\left(0, 0, \frac{c}{1 - a - b}, \frac{d}{1 - a - b}, \cdots, \frac{r}{1 - a - b}; (1 - a - b)s\right), \tag{37}$$

where $h(a, b, c, \cdots, r; s)$, defined for $a + b + c + \cdots + r = 1$, is the generating function of $Q_{t,j}$.

The principles are best illustrated by an example. Consider the problem of finding $E_{1k}^*(y)$, the expected number of the trial of the first occurrence of k consecutive points above $\bar{X}_0' + L_k(\sigma'/\sqrt{n})$ *or* a single point

above $\bar{X}_0' + L_1(\sigma'/\sqrt{n})$, where $L_k < L_1$. We use an asterisk to denote the fact that a function applies to a one-sided test. We let a be the probability of a point falling above both limits, b be the probability of falling between the two limits, and c be the probability of falling below both limits. Then we substitute $a + b$ for p, and c for q in (7) to obtain

$$Q_k^*(s) = \frac{1 - (a + b)^k s^k}{1 - s + c(a + b)^k s^{k+1}}. \tag{39}$$

Following (37), we find $Q_{1k}^*(s)$ by substituting 0 for a, $b/(1 - a)$ for b, $c/(1 - a)$ for c, and $(1 - a)s = (b + c)s$ for s in (39),

$$Q_{1k}^*(s) = \frac{1 - b^k s^k}{1 - (b + c)s + cb^k s^{k+1}}. \tag{40}$$

We set $s = 1$ in (40) to obtain

$$E_{1k}^*(s) = \frac{1 - b^k}{1 - (b + c) + cb^k}. \tag{41}$$

The properties of any test combining $T_1(L_1)$ with one other test whose properties are known can be determined in a similar way.

4.6 *Limits of $E(y)$ in Compound Tests*

A development similar to that in Section 4.4 will show, for example, that

$$\frac{1}{E_{123}(y)} \leqq \frac{1}{E_1(y)} + \frac{1}{E_2(y)} + \frac{1}{E_3(y)}, \tag{42}$$

where $E_{123}(y)$ pertains to recurrent event ϵ_{123}, whose set of outcome patterns is composed of those of recurrent events ϵ_1, ϵ_2, and ϵ_3.

It can also be shown that

$$E_{123}(y) \leqq E_{12}(y) \leqq E_1(y) \tag{43}$$

for example. Clearly we cannot increase the recurrence time of an event by increasing the different outcome patterns which define the event.

4.7 *Translating Limits to Obtain a Selected Probability of Type 1 Error*

By supplementing T_1 with other tests, we increase the probability of Type 1 errors. We can adjust the probability of Type 1 errors to any desired level by resetting the zone limits. With more than one set of limit lines, we have some freedom in setting the limits. A procedure that has the important attribute of simplicity translates all of the limit lines away from the central line $\bar{X}_0'$ by the same amount. The properties of the resultant test can be derived directly from the properties of the original test.

We first determine the properties of one-sided tests whose limits are translated; from these results we determine the properties of the corresponding two-sided test.

If $Q_t^*(L_{k_1}, L_{k_2}; \Delta; s)$ is the generating function of Q_j for test T^* with limits at $\bar{X}_0' + L_{k_1}(\sigma'/\sqrt{n})$ and $\bar{X}_0' + L_{k_2}(\sigma'/\sqrt{n})$ and with the process average $\bar{X}' = \bar{X}_0' + \Delta$, then (neglecting points from State 1):

$$Q_t^*(L_{k_1}, L_{k_2}; \Delta; s) \equiv Q_t^*\left(L_{k_1} + h, L_{k_2} + h; \Delta - h\frac{\sigma'}{\sqrt{n}}; s\right). \quad (44)$$

This equation says that the probabilities involved are identical if we translate the limits by a given amount or if we translate the process average in the opposite direction by the same amount. The truth of this stems from the fact that the probabilities depend on the position of the process average $\bar{X}'$ *relative* to the zone limits.

If we wish to set the limits so that the probability of a Type 1 error is $\frac{1}{500}$, say, for a two-sided test, we can proceed as follows:

(1) draw the curve of $E(y)$ versus Δ for the corresponding one-sided test (the abscissa is assumed to be in units of $\sigma'/\sqrt{n}$),

(2) translate this curve to the right (or left) until $E(y; 0) = 1000$,

(3) measure the amount h of the translation, and translate the zone limits away from (or toward) $\bar{X}_0'$ by an amount $h(\sigma'/\sqrt{n})$ (control chart units).

The translated $E(y)$ versus Δ curve represents the new one-sided test. The curve for the corresponding two-sided test can be derived using (34); it will have a value $E(y; 0) = 500$.

4.8 *Monte Carlo Techniques to Determine the Properties of Zone Tests*

We can determine approximately the properties of zone tests by using Monte Carlo techniques on modern high-speed computers. First we generate a random series of numbers with a known distribution. Then, using the appropriate correspondence between limits within the distribution and zone limits, we translate the random numbers into a random sequence of zone designations, which we test for occurrences of the events in question. We keep score of the number of points until the event finally occurs. We then start counting again as though the sequence were just starting. By running through a great many cycles, we obtain an approximation to the distribution of the cycle length y, and an approximation to $E(y)$ for the particular value of Δ that applies to the limits we used. We repeat the process with different limits for different values of Δ.

Within the limitations of the computer, this technique can be used for any zone test. We used it to approximate the properties of $T_{12'4'8}$ for $\Delta = 0$, $\sigma'/\sqrt{n}$, $2(\sigma'/\sqrt{n})$, and $3(\sigma'/\sqrt{n})$.

V. CONCLUSIONS

If we supplement the standard control chart test with another zone test, we increase its sensitivity to process changes at the cost of more

frequent errors of Type 1 and a more complicated testing procedure; see Figs. 5 and 6. We can restore the original probability of Type 1 errors by changing the zone limits; in the following discussion we shall assume that this has been done, thereby simplifying the comparison of various tests. We shall say that a zone test T_t of the type we are concerned with is better than T_1 for a particular value of Δ if $E_t(y) < E_1(y)$ for that Δ.

In general, the curve of $E(y)$ versus Δ for a test T_t is below the corresponding curve for T_1 for Δ in a range $0 < \Delta < \Delta_t$, and above for $\Delta > \Delta_t$. The crossover point Δ_t in the cases we considered varied from 1.7 $(\sigma'/\sqrt{n})$ for $T_8(0.065)$ (Fig. 7) to over 3.5 $(\sigma'/\sqrt{n})$ for $T_{12'}(3.13, 2.13)$ (Fig. 9).

Consider a test T_{1t} that combines T_1 and T_t and has its zone limits set so that its probability of Type 1 errors is the same as for T_1 and for T_t. In the cases we have considered (see Figs. 8 and 9) T_{1t} essentially effects a compromise between T_1 and T_t — for small changes it is better than T_1 but not as good as T_t; for large changes it is better than T_t but not as good as T_1; for Δ near Δ_t it is better than T_1 and better than T_t.

In the cases we have considered, tests $T_{k'}(L_{k'})$ appear to be slightly better than tests $T_k(L_k)$ for small changes.

The reason that zone test T_t is better than T_1 for small changes seems to be due to the fact that it bases its decisions on a history of k consecutive points; in effect, it makes some use of a sample size kn rather than n. The cost of the increased effective sample size is paid during the first $k - 1$ points in State 2, where T_1 has a higher probability than T_t of detecting a change. The probability that a point falls outside of a 3σ limit remains fixed from sample to sample, and after the initial $k - 1$ points in State 2, this probability is less than the probability that T_t will detect a run. Large changes are likely to be detected by T_1 before T_t becomes effective; but when changes are small the corresponding values of $E(y)$ are large, and we can expect T_t to detect the change before T_1 does (see Fig. 7).

We have assumed that sample averages $\bar{X}_n$ are plotted on the control chart. In light of the above discussion the possibility of pooling data from k consecutive samples and plotting a statistic based on the kn measurements involved appears promising.

A preliminary study of zone tests on charts of moving averages of k consecutive equal-sized samples has been made. The statistic (or point)

$$Y_{kn,j} = (\bar{X}_{n,j} + \bar{X}_{n,j-1} + \cdots + \bar{X}_{n,j-k+1})/k$$

can easily be determined graphically in many cases. For example, the point $Y_{2n,j}$ is halfway between points $\bar{X}_{n,j-1}$ and $\bar{X}_{n,j}$ on the straight

line connecting them — vertical rulings on cross-section paper ordinarily used will spot points exactly. Points $Y_{4n,j}$ can be similarly derived from points $Y_{2n,j-2}$ and $Y_{2n,j}$. Fig. 12 shows curves of $E(y)$ versus Δ for T_1 used on points $Y_{kn,j}(k = 1, 2, 4)$; limit lines were assumed to be at $\bar{X}_0' \pm 3(\sigma'/\sqrt{kn})$. The curve for $k = 1$ is, of course, the curve T_1 of earlier figures; the curve for $k = 2$ was derived using tables of the bivariate normal distribution; the curve for $k = 4$ is an approximation based primarily on the results of a study making use of Monte Carlo

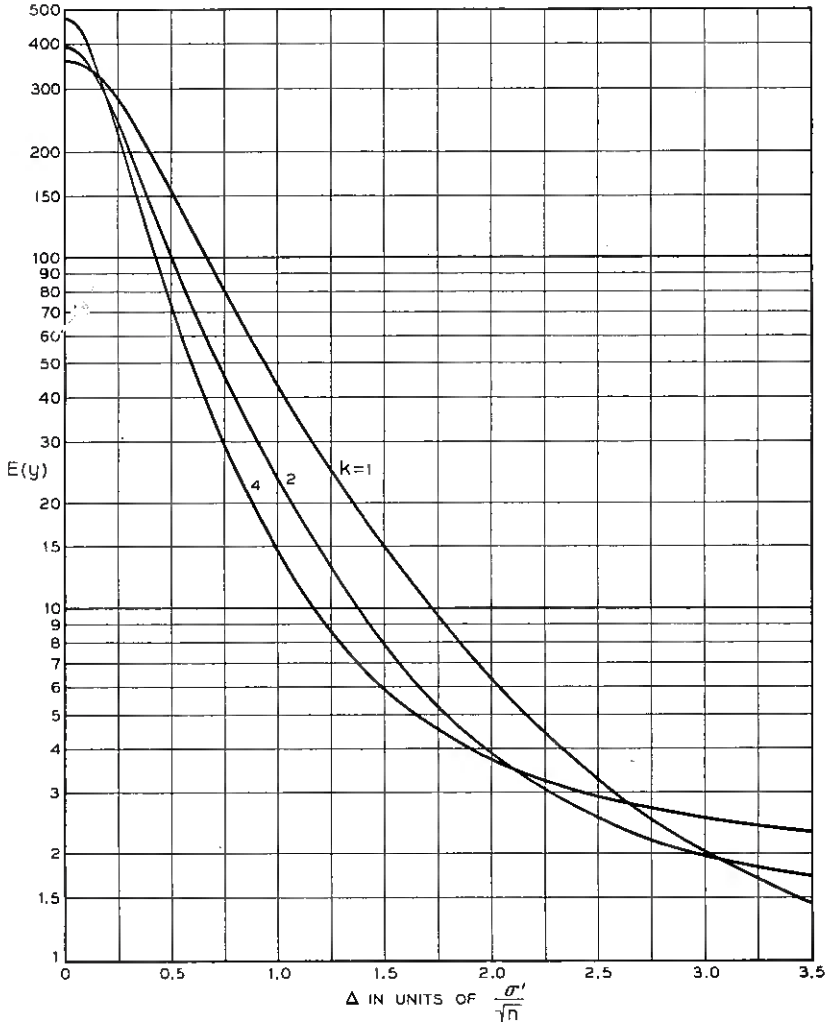

FIG. 12 — $E(y)$ versus Δ for T_1 applied to moving averages of $k(k = 1, 2, 4)$ consecutive sample averages.

techniques. We cover the possibility that the first point in State 2 will fall outside of its control limits by assuming the existence of prior points in State 1; all three curves approach $E(y) = 1$ as Δ approaches infinity.

In comparing Fig. 12 with Figs. 7, 8, and 9, it appears that T_1 used on moving averages provides an effective test for detecting shifts in process averages. Further study is required to determine the effectiveness of other run tests and of combinations of run tests applied to various moving averages.

In summary, it is possible to devise zone tests which — within the constraints of our model:

(1) indicate changes in process averages when none has occurred with the same average frequency as the standard control chart test T_1,

(2) detect small changes in process average — up to $1.3\sigma'$, say, for $n = 5$ — sooner on the average than T_1, and

(3) detect larger changes inappreciably later on the average than T_1.

Such tests require an appropriate setting of zone limits — generally at non-integral multiples of $\sigma'/\sqrt{n}$. If run tests are used to supplement T_1 without a compensating setting of zone limits, an increased frequency of false indications of process changes results.

The standard control chart test T_1 (or $T_1(L_1)$) is slightly more effective than alternative zone tests in detecting relatively large changes; in addition, it has the important virtue of simplicity — a virtue that extends the range of economic application of T_1 into areas where alternative tests have better statistical properties. It is difficult to recommend a single alternative test to T_1 for general application, though it is clear that alternative tests may be profitably used in many applications where early detection of relatively small changes is important.

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A Criterion to Limit Inspection Effort in Continuous Sampling Plans

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In continuous sampling plans of the type known as CSP-1, the amount of screening has an important bearing on the total inspection effort. To limit this effort an inspector may be required to take special action if the number of inspected units in one screening sequence exceeds some specified value or "critical length". The aim of the special action is to bring about improvement in the production process. This effect is possible also when the producing shop is required to do any screening called for by the inspection plan.

A procedure for calculating critical lengths may be based on simple approximations derived from the theory of runs.

1. INTRODUCTION

1.1 *Continuous Sampling Plans*

The CSP-1 continuous sampling plans introduced by H. F. Dodge[1] are designed for continuous or "belt" production of discrete units of product. To apply such a plan, inspected units must be classified as either "defective" or "nondefective". The inspector begins by inspecting each unit made in succession until a specified number, i, of consecutive units are found nondefective. A sequence of units so inspected is called a screening sequence and the number i the clearing number. After the initial screening sequence has ended, the inspector samples a fraction f of the units presented to him. He continues to sample until he finds a defective unit. At this point he again resorts to screening, following the same procedure as before, so that he alternates between screening and sampling inspection. The inspector rejects (or sets aside for correction) any inspected unit found to be defective and accepts all others.

Two refinements of this plan, CSP-2 and CSP-3, have appeared[2] as well as generalizations of CSP-1[3, 4, 5] entailing two or more levels of sampling inspection. In addition, various sequential continuous inspection plans have been proposed.[6]

The characteristics of these different sampling plans — such as AOQL, fraction inspected, or characteristic curves — have been explored under a variety of assumptions. Of these assumptions, the statistical behavior of the production process has the greatest effect on the results. There are three alternatives which have been used (but which may not cover all plausible situations):

(I) The production process is Bernoullian: each unit has the same probability of being defective independent of any other unit; the proportion of defective units converges almost certainly to this value as the number of units produced increases. It is therefore known also as the process average.

(II) The production process represents a stationary Markov chain; each unit has a probability of being defective which depends only on the defectiveness or non-defectiveness of the previous $k(\geqq 1)$ units produced and is otherwise independent of time.

(III) The production process represents a discrete stochastic process of an arbitrary nature.

Not all the continuous sampling plans introduced have been examined under each assumption.

Assumption (I) leads to the simplest mathematics and will be adopted here. Its use does not imply that the CSP-1 plans — with or without the criterion proposed below — are invalid if the production process goes out of control. These plans were designed with this condition in mind. The effect of lack of control is to alter the stated characteristics of such plans, but the author has no evidence from actual production processes that such deviations are wide.

Another factor that influences the characteristics of continuous sampling plans is the kind of sampling used when sampling is required. Again there are three alternatives commonly used:

(i) The sampling is Bernoullian; each unit bears a probability f of being sampled independent of any other unit; in this case and in (iii) below screening is usually required to begin with the next unit after a known defective.

(ii) One unit in each (disjoint) set of $1/f$ consecutive units produced is randomly chosen from the set for inspection. Screening, when required, may begin within the same set in which a defective unit is found, or it may begin with the first unit of the next set. One or the other method of starting to screen is usually specified.

(iii) Every $1/f$th unit is inspected.

In most characteristics of CSP-1 it makes little difference whether (i), (ii), or (iii) is used provided (I) is assumed. Again the mathematics is simpler with (i), and accordingly we shall follow it.

A third assumption is sometimes made about the operation of CSP-1: each defective unit inspected is replaced by a nondefective one. This assumption affects only the character of outgoing quality. It will have no bearing on the criterion for inspection effort discussed below.

1.2 *Inspector's Risk*

With this brief background we may take up the main subject of this paper. In using any inspection plan there are three areas of risk: one area pertains to the consumer's operations, another to the producer's operations, and the third to the inspector's operations. One risk in the third area is that the inspector may be called upon to perform an excessive amount of inspection for the amount of protection he furnishes. The CSP-1 plans, although admitting the necessity of high inspection rates on occasion, are not really intended to be used when inspection will continue indefinitely at a high rate. In general such a high rate would not lead to economical and effective inspection nor to economical manufacture: screening alone does not guarantee that the level of incoming quality will improve enough to diminish the amount of screening significantly in the future. Neither is there so much confidence in the outgoing quality, which poor incoming quality may affect adversely in spite of intensified screening. Indeed, the existence of such a situation may imply some basic difficulty in the process of design or manufacture that cannot be properly handled by inspection methods alone. Not only the inspector but the customer may be undergoing a special risk. Furthermore, the producer often does any screening required (as Dodge originally recommended[1]). He too might find an appropriate special action economically preferable to a great deal of screening.

Thus the inspector needs a special alarm signal to indicate that unless he takes special action a high rate of screening may continue. The following sections show how such a special alarm signal for CSP-1 plans may be devised on the basis of the number of units inspected in any one screening sequence. If this number exceeds a "critical length", n^*, chosen in advance, the inspector is to take an appointed special action.

A similar type of criterion could be evolved for other types of continuous sampling plans. The effectiveness of this type of criterion alone might be lessened if it were applied to other types of plans in which screening is not so promptly reinstated after a defective is found as in the CSP-1 plans. It seems certain that the "most sensitive criterion" for any of these plans, including CSP-1, would take account somehow of the observed per cent defective. On the other hand simplicity and convenience would have to be sacrificed to some extent to do so. For the

CSP-1 plans it is hoped that the proposed criterion of critical length is a satisfactory compromise between theoretical and practical requirements.

The special action to be taken when required by this criterion should depend upon the situation. It might be to notify the customer's purchasing or contracting department; it might go so far as to cause the inspector to stop inspection, effectively halting purchase of product. If such a severe action is specified, the manufacturing unit may rightly feel entitled to be informed in advance whenever such action appears imminent so that it may begin to adjust the process and to screen product ahead of the inspector. Using a different criterion from the one proposed here, an existing government inspection plan[7] does, in fact, require the inspector to stop inspecting. It is not our purpose, however, to discuss in detail any particular special action since its wisdom could be confirmed only by reference to the nature of the application. It is intended only to point out that such actions have already been devised and used.

There is no reason to adjust published AOQL figures for CSP-1 plans because of the addition of this special action criterion to their operation. If the special actions are suitable, there is no reason to expect anything but an improvement in the outgoing quality level.

II. THE CRITICAL LENGTH OF A SCREENING SEQUENCE

2.1 *The Basis for Choosing Critical Lengths*

It is generally possible for an inspection agency to state what it considers a reasonable upper limit to the amount of inspection it should be required to perform under a given CSP-1 plan. Let us call this limit F^*. Under our assumptions Dodge[1] has shown that, when the probability of a defective unit is p, the average amount of inspection (i.e., limiting fraction of units inspected) is

$$F = \frac{f}{f + (1 - f)q^i}, \qquad (q = 1 - p), \tag{1}$$

if an inspector uses a CSP-1 plan with clearing number i and sampling frequency f. It is clear that placing an upper limit F^* on F is equivalent to placing an upper limit p^* on p. Indeed, if

$$(1 - p^*)^i = \frac{f}{1 - f} \cdot \frac{1 - F^*}{F^*} = K, \tag{1'}$$

the inequality $F \leqq F^*$ implies and is implied by $p \leqq p^*$ according to (1).†

Having specified F^* as the upper limit to the amount of inspection, we need a measure of the price the inspection agency should be willing to pay to enforce it. For our purposes it will be convenient to choose as a measure the maximum probability α^* of taking special action when $F \leqq F^*$. It is equivalent to say that α^* shall be the fraction of all screening sequences in which the inspector takes special action when $F = F^*$ or when $p = p^*$. In practice the choice of F^* or α^* or both may be somewhat arbitrary. In the author's experience, the choice of $F^* = 0.5$ and $\alpha^* = 0.10$ has proved reasonable.

We may now choose a critical length n^* so that special action is taken in accordance with the risk specified above. First, the inspector is to take special action whenever a screening sequence has not terminated after the n^*th consecutive unit in the sequence has been inspected. Second, n^* is to be chosen so that when $F = F^*$ the fraction α^* of all screening sequences have not terminated after n^* units.

This second condition cannot in general be fulfilled exactly. Instead, if we call the probability that a screening sequence has not terminated after n units $T_n(p, i)$, we shall find n^* satisfying

$$T_{n^*}(p^*, i) \leqq \alpha^* < T_{n^*-1}(p^*, i). \tag{2}$$

It can be easily demonstrated that for any α^* and p^* satisfying $0 < \alpha^* < 1$ and $0 < p^* < 1$, there is a solution, n^*, to (2).

It is sensible to desire that the higher the "true" limiting fraction F of units inspected, the more likely it is that a screening sequence will exceed its critical length. The truth of this statement can be easily shown also. This guarantees that α^* is a *maximum* probability of taking the special action incorrectly.

The mathematical problem, as we have stated it, is covered by the theory of runs. Its solution has long been known[8] and will be discussed in the following section, as well as in the Appendixes. Briefly, in terms

† Certain variations of CSP-1 lead to different expressions from those given here. For instance, if the producer does *all* screening, the inspector will often inspect a fixed proportion, f, of all units — including those already screened. It is then more sensible to apply F^* as an upper limit only to the average amount of *screening*, which is the product of $1 - q^i$ and the right side of (1). Solving for p^* or K then requires that f be added to the denominator $(1 - f)F^*$ in (1′). Another variation arises when defective units inspected are repaired and reinspected. If it were assumed that the proportion defective among repaired units is again p, it would be necessary only to divide the right side of (1) by $1 - p$ and to replace F^* by $F^*(1 - p^*)$ in (1′). Then p^* or K would be found by iteration. The effects of these two variations may be combined but not without further assumptions.

of this theory, we may restate our problem as follows: Given a Bernoullian process with "success" probability $q^* = 1 - p^*$, to find the least number of trials, n^*, in which the probability of having had no runs of i (or more) "successes" is less than or equal to α^*.

Common sense demands that the special action never be taken until there has been some chance to complete the screening sequence. That is, the critical length of a screening sequence must be larger than the clearing number. It is shown in Appendix A that it is equivalent to require

$$\alpha^* < \frac{F^* - f}{(1 - f)F^*}. \tag{3}$$

This restriction is usually minor. For instance, if, as above, $F^* = 0.5$ and $\alpha^* = 0.10$, then according to (3) $f < 9/19 = 0.474^-$. For $\alpha^* \leqq 0.5$ it is more convenient generally to use the inequality

$$F^* \geqq f + 0.70\alpha^*. \tag{3'}$$

As is shown in Appendix A, (3) is satisfied whenever (3') is.

In any case the value computed for n^* will depend upon the assumptions discussed in Section I. If these are inexact, the probability statements outlined above will generally be inexact also. Nevertheless, the same value of n^* may still be used with good prospect of limiting screening effort without added penalty to the manufacturer.

2.2 *Computation of n^**

As noted above, the exact relation between n^*, i, p^* and α^* is known, but it is difficult to use. To simplify computation it has been found advisable to resort to an approximation for n^*, which assumes the form

$$n^* \doteq a_1 i + a_0 , \tag{4}$$

where a_0 and a_1 depend only on K, defined in (1'), and α^*. A derivation of this approximation is given in Appendix B. It is based on asymptotic results for large n^* and i.

It is interesting to note that K, when defined in terms of p^* and i, is the probability of terminating a screening sequence in exactly i trials. For the purposes of this paper we shall determine K in terms of f and F^*.

For convenience the coefficients a_0 and a_1 are presented in graphical form in Fig. 1 with values of K on the abscissa and with separate curves for $\alpha^* = 0.01, 0.05, 0.10$. The requirement (3) is observed by plotting these curves only over the interval of values of K satisfying $\alpha^* < 1 - K < 1$. While the immediate field of interest is inspection, the

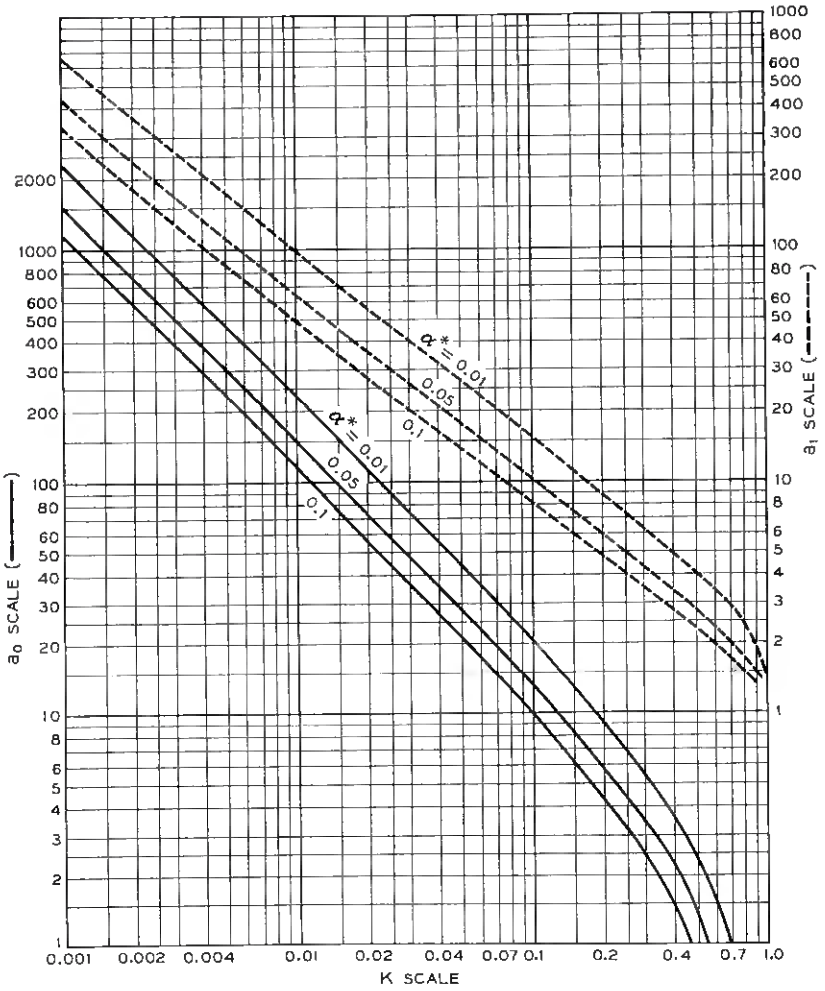

Fig. 1 — The coefficients a_0 and a_1 as functions of $K = f(1 - F^*)/F^*(1 - f)$ for $\alpha^* = 0.01, 0.05, 0.10$. The critical length n^* is approximated by $a_1 i + a_0$.

values of a_0 and a_1 read from this chart obtain equally well in other uses of the theory of runs. Therefore, the range of K in Fig. 1 is considerably larger than would be necessary to handle this particular problem alone. Given the values f, i, F^* and α^*, the value K may be computed from (1′). If $\alpha^* = 0.01, 0.05$, or 0.10, we may choose the proper curve for a_0, read off its ordinate at the computed value of K, follow a similar procedure to find a_1, and compute n^* from (4).

In order to compute the coefficients a_0 and a_1 for any values of α^* and F^* satisfying (3), it is necessary to compute

$$w = -\ln K = \ln \frac{F^*}{1 - F^*} - \ln \frac{f}{1 - f} \tag{5}$$

and to solve

$$we^{-w} = ve^{-v}, \qquad (v \neq w \text{ for } w \neq 1), \tag{6}$$

for v, the letters "ln" indicating the logarithm to the base e. It is usually easiest to solve (6) by the following convergent iterative procedure: If $w \geqq 1$, put

$$v_0 = we^{-w}, \qquad v_{m+1} = v_0 e^{v_m}; \tag{7}$$

if $w \leqq 1$, put

$$v_0 = w - \ln w, \qquad v_{m+1} = v_0 + \ln v_m . \tag{7'}$$

In either case v may be obtained with as much accuracy as desired by simple iteration with formulas (7) or (7').

The coefficients a_0 and a_1 then may be expressed as

$$a_1 = \frac{1}{v}\left[\ln \frac{w - v}{2(1 - v)} - \ln \frac{w\alpha^*}{2}\right], \tag{8}$$

$$a_0 = a_1 \cdot \frac{w - v}{2(1 - v)} - \frac{v + w - 2}{2(1 - v)^2} - 1. \tag{9}$$

The limiting values of a_1 and a_0 as w and v approach unity are given by (B9), (B11), and (B12) in Appendix B.

The accuracy of the approximation (4) has been investigated and found to be adequate. For small i and f and large F^* slightly greater precision is possible with the Uspensky approximation[9], computation of which is simplified by Feller's iterative procedure[10] (see Appendix B). Both approximations lose accuracy as α^* increases.

For $F^* = 0.5$ and $\alpha^* = 0.1$, Table I presents a comparison of the exact integral value satisfying (2) with the two approximations, in which the value of n^* satisfies the equation concerned as precisely as possible and is therefore not integral. For this table the exact recursion formulas (A3) and (C3) were used. The latter was found by Miss M. N. Torrey and the writer and appears to be new.

2.3 *Some Properties of the Criterion of Critical Length*

According to the previous discussion, it is proposed to take special action whenever a screening sequence exceeds its critical length. Since

TABLE I — TABLE OF CRITICAL NUMBERS n^* FOR UPPER LIMIT OF FRACTION INSPECTED $F^* = 0.5$ AND MAXIMUM PROBABILITY OF ERROR $\alpha^* = 0.1$†

Clearing Number, i	Sampling Frequency, f								
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
5	88	47	32	24	19	15	12	10	8
	87.8	46.6	31.5	23.4	18.2	14.5	11.7	9.4	7.4
	85.7	45.9	31.2	23.3					
10	153	84	58	44	35	28	23	19	16
	152.2	83.3	57.5	43.5	34.4	27.8	22.8	18.6	14.8
	151.2	83.0−	57.4						
20	283	158	110	84	67	55	45	38	31
	282.5	157.3	109.8	83.7	66.7	54.4	44.8	36.9	29.8
	282.0−	157.1							
50	675	380	267	205	164	135	111	94	75
	674.7	379.6	266.8	204.5	163.6	134.1	111.0+	91.9	74.6
	674.5								
100	1329	751	529	406	326	268	221	187	150
	1329	750.3	528.6	405.8	325.2	266.8	221.4	183.5	149.3
	1329	750.2				267.0−			
300	3946	2233	1576	1212	973	800	661	560	450
	3947	2234	1577	1212	970.9	797.0−	662.7	550.1	448.0+
	3946	2233	1576	1212	971.9	798.4	662.9		

† The triad of numbers appearing for f and i combinations are, reading down, the exact value, the Uspensky approximation, and the approximation (4) to (9). The last is omitted if it agrees with the second to 0.1. Approximate values less than 1,000 should be rounded to the next higher integral value to obtain the result corresponding to the exact value of n^*. This method was followed in rounding approximate values greater than 1,000.

the aim of such action would be to bring about improvement in the process, it might be justifiable to resume inspection with sampling after the special action has been taken. There is a question in any case whether the screening sequence, once interrupted by special action, should be resumed at the point it was stopped. A cautious procedure would be to resume inspection with a new screening sequence not involving any previously inspected units. This course would lead to a lower AOQL but a higher fraction inspected, F^c, than the original plan.‡ Resumption with sampling would have the opposite effect, but the changes in either case should be slight in practice.

We shall consider in detail only the effect of increasing the total amount of inspection when inspection is resumed with a new screening sequence after special action has been taken. With this alteration in the CSP-1 inspection plan, the limiting fraction inspected, F^c, according to our

‡ There is no change from the original values if the inspector takes special action as soon as he finds a defective unit after $n^* - i$ units in a single screening sequence and before that sequence is ended.

mathematical model, is given by (D24) in Appendix D. Furthermore, the upper limit to this fraction is according to (D34)

$$F^{C*} = \frac{F^*}{F^* + (1 - F^*)\beta^*}, \tag{10}$$

where

$$\beta^* = (1 - \alpha^*)/(1 - \alpha^* K - T_{n^*+i}(p^*, i)) \tag{11}$$

and all other quantities are as defined previously. The comparatively minor change from F^* to F^{C*} in the range of interest can be illustrated by noting that if $F^* = 0.5$, $\alpha^* = 0.1$, and $f \geqq 0.05$, we will have $0.5 < F^{C*} < 0.51$.

Under these same assumptions two other characteristics of the modified CSP-1 plan can be readily computed: The average number of special actions per 10,000 units produced and the average number of special actions per 10,000 units inspected. These may be computed by multiplying C in (D27) and C^I in (D32) respectively by 10^4. The first of these two averages may be the more useful to the practitioner, who can use the value of this average at $p = p^*$ as an added measure of the price paid for using the criterion of critical length. In some cases he may prefer it to α^*. For $F^* = 0.5$, $\alpha^* = 0.1$, and $p = p^*$ this number varies from about 0.4 for $f = 0.05$ and $i = 5$ to about 15 for $f = 0.45$ and $i = 100$.

Another more theoretical use may be made of these two averages. We may wish to compare the operation of the criterion of critical length with that of any other criterion adopted for the same purpose. The parameters of the criterion to be compared to the present one could be adjusted so that one or both of these two averages agree for the two schemes when $p = p^*$. Then the average number of special actions per 10,000 units produced could be plotted against p or F in both cases. On the other hand one may wish the fraction inspected to be the same at $p = p^*$ for the two schemes. However, criteria calling for special action at certain times when the last inspected unit was defective lead to the same fraction inspected as found in the original CSP-1 plan. In such cases it is not possible to obtain equal fractions inspected, since $F^C > F$. It appears better in general to deal with the two averages C and C^I for the purpose of comparing criteria. At any event, as has been mentioned above, such formal comparisons are not complete measures of practical value in themselves.

III ACKNOWLEDGMENTS

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APPENDIX A

Some Properties of the Run Probability $T_n(p, i)$

As before let $T_n(p, i)$ be the probability that a screening sequence with clearing number i and process average p has not terminated after the nth consecutive unit has been inspected. This is the same as the probability of no run of i or more "successes" each having probability $q = 1 - p$ in n independent trials. Except when necessary for clarity to do otherwise, we shall abbreviate $T_n(p, i)$ by T_n.

It is easy to see that

$$T_n = 1, \qquad n = 0, 1, \cdots, i - 1, \tag{A1}$$

$$T_i = 1 - q^i, \ (q = 1 - p), \tag{A2}$$

$$T_{n-1} - T_n = pq^i T_{n-i-1}, \qquad (n > i). \tag{A3}$$

From these relations it appears that the generating function of T_n,

$$T(x) = \sum_{n=0}^{\infty} x^n T_n,$$

satisfies

$$T(x) = \frac{1 - q^i x^i}{1 - x + pq^i x^{i+1}}, \tag{A4}$$

in which both numerator and denominator have the common factor $1 - qx$.

If it is required that $T_i > \alpha^*$, we have directly from (A2)

$$1 - q^{*i} > \alpha^*. \tag{A5}$$

From (1′) and (A5) the inequality (3) follows. In turn $F^* - f$ is seen from (3) to exceed $\alpha^* f(1 - f)/[1 - \alpha^*(1 - f)]$. Maximizing this quantity with respect to f, we have $2(1 - \sqrt{1 - \alpha^*})/\alpha^* - 1$, which is less than α^* for $0 < \alpha^* < 1$. This maximum and its derivative with respect to α^* are increasing in this interval, and the former assumes the value $3 - 4/2^{-1/2} < 0.35$ at $\alpha^* = 0.5$. The inequality (3′) follows immediately.

APPENDIX B

Derivation of an Approximation for Critical Length

From the expansion of (A4) in partial fractions, Uspensky has shown that as n approaches infinity,

$$T_n \sim \frac{1 - q\xi}{p(i + 1 - i\xi)} \cdot \frac{1}{\xi n^{+1}} \tag{B1}$$

with ξ the unique positive root of

$$\sum_{s=0}^{i-1} (qx)^s = \frac{1}{px}. \tag{B2}$$

The Uspensky approximation for T_n leads to the approximation (4) for n^* satisfying (2) for any given α^* $(0 < \alpha^* < 1)$, i, and $p^*(0 < p^* < 1)$. It can, in fact, be shown that

$$n^* \sim a_1 i + a_0 + a_{-1} i^{-1} + a_{-2} i^{-2} + \cdots .$$

If in (B1) and (B2), we put $p = p^*$, $q = q^*$, and $n = n^*$, it follows from (1′) and (B2) that

$$T_{n*} \sim \frac{1 - K\xi^i}{(i + 1 - i\xi)} \cdot \xi^{-n^*} . \tag{B3}$$

Likewise, making the same substitutions in (B2) it follows that

$$K(1 - K^{1/i})x^{i+1} - x + 1 = 0 \tag{B4}$$

has two and only two positive real roots, $x_1 = 1/q^*$ and $x_2 = \xi$.

We shall consider a system of equations in five variables equivalent to the system (B3) and (B4) in the five variables K, i, ξ, n^*, and T_{n^*}. We shall call the new variables w, z, v, φ, and α^*. The new system of equations is

$$\alpha^* = \frac{1 - e^{-w}(1 - vz)^{-1/z}}{1 - v - vz} \cdot (1 - vz)^{\varphi/z}, \tag{B5}$$

and

$$e^{-w}(1 - e^{-wz})(1 - vz)^{-1/z} = vz, \tag{B6}$$

where only finite positive values for all variables are going to be considered with $vz < 1$ and $0 < \alpha^* < 1$. If in (B5) and (B6) we put $w = -\ln K$, $z = i^{-1}$, $v = i(\xi - 1)/\xi$, $\varphi = i^{-1}(n^* + 1)$, and $\alpha^* = T_{n^*}$, the result is (B3) and (B4) with the symbol "$\sim$" replaced by "$=$" in (B3) and with $x = \xi$ in (B4).

Like (B4), (B6) has one "extraneous" root, $v_1(w, z) = (1 - e^{-wz})/z$. Either positive root, $v_1(w, z)$ or $v_2(w, z)$, is such that there exists a finite function $v_{s0}(w) = \lim_{z \to 0} v_s(w, z)$, $(s = 1, 2)$. Indeed, $v_{10}(w) = w$ and $v_{20}(w) = v_0(w)$ satisfy the limiting form of (B6) given by (6). Clearly either $0 < w \leqq 1 \leqq v_0(w) < \infty$ or $0 < v_0(w) \leqq 1 \leqq w < \infty$, the equality signs holding simultaneously.

Taking logarithms of both sides of (B5), we have

$$\varphi = \left(\ln \alpha^* + \ln \left[\frac{1 - v - vz}{1 - e^{-w}(1 - vz)^{-1/z}} \right] \right) \Big/ \frac{1}{z} \ln(1 - vz).$$

As z approaches zero, φ approaches

$$\varphi_0(w, \alpha^*) = -\frac{1}{v_0(w)} \left(\ln \alpha^* + \ln \left[\frac{w(1 - v_0(w))}{w - v_0(w)} \right] \right). \tag{B7}$$

We may differentiate (B6) and (B5) to obtain respectively

$$\left. \frac{\partial v(w, z)}{\partial z} \right|_{z=0} = \frac{v_0(w)}{2} \cdot \frac{(v_0^2(w) - w)}{(1 - v_0(w))}$$

and

$$\begin{aligned} \varphi_0'(w, \alpha^*) &= \left. \frac{\partial \varphi}{\partial z} \right|_{z=0} \\ &= \frac{w - v_0(w)}{2(1 - v_0(w))} \cdot \varphi_0(w, \alpha^*) - \frac{v_0(w) + w - 2}{2(1 - v_0(w))^2}. \end{aligned} \tag{B8}$$

Substituting the values $w = -\ln K$, $z = i^{-1}$, and $\varphi = i^{-1}(n^* + 1)$ in $\varphi = \varphi_0(w, \alpha^*) + z\varphi_0'(w, \alpha^*) + \cdots$ and putting

$$v_0(w) = v, \qquad a_1 = \varphi_0(w, \alpha^*), \qquad a_0 = \varphi_0'(w, \alpha^*) - 1, \tag{B9}$$

we have the approximation given by equations (4) through (9).

If both sides of equation (6) are divided by $d = v_0(w) - w$, we get

$$w = d/(e^d - 1). \tag{B10}$$

From (B7), (B8), and (B10) we find

$$\varphi_0(w, \alpha^*) = \ln 2 - \ln \alpha^* + 0_1(d), \tag{B11}$$

$$\varphi_0'(w, \alpha^*) = \varphi_0(w, \alpha^*) - \tfrac{1}{3} + 0_2(d). \tag{B12}$$

The related approximation for T_n, as n and i approach infinity, is of the form

$$T_n \sim A e^{-B(n+C)/(i+D)}, \tag{B13}$$

where A, B, C, and D depend only on q^i.

APPENDIX C

A Recursion Formula for $T_n(p, i)$

In order to investigate the error in n^* computed from the Uspensky approximation (B1) or the approximation (4), a convenient form of the exact value of $T_n(p, i)$ was needed. Such an expression is

$$T_n = \sum_{s=0}^{\min(k,r)} (-1)^s \binom{r}{s} (pq^i)^s T_{(k-s)i}, \tag{C1}$$

where $n = ki + r$ and $k \geqq 1$ and $0 \leqq r \leqq i$. This may be established easily by an inductive argument. Indeed, if $k = 1$ and $r = 0$, (C1) yields an identity. By adding successive expressions of the form of (A3), we obtain

$$T_{ki+r} = T_{ki} - pq^i \sum_{s=0}^{r-1} T_{ki-i+s}. \tag{C2}$$

For $k = 1$, (C2) yields with the aid of (A1)

$$T_{i+r} = T_i - rpq^i,$$

substantiating (C1) for $k = 1$ and $1 \leqq r \leqq i$. Next we assume (C1) to be true for some $k \geqq 1$ and some r, $(0 \leqq r \leqq i)$. We wish to show that (C1) is true for $n = (k + 1)i + r$. From (C2) and the induction assumption

$$T_{(k+1)i+r} = T_{(k+1)i} - pq^i \sum_{s=0}^{r-1} \sum_{t=0}^{\min(k,s)} (-1)^t \binom{s}{t} (pq^i)^t T_{(k-s)i}.$$

If the order of summation is reversed, the double sum becomes

$$-pq^i \sum_{t=0}^{\min(k,r-1)} (-1)^t (pq^i)^t T_{(k-t)i} \sum_{s=t}^{r-1} \binom{s}{t}$$

$$= \sum_{t=0}^{\min(k+1,r)-1} (-1)^{t+1} (pq^i)^{t+1} T_{(k-t)i} \binom{r}{t+1},$$

so that

$$T_{(k+1)i+r} = \sum_{s=0}^{\min(k+1,r)} (-1)^s \binom{r}{s} (pq^i)^s T_{(k+1-s)i}.$$

The special form of (C1) used in checking accuracy was that with $r = i$ and $1 \leqq k \leqq i$:

$$T_{(k+1)i} = \sum_{s=0}^{k} (-1)^s \binom{i}{s} (pq^i)^s T_{(k-s)i}. \tag{C3}$$

APPENDIX D

Some Characteristics of the CSP-1 Plan With and Without the Criterion of Critical Length

The use of generating functions is helpful in characterizing the original CSP-1 plans. The theory of Markov chains, applied somewhat as in Reference 4, also leads to some of the results found here. While this theory is convenient to show the validity of the strong law of large numbers as applied to fraction inspected and other ratios, the task of computing (to which we restrict ourselves here) appears generally simpler with the generating function technique. Let P_r be the probability that the rth unit produced is the first one in some sampling period, and let Q_r be the probability of being in a sampling period on that unit. Then

$$\begin{aligned} P_r &= 0, (0 \leqq r \leqq i), \qquad P_{i+1} = q^i, \\ P_r &= q^i p[1 - (1 - f)Q_{r-i-1}], \quad (r > i + 1), \end{aligned} \tag{D1}$$

and

$$Q_r = \sum_{s=0}^{r} P_s(1 - fp)^{r-s}, \qquad (r \geqq 0). \tag{D2}$$

If $P(x)$ and $Q(x)$ are the corresponding generating functions, we have from (D1) and (D2) respectively

$$P(x) = \frac{x^{i+1}q^i}{1 - x}[1 - qx - p(1 - f)(1 - x)Q(x)] \tag{D3}$$

and

$$Q(x) = P(x)/[1 - (1 - fp)x], \tag{D4}$$

whence

$$Q(x) = \frac{q^i x^{i+1}}{1 - x}\left[1 - p(1 - f)x \sum_{m=0}^{i-1} (qx)^m\right]^{-1}. \tag{D5}$$

With some manipulation of partial fractions we obtain

$$Q(x) = q^i x^{i+1}\left[\frac{(1 - x)^{-1}}{f + (1 - f)q^i} + \sum_{r=0}^{\infty} e_r x^r\right], \tag{D6}$$

where it can be shown that e_r approaches zero as r approaches infinity. It follows that

$$Q = \lim_{r \to \infty} Q_r = \frac{q^i}{f + (1 - f)q^i}, \tag{D7}$$

and, therefore, the limit of P_r exists as does the limit of

$$F_r = 1 - (1 - f)Q_r, \tag{D8}$$

the probability that the rth unit produced will be inspected. From (D7) and (D8) we may obtain (1).

There are similar results in terms of units inspected. Let P_r^I and Q_r^I correspond to P_r and Q_r with r in the former pair indicating the ordinal number of the unit inspected. Thus

$$\begin{aligned} &P_r^I = 0, (0 \leqq r \leqq i), \qquad && P_{i+1}^I = q^i, \\ &P_r^I = q^i p, && (r > i + 1), \end{aligned} \tag{D9}$$

and

$$\begin{aligned} &Q_r^I = 0, (0 \leqq r \leqq i), \\ &Q_r^I = \textstyle\sum_{s=0}^{r} P_s^I (1 - p)^{r-s} = q^i, \quad (r \geqq i + 1). \end{aligned} \tag{D10}$$

Before passing to the modified CSP-1 plan, we observe that we may write the expression for P_r in (D1) and (D3) in a different way. First, let R_r be the probability that screening is stopped for the first time after the rth unit produced (i.e., a run of i nondefectives has been completed for the first time with that unit). The generating function of R_r, $R(x)$, then satisfies

$$R(x) = 1 - (1 - x)T(x) = \frac{q^i x^i (1 - qx)}{1 - x + pq^i x^{i+1}}, \tag{D11}$$

where $T(x)$ is given by (A4). Next we may put

$$P_r = R_{r-1} + fp(Q_0 R_{r-1} + \cdots + Q_{r-1} R_0), \tag{D12}$$

or

$$P(x) = xR(x)[1 + fpQ(x)], \tag{D13}$$

and

$$Q(x) = xR(x)/[1 - x + fpx(1 - R(x))]. \tag{D14}$$

We now take up the case in which a criterion for the length of screening sequences is applied in the operation of the CSP-1 plan. To simplify

notation a little, we will hereafter call the (fixed) critical length n rather than n^*. Also the probabilities corresponding to P_r and Q_r will be denoted by $P_r{}^C$ and $Q_r{}^C$. We shall treat the occurrence of an incomplete screening sequence of length n as a recurrent event; as soon as such a sequence occurs, the whole inspection procedure begins anew with the next unit produced.

Now let $R_r{}^C$ be the probability that screening is stopped for the first time after the rth unit produced. Then

$$R_{r+1}{}^C = T_n{}^{[r/n]} R_{r-n[r/n]}, \tag{D15}$$

and the generating function is

$$R^C(x) = R^*(x)/(1 - T_n x^n), \tag{D16}$$

where

$$R^*(x) = \sum_{r=0}^{n} R_r x^r. \tag{D17}$$

If the superscript C is attached to the symbols P, Q, and R in (D2) and (D12), we arrive at valid equations. Therefore, we may write down the generating function $Q^C(x)$ by placing the superscript C on the same letters in (D14). Using the identity

$$R^*(x) = 1 - T_n x^n - (1 - x)T^*(x), \tag{D18}$$

where

$$T^*(x) = \sum_{r=0}^{n-1} T_r x^r, \tag{D19}$$

we have

$$Q^C(x) = \frac{xR^*(x)}{1 - x} [1 - T_n x^n + fpxT^*(x)]^{-1}. \tag{D20}$$

Again we find by the use of partial fractions

$$Q^C(x) = xR^*(x) \left[\frac{(1 - x)^{-1}}{1 - T_n + fpT^*(1)} + \sum_{r=0}^{\infty} e_r' x^r \right], \tag{D21}$$

where it can be shown as before the e_r' approaches zero as r approaches infinity. Hence,

$$Q^C = \lim_{r \to \infty} Q_r{}^C = \frac{1 - T_n}{1 - T_n + fpT^*(1)}, \tag{D22}$$

and the limit of the probability of inspecting the rth unit produced,

$$F_r^{\,C} = 1 - (1 - f)Q_r^{\,C}, \tag{D23}$$

is

$$F^C = \lim_{r\to\infty} F_r^{\,C} = \frac{f(1 - T_n) + fpT^*(1)}{1 - T_n + fpT^*(1)}. \tag{D24}$$

We may likewise compute the probability C_r that the criterion of critical length will be applied on the rth unit produced:

$$C_r = D_r + fp(Q_{r-n[r/n]}{}^{C}T_n^{[r/n]} + \cdots + Q_{r-2n}{}^{C}T_n^{\,2} + Q_{r-n}{}^{C}T_n), \tag{D25}$$

where $D_r = T_n^{\,k}$ if $r = nk$ and is zero otherwise. The generating function is

$$C(x) = \frac{T_n x^n}{1 - T_n x^n}(1 + fpQ^C(x)), \tag{D26}$$

so that

$$C = \lim_{r\to\infty} C_r = \frac{fpT_n}{1 - T_n} Q^C = \frac{fpT_n}{1 - T_n + fpT^*(1)}. \tag{D27}$$

As with the original CSP-1 plan we may find the probabilities $P_r^{\,IC}$ and $Q_r^{\,IC}$ corresponding to $P_r^{\,C}$ and $Q_r^{\,C}$ in terms of inspected rather than produced units. We may also obtain C_r^I, the probability of applying the criterion on the rth unit inspected. The generating functions are easily seen to satisfy

$$P^{IC}(x) = xR^C(x)[1 + pQ^{IC}(x)], \tag{D28}$$

$$Q^{IC}(x) = P^{IC}(x)/(1 - qx), \tag{D29}$$

and

$$C^I(x) = \frac{T_n x^n}{1 - T_n x^n}(1 + pQ^{IC}(x)). \tag{D30}$$

We find, using the previous methods,

$$Q^{IC} = \lim_{r\to\infty} Q_r^{\,IC} = \frac{1 - T_n}{1 - T_n + pT^*(1)}, \tag{D31}$$

$$C^I = \lim_{r\to\infty} C_r^{\,I} = \frac{pT_n}{1 - T_n + pT^*(1)}. \tag{D32}$$

It is possible to obtain the limiting probabilities (D27) and (D32) indirectly as the reciprocals of expectations of recurrence times, although the method of generating functions allows a more complete characterization. If m is the number of units produced until the criterion of critical length is applied and m_I is the number of units inspected until the same event, it is interesting to note from (D24), (D27), and (D32) that

$$F^C = C/C^I = \mathrm{E}(m_I)/\mathrm{E}(m), \tag{D33}$$

where E is the expectation operator.

The values of these same limiting probabilities are of particular interest when $p = p^*$. Then $T_n = \alpha^*$ and

$$p^*KT^*(1) = 1 - K - T_{n^*+i},$$

assuming an exact solution to (2). From (D24), for instance, we have for $p = p^*$

$$F^C = F^{C*} = \frac{f}{f + (1 - f)K\beta^*}, \tag{D34}$$

and

$$C^I = C^{I*} = \alpha^*\beta^*p^*K/(1 - \alpha^*), \tag{D35}$$

where β^* is given by (11). The denominator of β^* can be fairly well approximated by

$$1 - \alpha^*(K + e^{-v}),$$

where v is defined by (6). Finally (D33) may be used to find C^*.

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Nonparametric Definition of the Representativeness of a Sample—with Tables

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The problem is to determine how large a random sample is needed in order to attain a preassigned probability $P^*(\frac{1}{2} \leqq P^* < 1)$ *that the sample will possess a certain amount (or degree) of representativeness of the true unknown (cumulative) distribution* F *under study. The definition of representativeness involves two preassigned constants* k *and* $\beta^*(k \geqq 2$ *is an integer). For example, for* $k = 2$ *and any* $\beta^*(0 < \beta^* \leqq \frac{1}{2})$ *the sample is defined to be representative if the proportion of the total sample size falling on each side of the population median differs from* $\frac{1}{2}$ *by at most* β^*. *In this case the degree of representativeness is defined as* $d_g^* = 1 - 2\beta^*$.

This idea can be extended to any number k *of disjoint, exhaustive cells equi-probable under* F; *tables and graphs are given for finite and infinite populations for selected values of* k, β^* *and* P^*. *The definition is also extended to cases in which the experimenter is particularly interested in parts of* F *which are not equi-probable and/or parts of* F *which do not exhaust the whole sample space; tables and graphs accompany each application.*

These results are non-parametric, i.e., if the prescribed sample size is used then the experimenter's requirements for representativeness will be satisfied whatever the unknown distribution. Derivations of exact and approximate formulae used in computing tables are given in the Appendices.

I. INTRODUCTION

This paper deals with the problem of determining how large a random sample is needed in order to guarantee with preassigned probability P^* that the sample will have a specified amount (or a specified degree) of representativeness of the true, unknown (cumulative) distribution F under study. No à priori information is given about F and no assumptions are made about the form of F. The solution given is nonparametric (i.e., distribution-free) so that the results obtained and the tables and graphs

constructed are valid for any true underlying distribution. The case of a finite population as well as that of an infinite population is considered; in the latter case it is assumed only for ease of exposition that those percentiles of F which enter the discussion are uniquely defined and have probability zero under F. (This will, in particular, be the case when F has a density function without zero-stretches between points having positive density.)

A definition of representativeness (and also a degree of representativeness) is given with respect to those parts of F which are between certain percentiles which we denote by $F^{-1}(p_i)$, the values of p_i being preassigned. The *intervals* between these percentiles will be called *cells* and we shall only consider collections of *pairwise disjoint* cells. For example the experimenter may want to guarantee with probability at least $P^* = 0.90$ that between 40 per cent and 60 per cent of his sample will lie on *each* side of the population median. In this case we are interested in the part of F (or the cell) between $F^{-1}(0)$ and $F^{-1}(0.5)$ and also the part of F (or the cell) between $F^{-1}(0.5)$ and $F^{-1}(1)$. By the definitions below the common allowance β^* is 0.10 and the degree of representativeness $d_q{}^*$ is 0.80 (or 80 per cent). Then we enter Table I (or II) with $k = 2$, $P^* = 0.90$ and $\beta^* = 0.10$ and find that the smallest sample size needed to satisfy the experimenter's requirement for representativeness is $n = 60$. (It is instructive to note that the same solution would hold for any two disjoint, exhaustive *subsets* of the sample space having a common probability of $\frac{1}{2}$ under F. However, the cases in which we consider disjoint *cells* and, in particular, disjoint cells which start from *one end* or *both ends* of the distribution are of considerably more practical interest. The *cell* terminology will be used in the body of the paper while the *subset* terminology will be used in the appendices.)

In the above example the sample space is broken up into two disjoint, exhaustive cells which are equi-probable under F. This idea of representativeness can be extended to any number k of pairwise disjoint, exhaustive cells equi-probable under F and in the numerical work the values $k = 2, 3, 4, 5$ and 10 are considered. The idea of representativeness can also be used with cells that are not equi-probable and/or with cells that do not exhaust the whole sample space. As an example of the first type (cells not equi-probable) we might be concerned about whether a sample is large enough to be *simultaneously* representative of a single tail with preassigned probability $p < \frac{1}{2}$ under F *and* of its complement which has probability $(1 - p) > \frac{1}{2}$ under F. As an example of the second type (non-exhaustive cells) we might be concerned about whether a sample

is large enough to be representative of both tails (each having (say) a common preassigned probability $p < \frac{1}{2}$ under F), without any concern about the middle cell between the two tails. For each problem tables and graphs throughout this paper give the smallest required sample size for selected values of P^* and specified amounts (or specified degrees) of representativeness.

Assuming for the moment that the density of F is known and that all of its deciles are finite then we can plot an observed bar diagram (i.e., rectangles with different widths under the *dashed lines* in Fig. 1) and the true density on the same diagram as shown in Fig. 1 to illustrate the idea of a representative sample. By definition of a decile each of the vertical strips bounded above by the *curve* has an area (or probability under F) of 0.1. The observed sample is considered representative relative to this pattern of ten disjoint, exhaustive and equi-probable cells to within a common allowance β^* if *simultaneously* the areas of *all* vertical *rectangles* differ from the theoretical value of 0.1 by at most $\beta^*(0 < \beta^* \leqq 0.1)$. Then the degree $d_\beta{}^*$ of representativeness as defined in Section III is equal to $1 - 10\beta^*$. We are interested in finding the smallest sample size needed to guarantee a probability of at least P^* that the above condition will hold in a sample drawn at random from F.

This problem is related to the well-known problem[1] of Kolmogorov-Smirnov since they both have the common purpose of determining the sample size required to obtain a representative sample. Since their definition of representativeness is different from the one treated here, it is difficult to make a proper comparison of the two procedures. Another remark on this comparison is made in Appendix IV.

II. DEFINITION OF REPRESENTATIVENESS

Let F denote the true unknown cumulative distribution and let $F_n{}^*$ denote the observed sample distribution based on n observations. For any *given* k let $C_1, C_2, \cdots, C_k$ denote pairwise disjoint cells (not necessarily exhaustive or equi-probable under F) which are defined by certain percentiles. The cells $C_1, C_2, \cdots, C_k$ are not known but their probabilities under F are given positive numbers; let $F(C_i)$ denote the probability assigned to C_i by the distribution $F(i = 1, 2, \cdots, k)$. (We are using F and $F_n{}^*$ as symbols for both point functions and probability measures which are set functions; clearly, the nature of the argument will prevent any confusion.) Let $\beta_i{}^*$ denote specified positive numbers (which we shall call allowances) such that

$$0 < \beta_i{}^* \leqq F(C_i) \qquad (i = 1, 2, \cdots, k). \quad (1)$$

We shall be particularly interested in the special case $\beta_1^* = \beta_2^* = \cdots = \beta_k^* = \beta^*$ (say), whether or not the quantities $F(C_i)$ are all equal. Then a sample is defined to be representative relative to a fixed pattern of k disjoint cells $C_1, C_2, \cdots, C_k$ to within the allowances $\beta_1^*, \beta_2^* \cdots, \beta_k^*$, respectively, if we have *simultaneously*

$$| F_n^*(C_i) - F(C_i) | \leqq \beta_i^* \qquad (i = 1, 2, \cdots, k). \quad (2)$$

III. DEFINITION OF DEGREE OF REPRESENTATIVENESS

Although the quantities $\beta_i^* (i = 1, 2, \cdots, k)$ are basic to the idea of representativeness it may be useful, in a given problem, to combine them to define a measure of the *degree* of representativeness. We define

$$d_g^* = \left\{ \prod_{i=1}^{k} \left[1 - \frac{\beta_i^*}{F(C_i)} \right] \right\}^{1/k} \quad (3)$$

where the subscript g denotes the fact that d_g^* is a *geometric* mean. It follows from (1) that $0 \leqq d_g^* < 1$ and that d_g^* can take on all the values in this interval.

It should be noted that for any fixed set of values of $F(C_i)$ $(i = 1, 2, \cdots, k)$ if there is a common β^* then the right hand member of (3) is a strictly decreasing function of β^* for $\beta^* \leqq \min F(C_i)$. Hence, if there is a common β^* the values of d_g^* and β^* uniquely determine each other. When this is the case we may be interested sometimes in specifying d_g^* (instead of β^*) and then using (3) to solve for the common β^*.

We shall say that a random sample is representative relative to a fixed pattern of k disjoint cells $C_1, C_2, \cdots, C_k$ to a degree d_g^* if for *the common* $\beta^* = \beta^*(d_g^*)$ satisfying (3) we have

$$| F_n^*(C_i) - F(C_i) | \leqq \beta^* \qquad (i = 1, 2, \cdots, k). \quad (4)$$

It should be emphasized that the chief interest of this paper is in the concept of representativeness as formulated in Section II and that the present definition of the *degree* of representativeness is to be regarded as supplementary.

One possible criticism of the definition of d_g^* is that it may require a positive (and sometimes substantial) number of observations to attain a *zero* degree of representativeness (see, for example, the last and third from last columns in Table III). However, since the practical use of the concept of *degree* of representativeness is mainly for *large* values of d_g^* this objection is not serious.

It is possible also to define the *degree* of representativeness as an *arithmetic* mean d_a^* of the bracketed quantities in (3) but then for a common β^* and different $F(C_i)$, because of (1), the value of d_a^* is restricted to an interval $J \leqq d_a^* < 1$ where J is *positive* and depends on the values of the $F(C_i)$ $(i = 1, 2, \cdots, k)$. Clearly, if the $F(C_i)$ are all equal and there is a common β^* then $d_a^* = d_g^*$.

IV. CONSTRUCTION OF TABLES

The problem is to find the *smallest* sample size n such that the joint probability of all the inequalities (2) [or (4)] is at least equal to a specified value $P^* < 1$, i.e., such that

$$P\{ \mid F_n^*(C_i) - F(C_i) \mid \leqq \beta_i^* (i = 1, 2, \cdots, k)\} \geqq P^*. \tag{5}$$

The reader is cautioned that it does not necessarily follow that (5) holds for any integer greater than n; however, since F_n^* converges almost certainly to F (see page 20 of Reference 2), it follows that there exists in each case a smallest number $n' \geqq n$ such that (5) holds for *every integer* greater than or equal to n'. For example, with $k = 2$, a common $\beta^* = 0.20$ and $P^* = 0.75$ the condition (5) is satisfied for $n = 3$, for 6 and for any integer greater than or equal to $n' = 9$.

Since the cells C_i are pairwise disjoint and the values of $F(C_i)$ are given $(i = 1, 2, \cdots, k)$ the left member of (5) is determined for any particular sample size whatever the unknown distribution F. In the case of an infinite population we use the multinomial distribution with k or $k + 1$ disjoint cells depending on whether or not the k disjoint cells are exhaustive, i.e., on whether or not $\sum_{i=1}^{k} F(C_i) = 1$. For the case of two disjoint, exhaustive cells this clearly reduces to a problem of the binomial distribution which is closely related to the problem of finding confidence limits on a population percentile by the use of order statistics. Similarly in the case of a finite population we use the hypergeometric distribution with k or $k + 1$ categories depending on whether or not $\sum_{i=1}^{k} F(C_i) = 1$. The exact and approximate formulae for computing the left member of (5) are given in Appendices I and II, respectively. The approximate calculation involves several interesting geometrical digressions which are discussed in Appendix III.

Table I gives for $k = 2$ and selected values of β^* and P^* the required sample sizes n *and* n' and also the maximum drop in probability below the specified P^* for all sample sizes between n and n'. In the remaining tables only the values of n are given. Table II gives the required sample size for $k = 2$, $F(C_1) = p$, $F(C_2) = 1 - p$ for $p = 0.5$, 0.2 and 0.1 (for

TABLE I

Sample size required to attain a probability P^* that a sample will be simultaneously representative to within a common allowance β^* of two disjoint and exhaustive cells separated by the median for any true distribution.

In each set the first entry is the smallest sample size required to satisfy (4); the second entry is the smallest size required such that for *all* sample sizes at least as large, (4) is satisfied; the last entry is the maximum deviation in probability below P^* obtained for all sample sizes between the first two entries.

P^* \ β^*	0.01	0.05	0.10	0.15	0.20	0.25	0.40
0.50	1051 1199 (0.0264)	31 59 (0.1271)	5 14 (0.2266)	5 10 (0.1875)	2 5 (0.1250)	2 2 (0)	2 2 (0)
0.60	1700 1850 (0.0162)	60 79 (0.0704)	5 24 (0.3266)	5 10 (0.2875)	3 8 (0.2250)	3 3 (0)	3 3 (0)
0.70	2600 2750 (0.0124)	100 119 (0.0382)	20 29 (0.1049)	8 16 (0.2078)	3 8 (0.3250)	3 6 (0.0750)	3 3 (0)
0.75	3251 3399 (0.0077)	120 150 (0.0407)	25 39 (0.0769)	11 16 (0.1377)	3 9 (0.3750)	3 6 (0.1250)	3 3 (0)
0.80	4051 4199 (0.0058)	151 179 (0.0328)	35 44 (0.0430)	14 24 (0.0518)	9 12 (0.0266)	4 7 (0.0750)	4 4 (0)
0.85	5100 5250 (0.0052)	191 219 (0.0269)	45 54 (0.0434)	17 27 (0.0879)	10 15 (0.0766)	4 10 (0.1250)	4 4 (0)
0.90	6700 6850 (0.0029)	260 279 (0.0129)	60 74 (0.0299)	28 33 (0.0360)	13 18 (0.0796)	8 11 (0.0797)	5 5 (0)
0.95	9551 9699 (0.0012)	371 399 (0.0070)	90 99 (0.0114)	37 47 (0.0230)	20 28 (0.0284)	12 15 (0.0423)	6 6 (0)
0.99	16500 16650 (0.0003)	651 679 (0.0013)	160 169 (0.0022)	71 76 (0.0028)	39 42 (0.0015)	24 26 (0.0046)	8 12 (0.0017)

For $n \leqq 150$ the entries are all exact; for $n > 150$ the entries involve approximations. The pattern of increases and decreases of the probability as a function of n was also used to obtain the first two entries for large n.

selected values of β^* and P^*). Table III gives the required sample size for the case of k pairwise disjoint, exhaustive and equi-probable cells $(C_1, C_2, \cdots, C_k)$ for $k = 2, 3, 4, 5$ and 10 (for selected values of β^* and P^*). Table IV gives the required sample size for $k = 2$, $F(C_1) = F(C_2) = p$ for $p = 0.2, 0.1$ and 0.05 (here the cells are disjoint and equi-probable but not exhaustive). Table V considers the same problem as in Table III and compares the required sample sizes for infinite populations, $N = \infty$, with those for finite populations of size N for $N = 60$, 120, 360. Tables VI and VII give illustrations of the error involved in using the approximations used in Tables IV and V, respectively, instead of an exact probability calculation.

Fig. 2 shows for selected values of P^* that the sample sizes in Table I and in the first portion of Table II can be "linearized" for large n on a log-log plot of n versus β^*. Figs. 3 and 4 show the same result for the last and middle portion of Table II, respectively.

Table II

Minimum sample size required to attain a probability of at least P^* that a sample will be simultaneously representative to within a common allowance β^* of two disjoint and exhaustive cells separated by the 100 pth percentile for any true distribution. (The degree of representativeness is then defined as $d_\beta^* = \sqrt{\left(1 - \frac{\beta^*}{p}\right)\left(1 - \frac{\beta^*}{1-p}\right)}$.)

	50th Percentile (Median) ($p = 0.50$)					20th or 80th Percentile ($p = 0.20$ or 0.80)					10th or 90th Percentile ($p = 0.10$ or 0.90)		
$P^* \backslash \beta^*$	0.01	0.05	0.10	0.15	0.20	0.01	0.05	0.10	0.15	0.20	0.01	0.05	0.10
0.50	1,051	31	5	5	2†	662	12	7	6	1†	355	14	1†
0.60	1,700	60	5	5	3†	1,062	32	7	6	1†	500	14	1†
0.70	2,600	100	20	8	3†	1,662	52	10	9	1†	900	20	1†
0.75	3,251	120	25	11	3†	2,062	72	10	9	1†	1,100	40	1†
0.80	4,051	151	35	14	9	2,562	92	20	12	1†	1,400	40	1†
0.85	5,100	191	45	17	10	3,262	120	27	12	3†	1,800	60	1†
0.90	6,700	251	60	28	13	4,262	160	37	15	5	2,355	80	1†
0.95	9,551	371	90	37	20	6,100	232	50	20	10	3,400	120	10
0.99	16,500	651	160	71	39	10,562	420	100	40	20	5,900	220	15

For $n \leqq 150$ the entries are all exact; for $n > 150$ the entries are based on approximations together with a knowledge of the monotonicity pattern of the probability of representativeness as a function of n.

† Small entries for certain pairs (β^*, P^*) indicate a condition too weak for practical usage.

Table III

Minimum sample size required to attain a probability of at least P^* that a sample will be simultaneously representative to within a common allowance β^* of k equi-probable disjoint and exhaustive cells for any true distribution. (The degree of representativeness is then defined as $d_o^* = 1 - k\beta^*$).

	$k = 2$			$k = 3$			$k = 4$			$k = 5$			$k = 10$	
P^* \ β^*	0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10
0.50	31	5	2	102	21	6	120	26	9	120	30	5	100	20
0.60	60	5	3	141	30	6	140	38	9	140	30	5	100	20
0.70	100	20	3	180	47	12	180	43	12	180	40	5	120	30
0.75	120	25	3	222	51	14	200	52	14	200	50	10	120	30
0.80	151	35	9	240	60	15	240	60	14	220	50	10	140	30
0.85	191	45	10	300	72	15	280	66	16	240	60	15	160	30
0.90	251	60	13	360	90	21	320	80	18	280	70	15	160	40
0.95	371	90	20	480	120	29	400	100	27	360	90	23	200	50
0.99	651	160	39	741	180	45	600	146	38	500	120	35	260	60

For $k \geqq 3$ probabilities were computed exactly only for $n \leqq (200/k)$; for $n > (200/k)$ the approximation in Appendix 2 was used together with a knowledge of the monotonicity pattern of the probability of representativeness as a function of n.

Table IV

Minimum sample size required to attain a probability of at least P^* that a sample will be simultaneously representative to within a common allowance β^* of any two disjoint equi-probable cells defined by percentiles and having a common probability p under the true, unknown distribution. (The degree of representativeness is then defined as $d_o^* = 1 - \beta^*/p$.)

Application	Below 20th and Above 80th Percentiles ($p = 0.20$)			Below 10th and Above 90th Percentiles ($p = 0.10$)			Below 5th and Above 95th Percentiles. ($p = 0.05$)	
P^* \ β^*	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05
0.50	1,700	52	10	900	20	1†	450	1†
0.60	2,262	72	10	1,255	40	1†	600	1†
0.70	3,000	112	20	1,655	54	1†	850	1†
0.75	3,500	132	30	1,955	60	1†	1,000	1†
0.80	4,100	152	30	2,300	80	1†	1,150	1†
0.85	4,900	180	40	2,700	100	10	1,400	1†
0.90	6,000	232	50	3,355	120	20	1,750	1†
0.95	7,900	300	70	4,455	160	35	2,250	80
0.99	12,562	492	120	7,000	274	65	3,650	130
Another Application	Between 30th and 50th percentiles and between 50th and 70th percentiles			Between 40th and 50th percentiles and between 50th and 60th percentiles			Between 45th and 50th percentiles and between 50th and 55th percentiles	

For $n \leqq 40$ the entires are exact; for $n > 40$ normal approximation theory was used.

† Small entires for certain pairs (β^*, P^*) indicate a condition too weak for practical usage.

TABLE V

Minimum sample size required to attain a probability of at least P^* that a sample from a population of size N will be simultaneously representative to within a common allowance β^* of k equi-probable disjoint and exhaustive cells for any true population. (The degree of representativeness is then defined as $d_g^* = 1 - k\beta^*$).

The four entries in each set below correspond to $N = 60, 120, 360, \infty$, respectively.

	$k = 2$			$k = 3$			$k = 4$			$k = 5$			$k = 10$	
$P^* \backslash \beta^*$	0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10
0.50	20	5	2	40	19	6	40	20	7	40	20	3	34	10
	20	5	2	55	21	6	60	20	7	60	20	5	54	15
	20	5	2	81	21	6	80	20	7	80	24	5	74	15
	31	5	2	102	21	6	120	26	7	120	30	5	100	20
0.75	40	15	3	47	28	12	47	26	12	45	27	8	40	20
	60	20	3	76	37	14	74	38	12	72	30	8	60	25
	91	25	3	136	49	14	130	40	14	120	40	10	94	25
	120	25	3	222	51	15	200	52	14	200	50	10	120	30
0.85	51	25	9	53	30	14	50	32	14	49	30	10	40	20
	71	30	10	84	49	15	80	40	14	80	40	10	60	25
	120	40	10	162	60	15	150	58	16	152	50	13	100	30
	191	45	10	300	72	15	280	66	16	240	60	15	160	30
0.90	51	30	10	54	37	15	50	38	16	51	30	13	40	25
	80	40	13	93	51	19	90	46	16	80	40	13	74	25
	151	50	13	180	72	21	170	60	18	160	60	15	114	35
	251	60	13	360	90	21	320	80	20	280	70	15	160	40
0.95	51	35	16	54	42	21	50	38	18	52	37	15	47	25
	91	50	19	94	60	25	90	58	20	92	50	15	74	30
	180	70	20	201	88	27	190	80	25	180	70	18	120	40
	371	90	20	480	120	30	400	100	27	360	90	20	200	50
0.99	60	45	23	55	48	27	57	43	25	53	40	20	49	30
	100	70	30	102	72	30	100	66	29	98	60	23	80	40
	231	110	36	240	120	42	220	100	34	212	90	25	154	50
	651	160	39	741	180	45	600	146	37	500	120	30	260	60

For finite populations all entries with $n \leqq 2/\beta^*$ are based on exact computations; the entries with $n > 2/\beta^*$ are based on the approximation in equation (A17) of Appendix II. Another simpler approximation is given in equation (A18) of Appendix II.

Table VI

Comparison between the exact value of and the normal approximation to the joint probability that in a sample of size n from an infinite population the number of observations falling in each of two tails with common probability p is between $n(p - \beta^*)$ and $n(p + \beta^*)$, inclusive.

		$p = 0.10$ $\beta^* = 0.05$	$p = 0.20$ $\beta^* = 0.05$	$p = 0.20$ $\beta^* = 0.10$
$n = 10$	Normal Approx.	0.1628	0.0973	0.5910
	Exact	0.1510	0.0941	0.6014
	Error	+0.0118	+0.0032	−0.0104
$n = 20$	Normal Approx.	0.5432	0.3654	0.7075
	Exact	0.5566	0.3648	0.7171
	Error	−0.0134	+0.0006	−0.0096
$n = 40$	Normal Approx.	0.6608	0.4655	0.8574
	Exact	0.6731	0.4669	0.8736
	Error	−0.0123	−0.0014	−0.0162

Table VII

Comparison between the exact value of and the normal approximation to the joint probability that in a sample of size n from a population of size N the number of observations falling in each of k equi-probable cells is between $n\left(\frac{1}{k} - \frac{1}{20}\right)$ and $n\left(\frac{1}{k} + \frac{1}{20}\right)$, inclusive.

$N = \infty$ (Infinite Population)

		$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 10$
$n = 20$	Normal Approx.	0.4977	0.1166	0.1600	0.1172	0.0698
	Exact	0.4966	0.1145	0.1618	0.0955	0.0669
	Error	+0.0011	+0.0021	−0.0018	+0.0217	+0.0029
$n = 40$	Normal Approx.	0.5708	0.2196	0.2388	0.1962	0.1775
	Exact	0.5704	0.2181	0.2363	0.1904	0.1478
	Error	+0.0004	+0.0015	+0.0025	+0.0058	+0.0297
$n = 60$	Normal Approx.	0.6338	0.3974	0.3230	0.2876	0.3325
	Exact	0.6338	0.3982	0.3174	0.2979	*
	Error	0.0000	−0.0008	+0.0056	−0.0103	*

$N = 120$ (Finite Population)

		$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 10$
$n = 20$	Normal Approx.	0.5357	0.1397	0.1984	0.1550	0.1092
	Exact	0.5368	0.1359	0.1801	0.1547	0.1011
	Error	−0.0011	+0.0038	+0.0183	+0.0003	+0.0081
$n = 40$	Normal Approx.	0.6651	0.2822	0.3705	0.3413	0.4291
	Exact	0.6670	0.3084	0.3679	0.3313	0.3357
	Error	−0.0019	−0.0262	+0.0026	+0.0100	+0.0934
$n = 60$	Normal Approx.	0.7969	0.6338	0.6115	0.6228	0.8507
	Exact	0.7989	0.6104	0.6003	0.5972	*
	Error	−0.0020	+0.0234	+0.0112	+0.0256	*

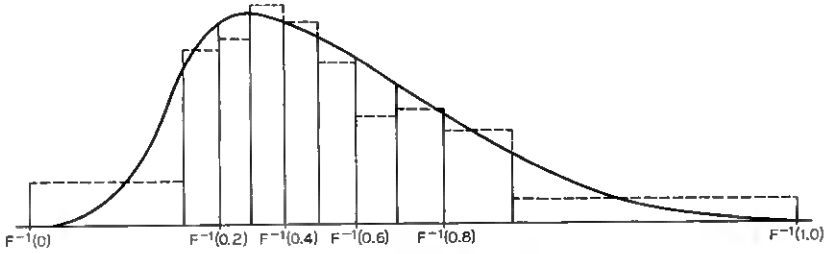

Fig. 1 — Pictorial diagram of representativeness using deciles ($k = 10$).

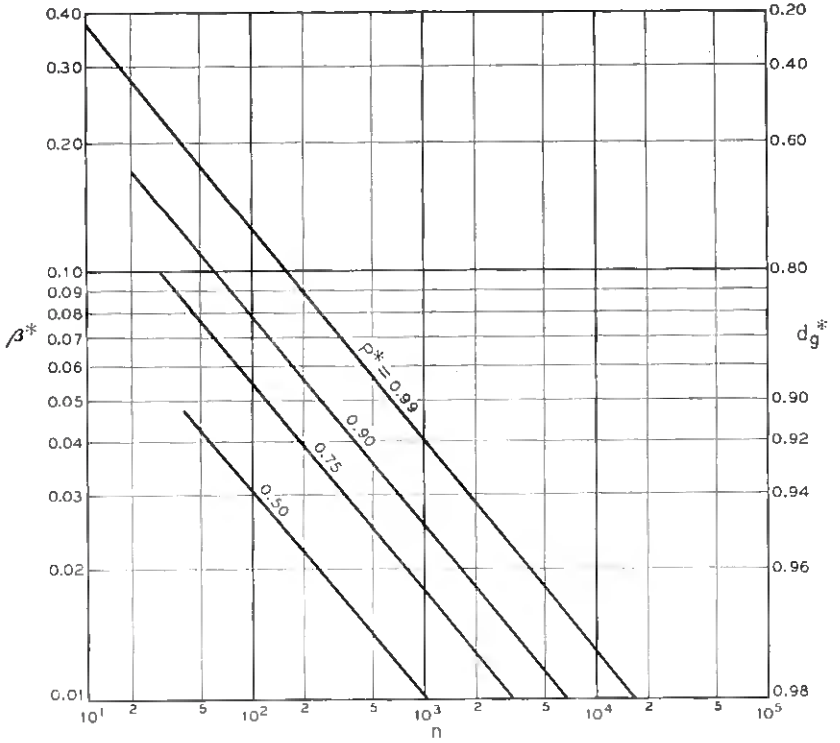

Fig. 2 — Minimum sample size n required to attain a probability of at least P^* that a sample is simultaneously representative to within a common allowance β^* of two disjoint and exhaustive cells each having probability $p = \frac{1}{2}$ under the true unknown distribution. (The degree of representativeness is $d_v^* = 1 - 2\beta^*$.)

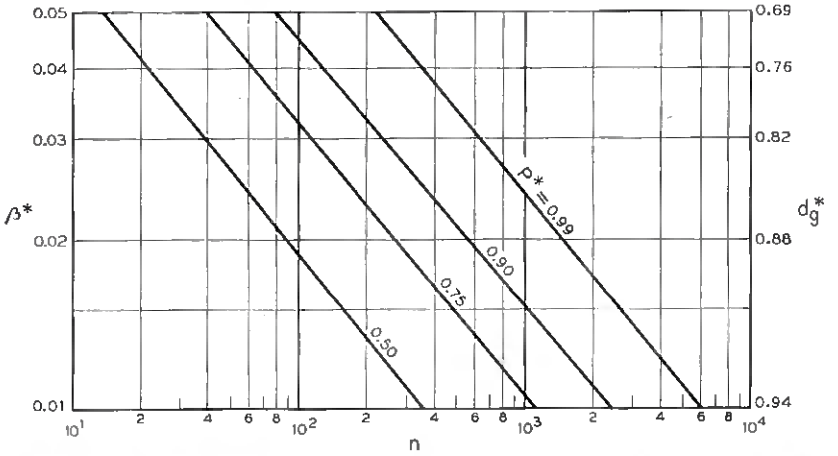

Fig. 3 — Minimum sample size n required to attain a probability of at least P^* that a sample is simultaneously representative to within a common allowance β^* of the two disjoint, exhaustive cells separated by the 10th (or the 90th) percentile for any true distribution. [The degree of representativeness is $d_g^* = (\frac{10}{3})\sqrt{(0.1 - \beta^*)(0.9 - \beta^*)}$.]

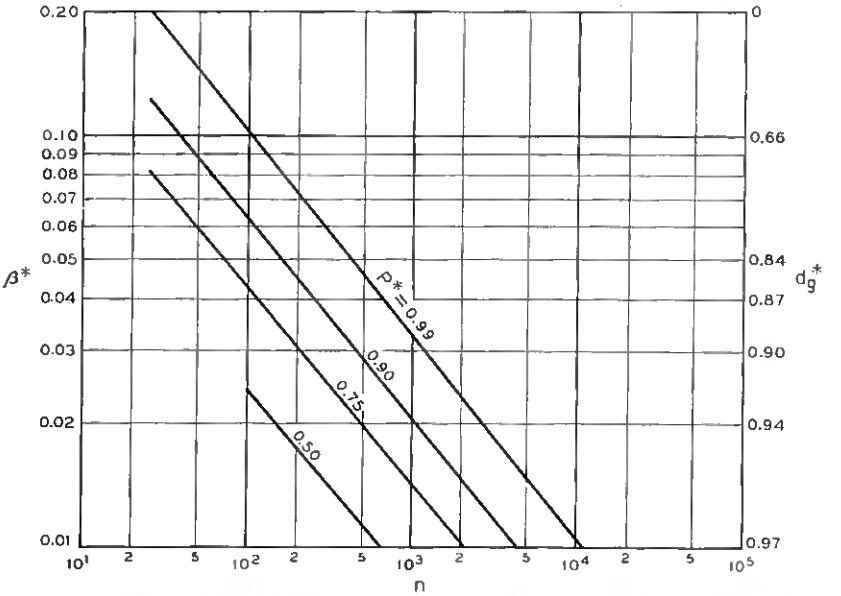

Fig. 4 — Minimum sample size n required to attain a probability of at least P^* that a sample is simultaneously representative to within a common allowance β^* of the two disjoint, exhaustive cells separated by the 20th (or the 80th) percentile for any true distribution. [The degree of representativeness is $d_g^* = (\frac{5}{2})\sqrt{(0.2 - \beta^*)(0.8 - \beta^*)}$.]

V. EMPIRICALLY OBSERVED MONOTONICITIES

It is interesting to note in Table III that for fixed β^* and increasing k the sample size n required is *not* monotonic but appears to reach a maximum and then decrease. As a result of this it becomes possible to to speak of the sample size n required for a sample to be representative for any specified β^* regardless of the number k of pairwise disjoint, exhaustive, equi-probable cells considered, provided only that $k \leqq 1/\beta^*$. For example, for $\beta^* = 0.1$ it appears likely from Table III that 90 observations would be sufficient to have a confidence of at least $P^* = 0.90$ that the sample is representative in the sense of (2) for *any one value* of $k(k = 1, 2, \cdots, 10)$.

Table VIII, some of whose entries are taken from Table III, shows *numerically* that *for fixed* d_0^* the required sample size is a monotonically non-decreasing function not only of P^* *but also of* k; *for fixed* β^*. Table III shows numerically that only the monotonicity with P^* holds. The former result is again shown in Figs. 5 and 6 which also emphasize the possibilities of interpolation on k.

The above monotonicities and lack of monotonicities have not been demonstrated mathematically.

TABLE VIII

Minimum sample size required to attain a probability of at least P^* that a sample will be simultaneously representative to a degree $d_0^* = 1 - k\beta^*$ of k equi-probable disjoint and exhaustive cells for any true distribution.

P^*	$d_0^* = 0.80$			$d_0^* = 0.90$		
	$k = 2$	$k = 4$	$k = 10$	$k = 2$	$k = 5$	$k = 10$
0.50	5	120	600	31	800	2500
0.60	5	140	700	60	950	2800
0.70	20	180	800	100	1150	3200
0.75	25	200	850	120	1250	3400
0.80	35	240	900	151	1400	3700
0.85	45	280	1000	191	1600	4000
0.90	60	320	1100	251	1850	4400
0.95	90	400	1250	371	2250	5100
0.99	160	600	1650	651	3150	6600

In comparing results for a *fixed degree* d_0^* it should be noted that the sample size appears to be a monotonically non-decreasing function of P^* *and* also of k; for a *fixed common* allowance β^* only the monotonicity with P^* holds as is evident in Table II. The remarks at the bottom of Table III apply here also.

VI. CONFIDENCE BANDS—INFINITE POPULATION CASE

The experimenter will usually be interested in the confidence statement that the above formulation allows him to make *after the observations are taken*. Suppose, for example, that he was interested in representativeness in each of $k = 10$ pairwise disjoint, exhaustive and equi-probable cells and that he specified $\beta^* = 0.02$ (so that $d_g^* = 0.80$) and $P^* = 0.85$ and that he has taken 1,000 observations in accordance with Table VIII.

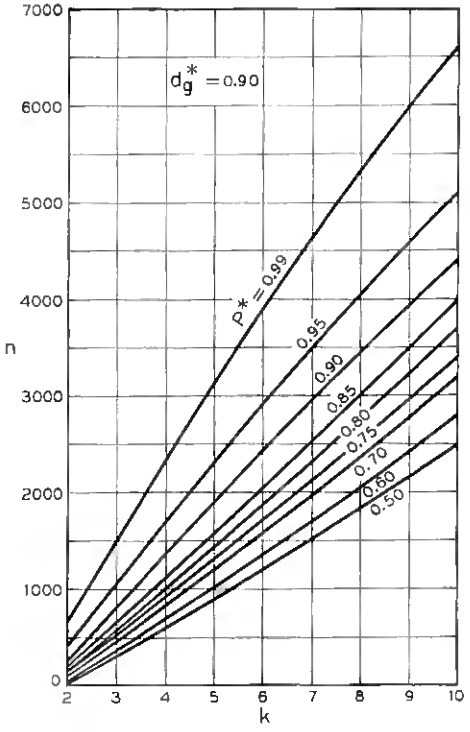

Fig. 5 — Minimum sample size n required to attain a probability of at least P^* that a sample will be simultaneously representative to a degree $d_g^* = 0.90$ of k equi-probable, disjoint and exhaustive cells for any true distribution. The common allowance β^* is given by $\beta^* = (1 - d_g^*)/k = 0.10/k$.

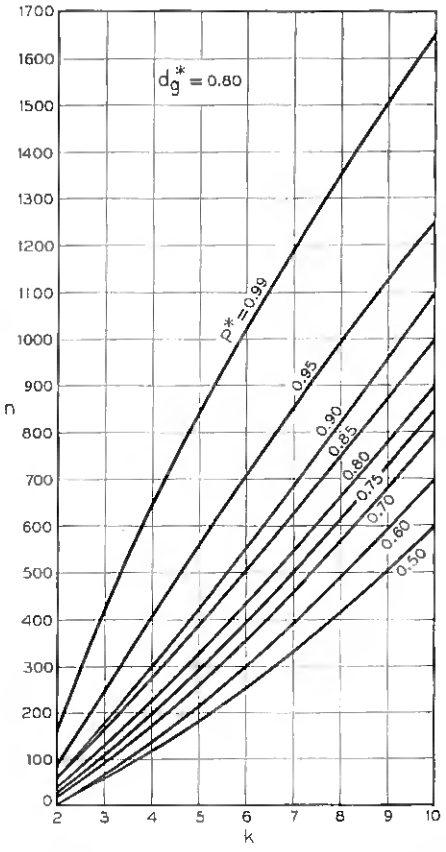

Fig. 6 — Minimum sample size n required to attain a probability of at least P^* that a sample will be simultaneously representative to a degree $d_g^* = 0.80$ of k equi-probable, disjoint and exhaustive cells for any true distribution. The common allowance β^* is given by $\beta^* = (1 - d_g^*)/k = 0.20/k$.

He can then make a number of confidence statements about the population deciles $F^{-1}(0.1)$, $F^{-1}(0.2)$, $\cdots$, $F^{-1}(0.9)$ (and also about $F^{-1}(0)$ and $F^{-1}(1)$ defined as the greatest lower bound of all x for which $F(x) > 0$ and the least upper bound of all x for which $F(x) < 1$, respectively). For example, if x_m denotes the mth (smallest) ordered observation, it follows from the condition of representativeness that we have *simultaneously* with joint confidence greater than P^* all of the inequalities

$$\left\{\begin{array}{lll} -\infty \leqq F^{-1}(0) & < x_1 \\ x_{80} \leqq F^{-1}(0.1) & < x_{121} \\ x_{160} \leqq F^{-1}(0.2) & < x_{241} \\ \cdots\cdots\cdots\cdots\cdots & \\ x_{640} \leqq F^{-1}(0.8) & < x_{961} \\ x_{720} \leqq F^{-1}(0.9) & < \infty \\ x_{1000} \leqq F^{-1}(1) & \leqq \infty \end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{lll} x_{1000} < F^{-1}(1) & \leqq \infty \\ x_{879} < F^{-1}(0.9) & \leqq x_{920} \\ x_{759} < F^{-1}(0.8) & \leqq x_{840} \\ \cdots\cdots\cdots\cdots\cdots & \\ x_{39} < F^{-1}(0.2) & \leqq x_{360} \\ -\infty < F^{-1}(0.1) & \leqq x_{280} \\ -\infty \leqq F^{-1}(0) & \leqq x_1 \end{array}\right\}. \tag{6}$$

For example, $F^{-1}(0.2)$ must be greater than or equal to x_{160} and less than x_{241} in the confidence statement since under the condition of representativeness all cells and, in particular, the last two cells on the left contain between 80 and 120 observations, inclusive.

The right hand set of inequalities are in reverse order since they are obtained by similar reasoning as the left hand set except that we start at the right end of the distribution and work backwards. If we keep only the stronger results in (6) for each decile and disregard the weaker ones, then we obtain eleven (finite or infinite) line segments as in Fig. 7. We can then state with joint confidence greater than P^* that the unknown distribution F has a (finite or infinite) point of contact with (or a saltus passing through) each of the line segments; the two end segments are

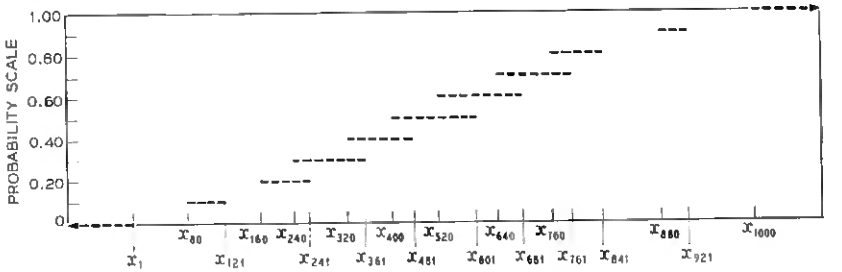

Fig. 7 — Confidence intervals for the deciles with joint confidence level $P^* = 0.85$ for $k = 10$, $n = 1000$ and $\beta^* = 0.02$ (which implies that $d_\varphi^* = 0.80$).

actually half-lines and in these cases we must allow $+\infty$ and $-\infty$ as possible "points" of contact.

The above result then gives rise to two "staircases", as in the middle diagram of Fig. 8, such that any distribution contacting every line segment in Fig. 7 must everywhere lie between (or on the boundary of) the two "staircases". Hence we can state with confidence *greater than* P^* (see explanation below) that the two "staircases" form a confidence band on the unknown distribution.

If we keep k and P^* fixed and decrease β^* (or increase $d_g^* = 1 - k\beta^*$)

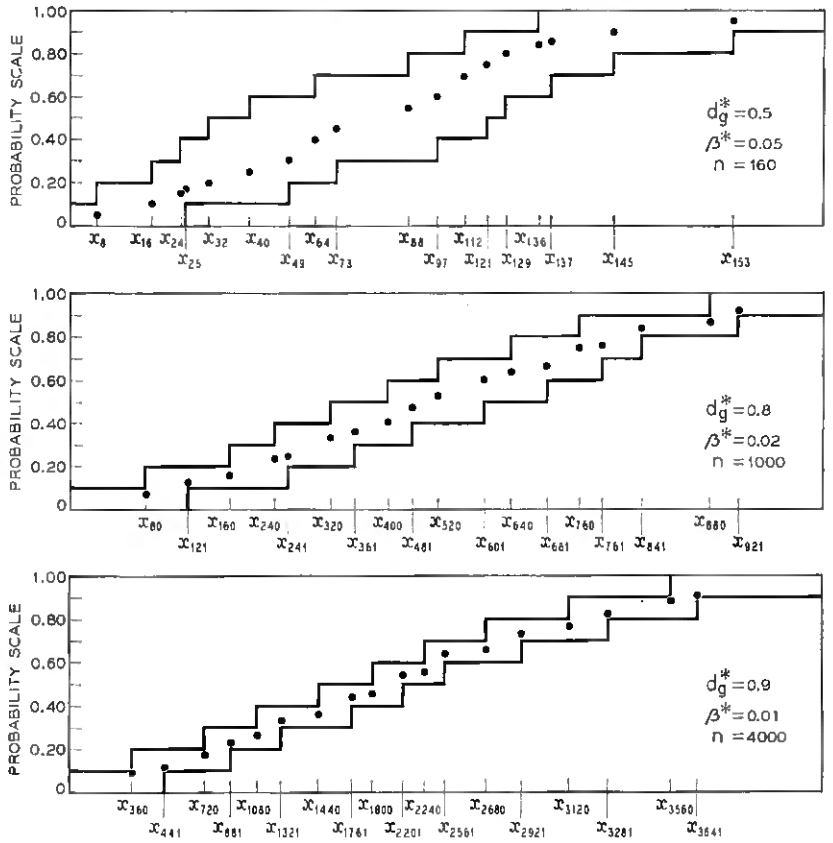

Fig. 8 — Confidence bands which include the true distribution function with confidence greater than $P^* = 0.85$ for $k = 10$ and $d_g^* = 0.5, 0.8, 0.9$. Small circles between the confidence bands represent ordinates of the sample distribution function. The three figures above were constructed with observations obtained from a table of random normal deviates (with different horizontal scaling applied in each case).

then the required sample size increases and the confidence band becomes narrower. This is illustrated in the three diagrams of Fig. 8.

It should be noted that the inequalities (6) are implied by but do not imply (i.e., they are not equivalent to) the condition of representativeness. Hence the confidence level associated with (6) is *greater than* the specified P^*. To illustrate this we note from (6) the stronger inequalities

$$x_{80} \leqq F^{-1}(0.1) < x_{121} \quad \text{and} \quad x_{160} \leqq F^{-1}(0.2) < x_{241} . \tag{7}$$

These inequalities (7) allow as few as 40 and as many as 161 observations between $F^{-1}(0.1)$ and $F^{-1}(0.2)$, including endpoints. On the other hand we have confidence P^*, under the condition of representativeness, that every such cell contains between 80 and 120 observations, inclusive. This shows that the confidence level associated with the confidence band is greater than the probability achieved for the representativeness of the sample.

This method of obtaining a confidence band for the unknown distribution would be more valuable if we could obtain a simple way of computing (or estimating more accurately) the actual confidence level attained. For example, with $k = 3$, $\beta^* = 0.10$ (so that $d_\varrho^* = 0.70$) and $P^* = 0.60$ we obtain $n = 30$ from Table III, the probability achieved for representativeness is 0.6369 and the confidence level associated with the two "staircases" is 0.6825. The latter is obtained by using inequalities similar to (6) and computing the probability exactly with a multinomial distribution. The reader should note that the idea of a confidence band containing the true, unknown distribution is not the main theme of this paper but only an interesting by-product of the idea of the representativeness of the sample.

APPENDIX I

Exact Formulae — Finite and Infinite Populations

The concept of the representativeness of a sample can be applied to finite as well as infinite populations. Let N denote the total size of a finite population; conceptually we may regard the population as being partitioned into k subsets S_i of size $F(S_i)(i = 1, 2, \cdots, k)$. We shall assume that the sets S_i are pairwise disjoint and, to simplify the discussion, we also assume that the quantities $N_i = NF(S_i)(i = 1, 2, \cdots, k)$ are *positive integers*.

Let $x_i \geqq 0$ denote the random integral number of observations in the observed sample of size n which fall in the set $S_i(i = 1, 2, \cdots, k)$. If

the k sets S_i are exhaustive then

$$\sum_{i=1}^{k} x_i = n \qquad \text{and} \qquad \sum_{i=1}^{k} N_i = N. \tag{A1}$$

We define for $i = 1, 2, \cdots, k$

$$c_i = n[F(S_i) - \beta_i^*] \quad \text{and} \quad d_i = n[F(S_i) + \beta_i^*], \tag{A2}$$

which are non-negative but need not be integers. Then for a finite population the probability corresponding to the left number of (5), using the hypergeometric distribution, is given *exactly* by

$$P_n^{(N)}[N_i, a_i, b_i \ (i = 1, 2, \cdots, k)] = \sum \prod_{i=1}^{k} \binom{N_i}{x_i} \Big/ \binom{N}{n} \tag{A3}$$

where $\binom{N}{n}$ is the usual binomial coefficient and the summation in (A3) is over all vectors $\vec{x} = \{x_1, x_2, \cdots, x_k\}$ for which

$$c_i \leqq x_i \leqq d_i \qquad (i = 1, 2, \cdots, k). \tag{A4}$$

If the k sets are *not* exhaustive then we define another set S_{k+1} which is the complement of the union of the k sets S_i and use (A3) with k replaced by $k + 1$ in (A1) and (A3) but *not* in (A4), i.e., no condition is applied to the $(k + 1)$th variable.

In the case of an infinite population we use the multinomial distribution. If the k sets S_i are exhaustive, then using (A2) and letting $p_i = F(S_i)(i = 1, 2, \cdots, k)$ the left hand member of (5) is given *exactly* by

$$P_n^{(\infty)}[p_i, \beta_i^* \ (i = 1, 2, \cdots, k)] = \sum \frac{n!}{\prod_{i=1}^{k} (x_i!)} \prod_{i=1}^{k} (p_i^{x_i}) \tag{A5}$$

where the summation is again over all vectors $\vec{x} = \{x_1, x_2, \cdots, x_k\}$ satisfying (A1) and (A4). If the k sets are not exhaustive then we define S_{k+1} as above and the same expression (A5) is obtained with k replaced by $k + 1$ in (A1) and (A5) but *not* in (A4), i.e., no condition is applied to the $(k + 1)$th variable.

It is interesting to note that the results for the infinite case ($N = \infty$) can be obtained from those of the finite case by letting N tend to infinity. Table V illustrates this numerically since the four entries in each set correspond to $N = 60$, 120, 360 and ∞, respectively.

APPENDIX II

Approximate Solutions — Infinite and Finite Populations

Let x_i denote the random integral number of observations in a sample of size n which fall in the ith cell $(i = 1, 2, \cdots, k)$. If we let

$y_i = x_i - (n/k)$, then the two conditions $\sum_{i=1}^{k} x_i = n$ and

$$\sum_{i=1}^{k} y_i = 0 \tag{A6}$$

are equivalent. Let $[x]$ denote the largest integer not greater than x. We shall consider only the case of the equi-probable exhaustive sets.

In the case of an infinite population we wish to compute

$$P = P\left\{n\left(\frac{1}{k} - \beta_i^*\right) \leqq x_i \leqq n\left(\frac{1}{k} + \beta_i^*\right) \quad (i = 1, 2, \cdots, k) \;\middle|\; \sum_{i=1}^{k} x_i = n\right\}. \tag{A7}$$

If we introduce a continuity correction and use (A6) then we obtain

$$P = P\{-b_i \leqq y_i \leqq a_i (i = 1, 2, \cdots, k) \mid \textstyle\sum_{i=1}^{k} y_i = 0\} \tag{A8}$$

where for each $i (i = 1, 2, \cdots, k)$

$$a_i = \frac{1}{2} + \left[n\beta_i^* + \frac{n}{k}\right] - \frac{n}{k} \quad \text{and} \quad b_i = \frac{1}{2} + \left[n\beta_i^* - \frac{n}{k}\right] + \frac{n}{k}. \tag{A9}$$

If n/k is an integer and β^* is the common value of $\beta_i^* (i = 1, 2, \cdots, k)$ then $a_1 = a_2 = \cdots = a_k = b_1 = b_2 = \cdots = b_k = a$ (say) and (A8) reduces to

$$P = P\{\,|\,y_i\,| \leqq a (i = 1, 2, \cdots, k) \mid \textstyle\sum_{i=1}^{k} y_i = 0\} \tag{A10}$$

where $a = \frac{1}{2} + [n\beta^*]$.

To compute (A10) *two approximations* are made. The k-variate multinomial probability is first transformed by an orthogonal transformation into a $(k-1)$-variate distribution with homoscedastic and uncorrelated variables and the *first approximation* is to replace the latter distribution by a multivariate normal distribution with independent variables. The region of integration is the *intersection* of the hypercube $|\,y_i\,| \leqq a$ centered at the origin with edge-length $2a$ *and* the hyperplane (A6); the orthogonal transformation merely rotates this intersection about the origin. These intersections are convex figures symmetric with respect to the origin; for example, it is a regular centered hexagon for $k = 3$. These intersections, called Stott figures, are discussed in Appendix III. The *second approximation* made in computing (A10) was to replace the Stott figure by a $(k-1)$-dimensional central sphere whose radius R is determined by equating the two hypervolumes. Values of R for $k = 2(1)12$ for any a are given in Table IX.

TABLE IX

Intersection $\mathcal{S}$ of the hypercube of edge-length $2a$ centered at the origin and the hyperplane $x_1 + x_2 + \cdots + x_k = 0$.

Dimension k of hypercube	$J(k)$ = Number of equally large simplices in $\mathcal{S}$	Radius R of sphere with content equal to that of $\mathcal{S}$
2	1	1.4142 a
3	6	1.2861 a
4	4	1.3655 a
5	230	1.4436 a
6	66	1.5225 a
7	23,548	1.5995 a
8	2,416	1.6733 a
9	4,675,014	1.7443 a
10	156,190	1.8126 a
11	1,527,092,468	1.8786 a
12	15,724,248	1.9422 a

The content $I(k)$ of $\mathcal{S}$ for all k is given by

$$I(k) = \frac{a^{k-1}\sqrt{k}}{(k-1)!}\left[\binom{k}{0}(k)^{k-1} - \binom{k}{1}(k-2)^{k-1} + \binom{k}{2}(k-4)^{k-1} - \cdots\right]$$

where the terms continue only as long as the arguments $k, k-2, \cdots$ are positive. The radius R of a $(k-1)$-dimensional sphere of equal content is obtained by equating $I(k)$ and $(R\sqrt{\pi})^{k-1} \Big/ \Gamma\left(\frac{k+1}{2}\right)$.

The orthogonal transformation referred to above is

$$y_i' = \frac{1}{\sqrt{i(i+1)}}(y_1 + y_2 + \cdots + y_i - iy_{i+1}) \qquad (i = 1, 2, \cdots, k) \tag{A11}$$

where y_{k+1} is defined to be identically zero. Then y_k' is identically zero by (A6). The remaining y_i' all have a common variance $\frac{n}{k}$ since for each $i(i = 1, 2, \cdots, k-1)$

$$\sigma_{y_i'}^2 = \frac{1}{i(i+1)}\left\{i(i+1)n\left(\frac{k-1}{k^2}\right) + 2\binom{i}{2}\left(-\frac{n}{k^2}\right) - 2i^2\left(-\frac{n}{k^2}\right)\right\} = \frac{n}{k} \tag{A12}$$

and are pairwise uncorrelated since for $i < j$

$$\sigma_{y_j' y_i'} = \sigma_{y_i' y_j'} = \frac{1}{\sqrt{i(i+1)j(j+1)}} \left\{ \frac{ni(k-1)}{k^2} + 2\binom{i}{2}\left(-\frac{n}{k^2}\right) + i(j-i)\left(-\frac{n}{k^2}\right) - i(j-1)\left(-\frac{n}{k^2}\right) - \frac{ni(k-1)}{k^2} + (ij-ij)\left(-\frac{n}{k^2}\right) \right\} = 0. \quad \text{(A13)}$$

If we let $\nu = k - 1$, let $r = R/\sigma = R\sqrt{k/n}$ and let S denote the central sphere of radius r then the approximate probability (dropping primes) is given by

$$P = \int_S \cdots \int \left(\frac{1}{2\pi}\right)^{\nu/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{\nu} y_i^2\right\} dy_1\, dy_2 \cdots dy_\nu \quad \text{(A14)}$$
$$= P\{\chi_\nu^2 \leqq r^2\}$$

where χ_ν^2 denotes a chi-square random variable with ν degrees of freedom.

In the case of a finite population of size N the only change in the above discussion is to replace (A12) by

$$\sigma_{y_i'}^2 = \frac{n}{k}\left(\frac{N-n}{N-1}\right) \quad (i = 1, 2, \cdots, k-1) \quad \text{(A15)}$$

thus increasing the value of r^2 and the value of P; this decreases n if P is held fixed at any P^*. If we let n_N and n_∞ denote the required values for a finite population of size N and an infinite population, respectively, for the same fixed k, β^* and P^* then we obtain from (A14) and (A15)

$$n_\infty \cong n_N\left(\frac{N-n_N}{N-1}\right), \quad \text{(A16)}$$

or, taking the smaller solution in n_N, we have for large N

$$n_N \cong \frac{N - \sqrt{N^2 - 4(N-1)n_\infty}}{2}. \quad \text{(A17)}$$

Replacing $N - 1$ by N in (A16) we easily obtain for large N the simpler result

$$\frac{1}{n_N} \cong \frac{1}{n_\infty} - \frac{1}{N}. \quad \text{(A18)}$$

The error in P involved in both of the above approximations (A14) and (A17) is evaluated in Table VII for $N = 120$ and $N = \infty$ for selected values of n, β^* and k.

If n/k is not an integer then the above discussion may not apply since

a_i may not equal b_i in (A9). Assuming again a common β^* then we have a common "a" and a common "b" in (A9). In this case, averaging the approximate probabilities obtained by using $2a$ and $2b$ alternately as the edge-length of the hypercube was found to be satisfactory for computing the tables of this paper.

APPENDIX III

Geometric Results and Eulerian (Diamond) Numbers

The problem here is to find the $(k-1)$-dimensional content (or hypervolume) of the intersection $\mathcal{I}$ of the centered k-dimensional hypercube $|y_i| < a (i = 1, 2, \cdots, k)$ and the $(k-1)$-dimensional hyperplane $y_1 + y_2 + \cdots + y_k = 0$. The geometry for even k and odd k is quite different. The number of vertices of $\mathcal{I}$ for even k and odd k, respectively, is

$$\binom{k}{k/2} \quad \text{and} \quad k\binom{k-1}{(k-1)/2}; \tag{A19}$$

for example, for $k = 3$ we obtain the $3\binom{2}{1} = 6$ vertices $(a, -a, 0)$, $(-a, a, 0)$, $(a, 0, -a)$, $(-a, 0, a)$, $(0, a, -a)$ and $(0, -a, a)$. The vertices are all equally distant from the origin. All the edges of $\mathcal{I}$ have a common length $d = d(k)$ which equals $2a\sqrt{2}$ for even k and $a\sqrt{2}$ for odd k. The intersection $\mathcal{I}$ is a convex figure which is symmetric with respect to the origin and is known as a Stott figure.[6] The Stott figure can be partitioned into an integral number $J(k)$ of $(k-1)$-dimensional simplices which are not necessarily regular but are such that each simplex has the same content as a regular $(k-1)$-dimensional simplex with edge-length d. Hence, using a result on page 125 of Reference 8, the content $I(k)$ of $\mathcal{I}$ is given by

$$I(k) = \left(\frac{d\sqrt{2}}{2}\right)^{k-1} \frac{\sqrt{k}}{(k-1)!} J(k). \tag{A20}$$

The integers $J(k)$ are given in the middle column of Table IX; for example, the integer 6 for $k = 3$ indicates that there are six equilateral triangles in the centered hexagon.

D. Slepian[7] has shown that for even k the integers $J(k)$ can be found by generating a "triangle" of numbers using the recurrence relation

$$S_{i,j} = jS_{i-1,j} + iS_{i,j-1} \quad (i, j = 1, 2, \cdots) \tag{A21}$$

with boundary conditions $S_{1,j} = S_{j,1} = 1$ for all j; then the desired

quantities are

$$S_{i,i} = J(2i) \qquad (i = 1, 2, \cdots). \quad \text{(A22)}$$

Similarly for odd k he showed that we can use the recurrence relation

$$T_{i,j} = (2j + 1)T_{i-1,j} + (2i + 1)T_{i,j-1} \quad (i, j = 1, 2, \cdots) \quad \text{(A23)}$$

with boundary conditions $T_{0,j} = T_{j,0} = 1$ for all j; then the desired quantities are

$$T_{i,i} = J(2i + 1) \quad (i = 1, 2, \cdots). \quad \text{(A24)}$$

Fig. 9 shows these numbers in two diamond-shaped patterns and explains another interesting way of obtaining these numbers.

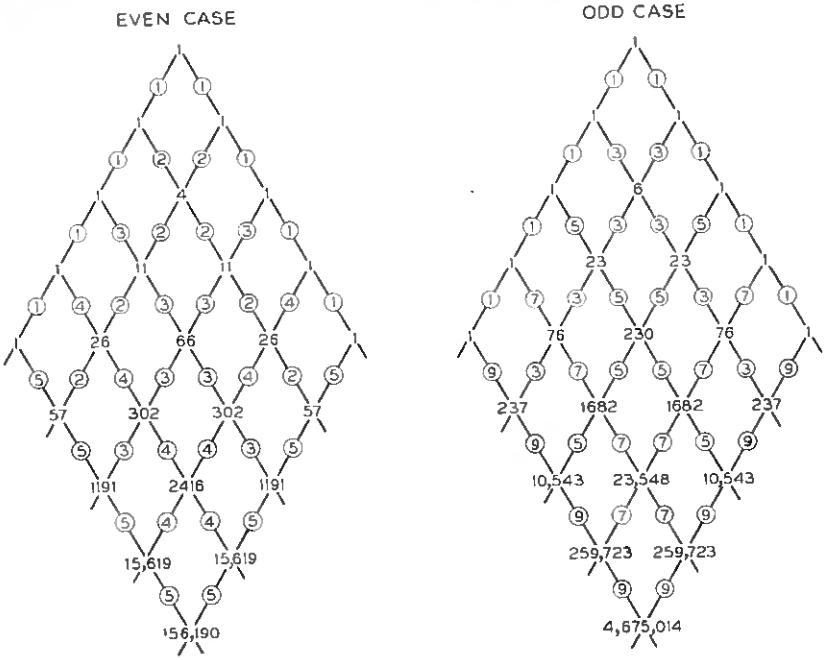

Fig. 9 — Combinatoric derivation of certain Eulerian (diamond) numbers. The number at any vertex V is obtained by considering any one path from the top vertex to V, multiplying the circled numbers encountered in this path, and summing the results obtained over all possible downward paths from the top vertex to V. In particular, the values on the vertical diagonal (of the diamond) are the values of $J(k)$ in Table IX. It is interesting to note that the sum of all the uncircled numbers in the mth row is $2^{m-1}(m-1)!$ for the odd case and $m!$ for the even case. This is shown above for $m = 1, 2, 3, 4, 5$ and would hold for all m if this pattern were continued indefinitely. The *circled* numbers are obtained by numbering the parallel diagonal lines starting with one at the "top," using all positive integers in the even case and only odd integers in the odd case.

The integers $J(k)$ arise in connection with combinatorial problems. As an example for even k, suppose we draw at random m balls in succession from an urn containing m balls marked $1, 2, \cdots, m$. Let X denote the number of times that the observed number increases, (say) always counting the first draw as an increase. Then it can be shown that

$$P\{X = j\} = S_{j,m+1-j}/m! \quad (j = 1, 2, \cdots, m), \quad \text{(A25)}$$

i.e., the mth row of the left diamond Fig. 9 divided by the sum $m!$ of that row gives the elementary probability distribution of X.

The problem of computing (A25) also arose in the work of V. H. Moore and W. A. Wallis[4] and M. MacMahon[3] who referred to it as Simon Newcomb's problem. J. Riordan[5] has studied the numbers $J(k)$ for even k and Carlitz and Riordan[5] call them Eulerian numbers (to be distinguished from the classical Euler numbers); an explicit formula as well as a generating function appears in these papers. The $S_{i,j}$ are related to the Eulerian numbers $A_{n,k}$ (defined in Reference 5) by $S_{i,j} = A_{i+j-1,j}$.

Explicit expressions for $J(k)$ for odd and even k are obtainable from (A22), (A24) and the more general results

$$S_{i,j} = \sum_{\alpha=0}^{j-1} (-1)^{\alpha} \binom{i+j}{\alpha} (j - \alpha)^{i+j-1} \quad \text{(A26)}$$

$$T_{i,j} = \sum_{\alpha=0}^{j} (-1)^{\alpha} \binom{i+j+1}{\alpha} [2(j - \alpha) + 1]^{i+j} \quad \text{(A27)}$$

due to D. Slepian.[7] It is easily shown that these formulae satisfy the corresponding recurrence relations as well as the boundary conditions. By an induction and symmetry argument applied to (A21) and (A23) and the boundary conditions it is easy to prove that

$$S_{i,j} = S_{j,i} \quad \text{and} \quad T_{i,j} = T_{j,i}. \quad \text{(A28)}$$

Substituting (A26) and (A27) in (A28) gives rise to interesting, nontrivial identities. For completeness we also give the generating functions derived by D. Slepian[7]

$$\sum_{i,j=1}^{\infty} \frac{S_{i,j} t^i u^j}{(i + j - 1)!} = \frac{tu(e^t - e^u)}{te^u - ue^t} \quad \text{(A29)}$$

$$\sum_{i,j=1}^{\infty} \frac{S_{i,j} t^i u^j}{(i + j)!} = \log_e \left[\frac{t - u}{te^u - ue^t} \right] \quad \text{(A30)}$$

$$\sum_{i,j=0}^{\infty} \frac{T_{i,j} t^i u^j}{(i + j)!} = \frac{(t - u)e^{t+u}}{te^{2u} - ue^{2t}}. \quad \text{(A31)}$$

The final result for the content $I(k)$ of $\mathcal{I}$ can, using the above be written as a *single* expression

$$I(k) = a^{k-1} \frac{\sqrt{k}}{(k-1)!} \sum_{\alpha=0}^{[(k-1)/2]} (-1)^{\alpha} \binom{k}{\alpha} (k - 2\alpha)^{k-1} \qquad \text{(A32)}$$

for all k where $[x]$ denotes the largest integer not greater than x. It has been pointed out by J. W. Tukey that (A32) can also be obtained by probabilistic considerations and that it appears in Laplace's "Theorie Analytique" (Book 2, page 260).

APPENDIX IV

Remarks on the Confidence Bands

It should be remarked that other assumptions on the true, unknown distribution can be used in conjunction with the confidence bands obtained in Section VI. It has been pointed out by J. W. Tukey, for example, that in the case of the first diagram in Fig. 8 the experimenter might be willing to assume that the true distribution is unimodal and that the mode x_m is such that $x_m \leqq x_{64}$. Then on purely geometrical considerations it can be shown that the confidence band can be modified as shown in the first diagram of Fig. 10. Briefly, if the true distribution enters any one of the three deleted triangles with any slope s then in order to get out again without leaving the confidence band the slope must get larger than s. But this contradicts the assumption that the density steadily decreases after x_{64}.

Similarly, with the same problem, if the experimenter assumes that the true distribution is unimodal and that $x_{73} \leqq x_m \leqq x_{88}$ then the first diagram of Fig. 8 can be modified as in the second diagram of Fig. 10. The assumption of unimodality is reasonable in many different practical applications but has not often been utilized in statistical techniques.

It is possible to formulate a problem for fixed P^* and n which requires the determination of that k which makes the *maximum* (or some average) *vertical width* of the confidence bands as small as possible. For example, for $P^* = 0.85$ and $n = 240$ the value $k = 10$ minimizes the maximum vertical width. It should be pointed out that if the experimenter's principal interest is in finding confidence bands with small vertical widths then this procedure appears to be quite inefficient compared with that based on the Kolmogorov statistic.[1]

A proper comparison is difficult since the nominal P^* is a lower bound and not the correct value of the confidence level associated with the pro-

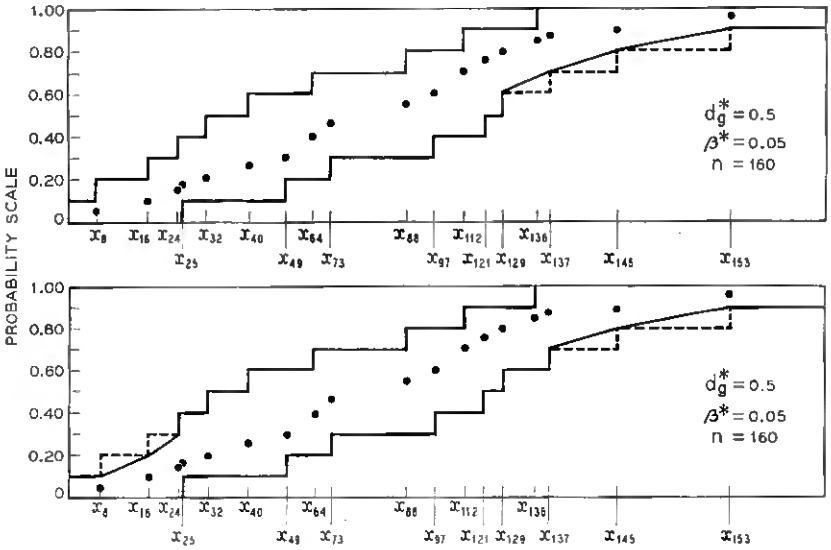

Fig. 10 — Modified confidence bands which include the true distribution function with confidence greater than $P^* = 0.85$ for $k = 10$ and $d_g^* = 0.5$.

posed confidence bands. As mentioned in the body of the paper the development of a confidence band is just a by-product of the main theme of this paper which is the representativeness of the sample.

VII. CONCLUSION

Definitions of representativness and of degree of representativeness are given and tables are included which give the sample size required to guarantee with preassigned probability P^* that a random sample will satisfy a condition of representativeness, the definition of which is agreed upon in advance. Thus, for experimenters who wish to know *in advance* how many observations will be needed for a distribution study, the problem has been given a precise nonparametric formulation and the solution has been found for some cases.

This formulation also leads to confidence bounds on the unknown distribution *after the observations are taken*. Examples are given to illustrate this.

The tables for the case of pairwise disjoint, equi-probable and exhaustive cells may also prove to be useful for the problem of determining the sample size required to obtain *simultaneous* confidence limits (on a preassigned level P^*) for *all* of the cell probabilities of a multinomial

distribution. Further investigation is needed to state precisely the conditions under which these tables can be used for this related problem.

VIII. ACKNOWLEDGEMENT

The authors wish to acknowledge the encouragement of the Bell Laboratories staff at Allentown and various helpful suggestions and aid received from D. Slepian, S. Monro, J. W. Tukey, J. Riordan, T. W. Anderson and others. The assistance of Miss Phyllis Groll in checking tables and proofreading is gratefully acknowledged.

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Fluctuations of Random Noise Power*

By D. SLEPIAN

(Manuscript received September 4, 1957)

The probability distribution of the power, y, of a sample of Gaussian noise of time duration T is considered. Some general theory is presented along with curves for the cumulative distribution and probability density of y for several different power spectra and values of T.

I. INTRODUCTION

A random quantity of interest in many communication and detection systems is the average power,

$$y = \frac{1}{T}\int_{-T/2}^{T/2} N^2(t)\, dt, \tag{1}$$

of a sample of finite time duration, T, of a Gaussian noise, $N(t)$. This quantity has been discussed in some detail by Rice in his classic paper[1] where he obtains expressions for the first few moments of y and an approximate probability density function.

In this paper the exact probability density function, $f(y)$, and the cumulative distribution function, $F(y)$, of the average power are computed for a number of ergodic Gaussian noises and for a number of values of T. The results are presented as a series of curves which are discussed in the next section. It is hoped that they will be of use to those designing specific systems.

II. SUMMARY OF COMPUTATIONAL RESULTS

Fig. 1 shows the probability density function, $f(y)$, for the random variable y of equation (1) when $N(t)$ has mean zero and power spectrum

$$w(f) = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}, \quad -\alpha \leqq f \leqq \infty .$$

Noise with this spectrum will be referred to as RC noise (see 5.1).

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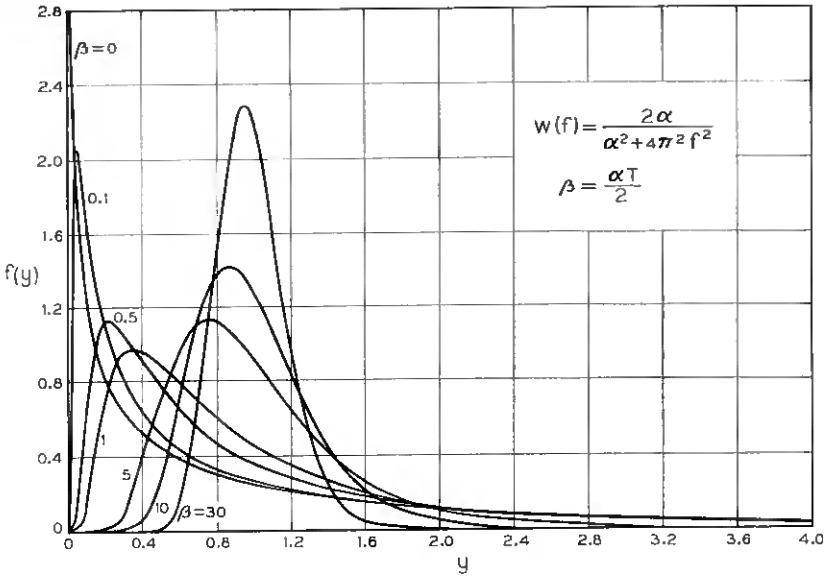

Fig. 1 — Probability density, $f(y)$, for RC noise.

The curves are labelled by values of $\beta = \alpha T/2$. The curve marked $\beta = 0$ is the probability density function for $y = N^2(t)$. Fig. 2 shows the corresponding cumulative distribution functions, $F(y)$.

For any $\beta > 0$, as y approaches zero, $f(y)$ and $F(y)$ approach zero more rapidly than any power of y.

As β becomes large, the density function $f(y)$ peaks up around unity which is the average power of $N(t)$. The variance of y is given by $(2\beta)^{-2}[4\beta - 1 + e^{-4\beta}]$. It approaches zero for large β like β^{-1}.

Figs. 3, 4 and 5 show $f(y)$ when $N(t)$ has mean zero and power spectrum

$$w(f) = \frac{2Q}{w_0} \frac{\omega^2}{\omega^2 + \left(\frac{Q}{\omega_0}\right)^2 (\omega^2 - \omega_0^2)^2}, \tag{2}$$

$$\omega = 2\pi f, \quad -\infty \leqq f \leqq \infty.$$

Noise with this spectrum will be referred to as RLC Noise (see 5.2). The figures are respectively for the cases $Q = 1$, 10 and 100. The curves are labelled by values of $s = \omega_0 T$. The curves marked $s = 0$ are the density function for $y = N^2(t)$. The corresponding cumulative density

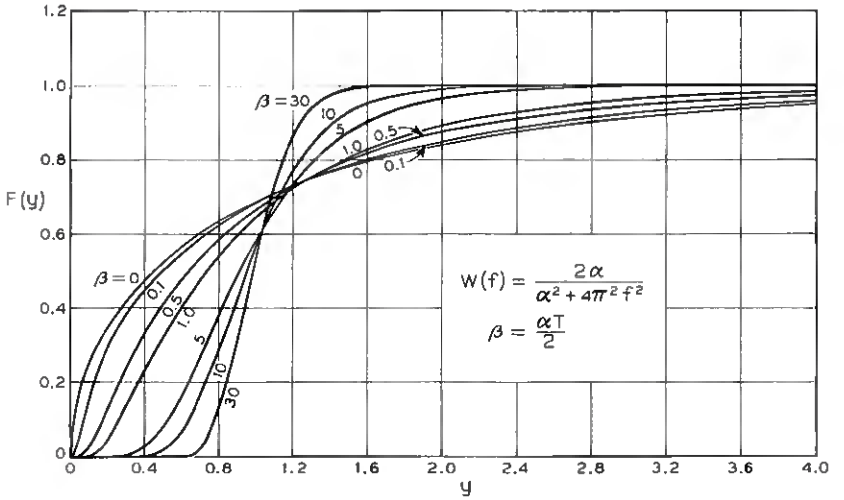

Fig. 2 — Cumulative distribution, $F(y)$, for RC noise.

functions, $F(y)$, are shown on Figs. 6, 7 and 8. The spectra for $Q = 1$, 10 and 100 are plotted on Fig. 9.

For any $s > 0$ and for any finite $Q > 0$, as y approaches zero, both $f(y)$ and $F(y)$ approach zero more rapidly than any power of y.

For any fixed Q, as s becomes large, the density function $f(y)$ peaks up around unity which is the average power of $N(t)$. The variance of y is given by

$$\sigma^2 = \frac{2}{\tau^2}\left[\tau - 1 + e^{-\tau} + \frac{2e^{-\tau}}{4Q^2 - 1}\sin^2\frac{\tau}{2}\sqrt{4Q^2 - 1}\right], \qquad \tau = \frac{s}{Q}. \tag{3}$$

For fixed Q, it approaches zero for large s like $2Q/s$.

If, however, $s = \omega_0 T$ is held fixed and Q is permitted to increase, Figs. 3, 4 and 5 show that $f(y)$ becomes less concentrated; that is, with fixed integration time and fixed resonant frequency, fluctuations in power become more pronounced as the relative width of the spectral peak is decreased. Indeed, one has

$$\lim_{Q\to\infty} \sigma^2 = 1 + \frac{\sin^2 s}{s^2},$$

so that

$$\lim_{s\to\infty}\lim_{Q\to\infty}\sigma^2 = 1,$$

whereas, as already noted,

$$\lim_{Q\to\infty}\lim_{s\to\infty}\sigma^2 = 0.$$

In the limit $Q = \infty$, the Gaussian noise can be taken to be the single frequency ensemble $N(t) = a\cos\omega_0 t + b\sin\omega_0 t$, where a and b are independent normal variates with mean zero and variance unity. The density for y in this case is

$$f(y) = \sec\varphi e^{-y\sec^2\varphi} J_0(iy\tan\varphi\sec\varphi)$$

where $\sin\varphi = \sin s/s$ and J_0 is the usual Bessel function (see Appendix 1). This density is plotted for several values of s in Fig. 10. It is to be noted that this limiting noise, although stationary, is not ergodic. It is this fact that causes the variance of y to be bounded away from zero as $s \to \infty$. Quite generally, if $N(t)$ has a purely continuous spectrum, the variance of y will approach zero as the integration time becomes infinite. If the spectrum of $N(t)$ has line components, this will not be the case.

It is not difficult to give a qualitative argument as to why power fluctuations in a fixed time interval increase as the power spectrum becomes

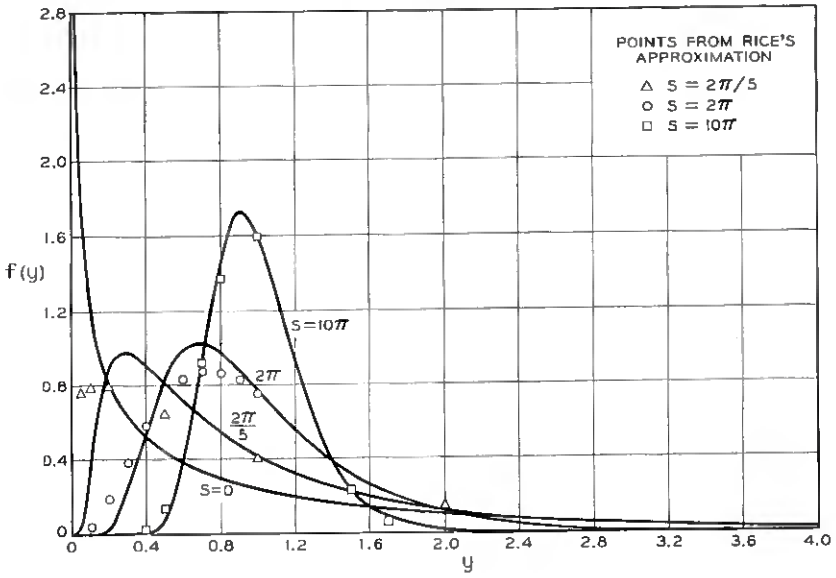

Fig. 3 — Probability density, $f(y)$, for RLC noise, $Q = 1.0$, $s = \omega_0 T$.

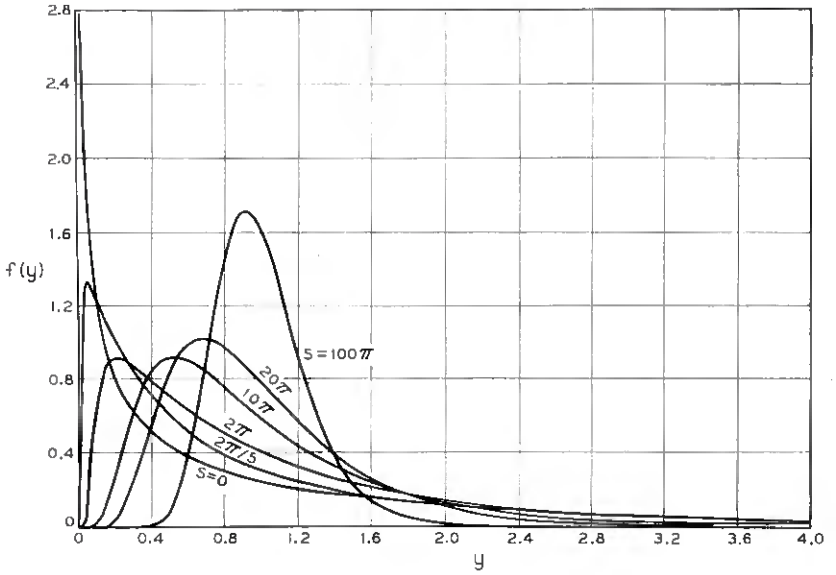

Fig. 4 — Probability density, $f(y)$, for RLC noise, $Q = 10$, $s = \omega_0 T$.

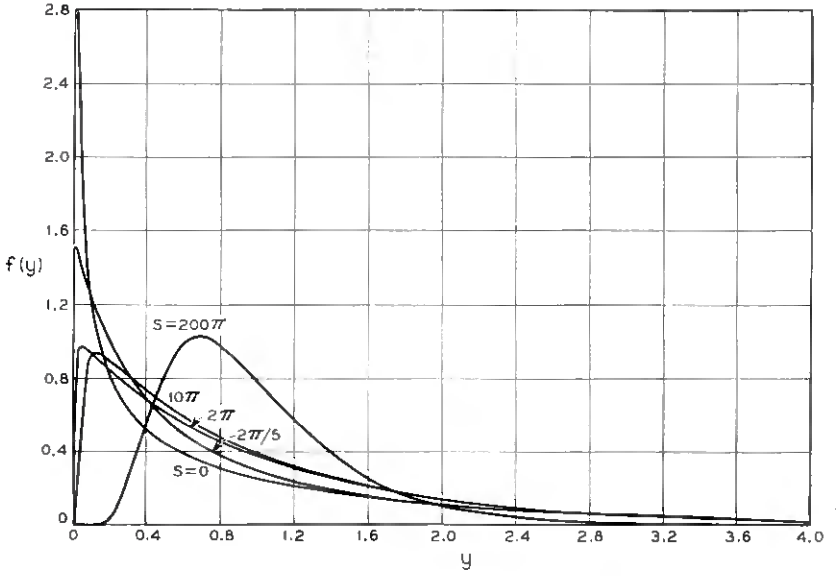

Fig. 5 — Probability density, $f(y)$, for RLC noise, $Q = 100$, $s = \omega_0 T$.

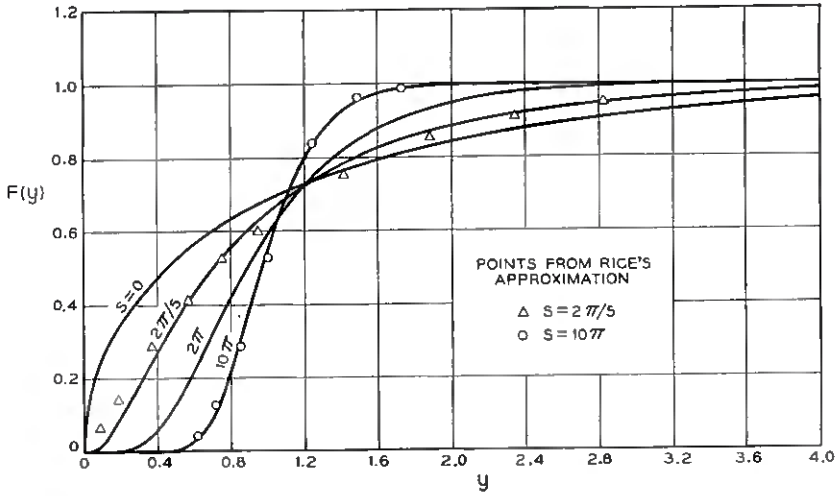

Fig. 6 — Cumulative distribution, $F(y)$, for RLC noise, $Q = 1.0$, $s = \omega_0 T$.

more peaked. Noise with the power spectrum (2) can be thought of as the noise voltage produced across the resistor in a series RLC circuit when the applied voltage to the circuit is white Gaussian noise. The larger the Q of the circuit, the more it tends to "ring" in response to an impulse input; i.e., the longer the transients persist. An atypical excur-

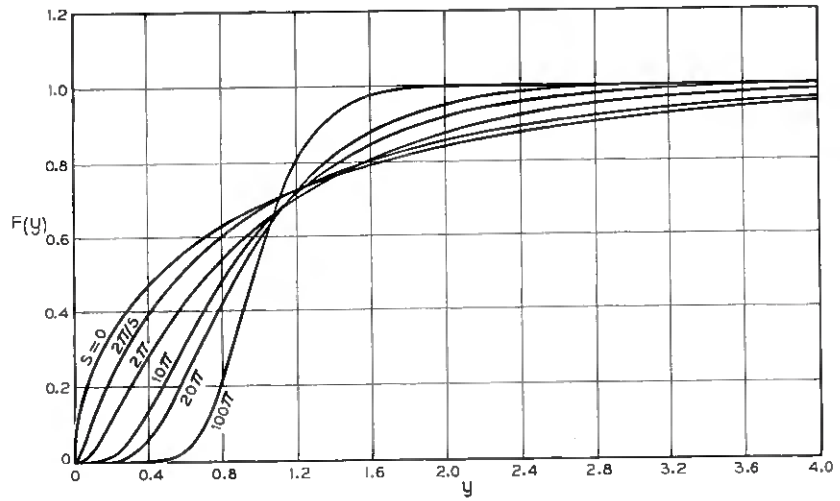

Fig. 7 — Cumulative distribution, $F(y)$, for RLC noise, $Q = 10$, $s = \omega_0 T$.

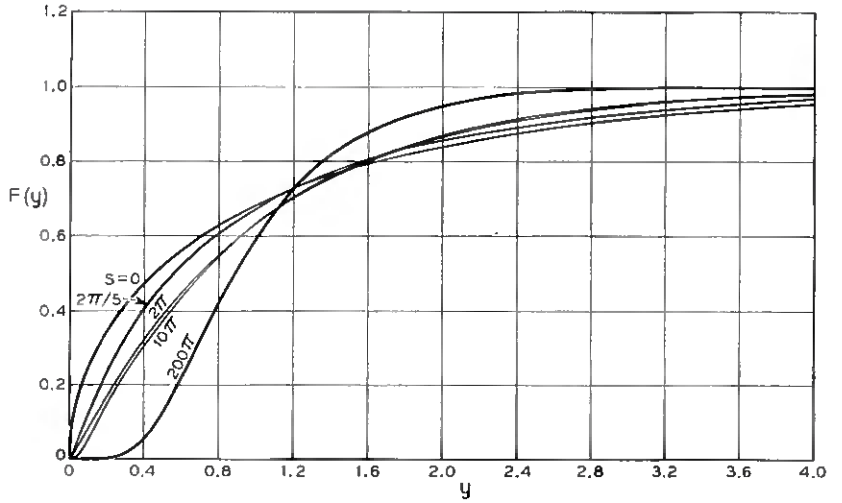

Fig. 8 — Cumulative distribution, $F(y)$, for RLC noise, $Q = 100$, $s = \omega_0 T$.

sion of the input voltage will therefore have a longer lasting effect in the output of a circuit with a large Q than in a circuit with a small Q. To obtain the same variance in power, then, the integration time must be longer in the circuit with the large Q value. It would seem reasonable to expect this argument to apply for any peaked spectrum, not solely for (2).*

If the Q of the spectrum (2) is increased, how much must the integration time be increased to maintain roughly the same power fluctuations? From (3), it is seen that for large Q, σ^2 is approximately $2\tau^2[\tau - 1 + e^{-\tau}]$, i.e., a function of

$$\tau = \frac{s}{Q} = \frac{\omega_0}{Q} T$$

alone. Now Q measures the relative sharpness of the spectral peak, so that ω_0/Q is a measure of the absolute width of the peak in radians/sec. As a rough rule, then, power measurements from different members of the family (2) will have the same fluctuations if their products "integration time" times "absolute spectral bandwidth" are the same. Fig. 11 shows σ^2 as a function of τ for $Q = 1$, 10, and 100. That τ is a good measure of the fluctuation in power can also be seen by comparing the f curves of equal τ value in Figs. 3, 4 and 5. They are almost identical.

* It seems to be very difficult to make any other qualitative statements regarding the relation between the shape of the noise spectrum and the density function for y.

On Fig. 11 the variance of y for bandpass noise with spectrum

$$w(f) = \begin{cases} \dfrac{1}{4\delta}, & | f \pm f_0 | \leqq \delta \\ 0, & | f \pm f_0 | > \delta \end{cases}$$

is plotted versus

$$\tau = \frac{s}{Q_b} = \frac{2\pi f_0 t}{Q_b}.$$

Here the Q of a bandpass circuit is defined by

$$Q_b = \pi \frac{f_0}{2\delta}$$

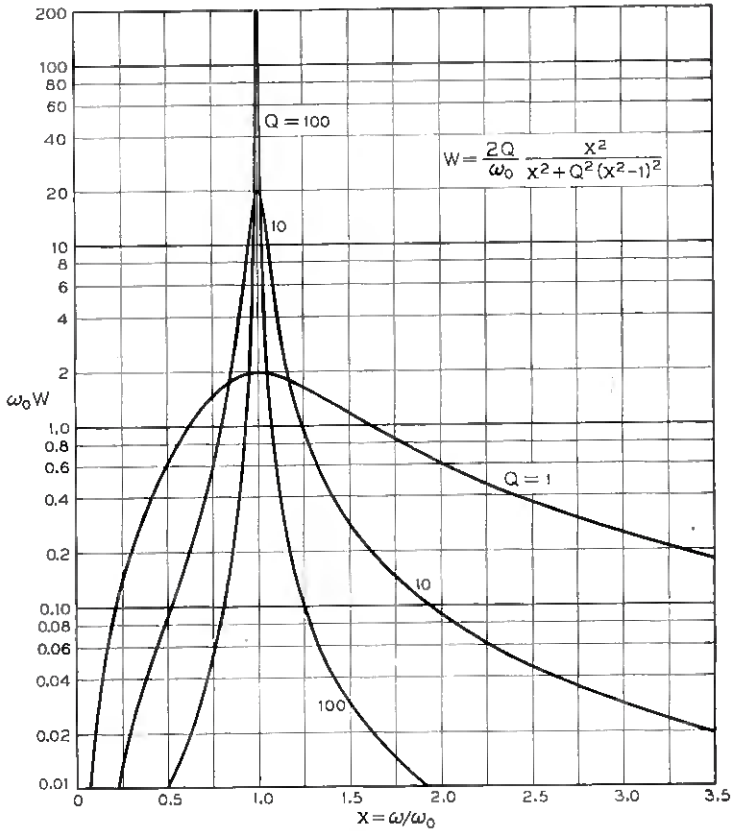

Fig. 9 — RLC spectra, Q = 1, 10, 100.

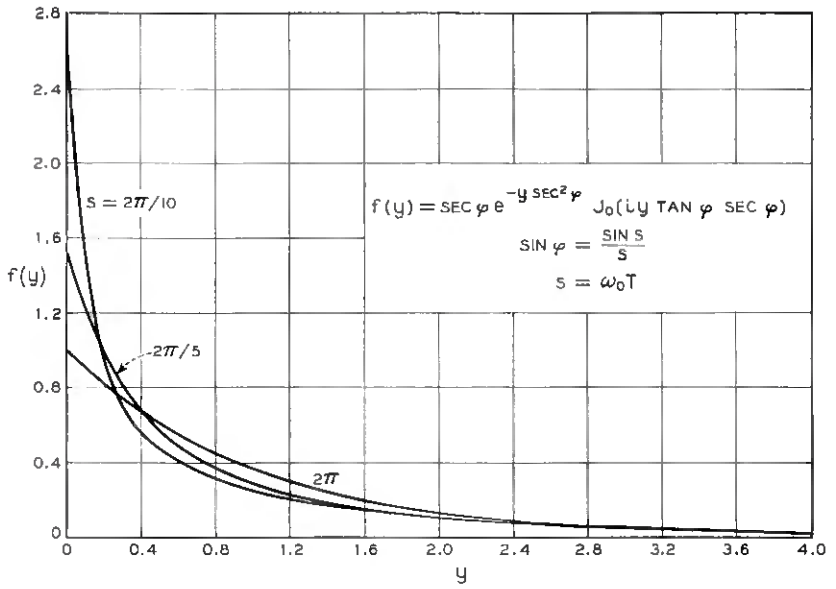

Fig. 10 — Probability density, $f(y)$, for RLC noise, $Q = \infty$.

and measures the relative width of the spectrum. This definition of Q_b causes the σ^2 curves of this noise power to agree asymptotically with those of the RLC noise power; namely $\sigma^2 \sim 2/\tau$ in both cases. Again, when it is not too small, τ seems to be a good measure of power fluctuations. The variance in this bandpass case is given by

$$\sigma_b^2 = 4 \int_0^1 (1 - y) \left[\frac{\cos Q_b \tau y \sin \frac{\pi \tau y}{2}}{\frac{\pi \tau y}{2}} \right]^2 dy$$

which can be readily evaluated in terms of Si and Ci functions. The curve for $Q_b = 100$ coincides so closely with the curve for $Q_b = 10$ it could not be shown on Fig. 11.

The asymptotic agreement of the variance of noise power for bandpass and RLC noise permitted defining the Q of the bandpass circuit as $Q_b = \pi(f_0/2\delta)$. These same considerations suggest defining the bandwidth W of the RLC spectrum by $W = \omega_0/2Q$. For, in the bandpass case, $\tau = 2(2\delta)T$ which is $2T$ times the bandwidth of the spectrum. For the RLC noise, $\tau = \omega_0 T/Q = 2T(\omega_0/2Q)$, whence the definition of W follows.

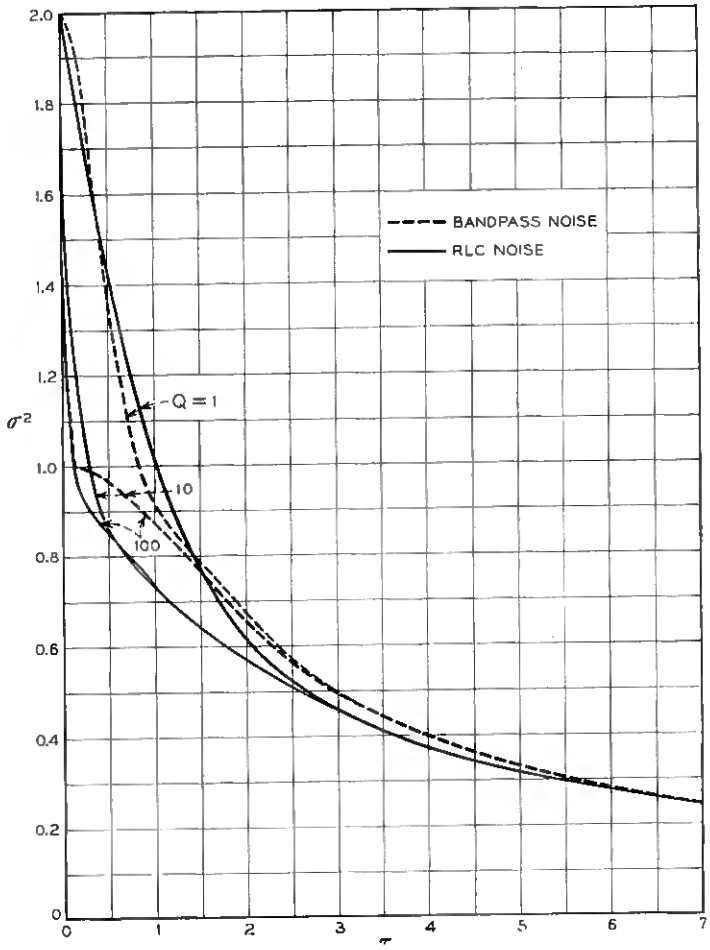

Fig. 11 — σ^2 for RLC noise and bandpass noise, $Q = 1, 10, 100$.

The curves shown in Figs. 1–8 are believed to be accurate to two significant figures. For comparison, some points computed from Rice's approximate formula for $f(y)$ (equation 3.9–20 of [Ref. 1]) are shown on Fig. 3. Rice's formula is seen to fit the tails of $f(y)$ well for large y, but the central portion of the distribution is given accurately only for large values of τ. However, the approximate cumulative distribution obtained by integrating Rice's formula agrees quite well with $F(y)$ for a wide range of τ values as is seen in Fig. 6.

The approximation in question assumes a χ^2 type distribution

$$\tilde{f}(y) = \frac{y^{(n/2)-1}e^{-(y/2q^2)}}{(2q^2)^{n/2}\Gamma\left(\frac{n}{2}\right)}.$$

The parameters q and n are chosen to make the first two moments of this density agree with the true first two moments of y. That is, for the normalization $Ey = 1$ adopted here, the equations $q^2n = 1$ and $2nq^4 = \sigma^2$ serve to determine q and n. These formulae give $n = 2/\sigma^2$. Since for large τ, $\sigma^2 \sim 2/\tau$ for bandpass noise, $n \sim \tau = 2(2\delta)T$. That is, for large τ, the bandpass noise acts like a χ^2 variate with $2(2\delta)T$ degrees of freedom in agreement with an argument easily derived from the sampling theorem.

III. GENERAL THEORY

Let $N(t)$ be a Gaussian noise with mean zero and covariance

$$\rho(t, t') = E[N(t)N(t')]$$

where as usual E denotes expectation. In studying properties of $N(t)$ in a finite time interval, say $(-T/2, T/2)$, it is convenient to make an expansion in terms of an orthonormal set of functions, $\varphi_n(t)$, $n = 0, 1, 2, \ldots$. We write

$$N(t) = \sum_0^\infty n_i\varphi_i(t), \qquad |t| \leqq \frac{T}{2}$$

where

$$n_i = \int_{-T/2}^{T/2} N(t)\varphi_i(t)\, dt, \qquad i = 0, 1, 2, \cdots$$

and

$$\int_{-T/2}^{T/2} \varphi_i(t)\varphi_j(t)\, dt = \delta_{ij}, \qquad i, j = 0, 1, 2, \cdots.$$

As is well known,[2] it is particularly convenient in this description of the noise to choose as the orthonormal set, φ_i, the solutions of the homogeneous Fredholm equation with $\rho(t, t')$ as kernel. That is, the φ's are chosen so that

$$\lambda_i\varphi_i(t) = \int_{-T/2}^{T/2} \rho(t, t')\varphi_i(t')\, dt', \qquad |t| \leqq \frac{T}{2}, \qquad i = 0, 1, 2, \cdots. \quad (4)$$

For, with this choice of the φ's, it is easily shown that the n_i are *independent* Gaussian variates with mean zero and variance $E(n_i^2) = \lambda_i$, $i = 0, 1, 2, \ldots$. We assume in all that follows that the λ's are so labelled that $\lambda_0 \geqq \lambda_1 \geqq \lambda_2 \geqq \ldots$.

Consider now the average power, y, of a finite sample of the noise. It follows that

$$\begin{aligned} y &= \frac{1}{T}\int_{-T/2}^{T/2} N^2(t)\,dt = \frac{1}{T}\sum_0^\infty n_i^2 \\ &= \sum_0^\infty a_i x_i^2, \end{aligned} \tag{5}$$

where

$$x_i = \frac{n_i}{\sqrt{\lambda_i}}$$

and

$$a_i = \frac{\lambda_i}{T}. \tag{6}$$

Equation (5) exhibits y as a linear combination of independent random variables. The x_i are independent Gaussian variables all with mean zero and variance unity. The characteristic function, $C(\xi)$, for y then follows readily. One has

$$\begin{aligned} C(\xi) &= Ee^{i\xi y} = Ee^{i\xi\Sigma a_j x_j^2} = \prod_{j=0}^\infty Ee^{i\xi a_j x_j^2} \\ &= \prod_{j=0}^\infty (1 - 2i\xi a_j)^{-1/2}. \end{aligned} \tag{7}$$

Here, as throughout this paper, the positive square root of a complex quantity is taken to have an angle between $-(\pi/2)$ and $+(\pi/2)$ radians (the cut line is along the negative real axis).

From the characteristic function (7), the semi-invariants of y can be calculated. By definition[3] of the semi-invariants, κ_j,

$$\log C(\xi) = \sum_1^\infty \frac{\kappa_\nu}{\nu!}(i\xi)^\nu.$$

From (7) and the expansion

$$\log (1 - x) = -\sum_{n=1}^\infty \frac{x^n}{n},$$

it follows that

$$\log C(\xi) = -\frac{1}{2}\sum_{j=0}^{\infty}\log(1 - 2i\xi a_j) = \frac{1}{2}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{(2i\xi a_j)^k}{k}$$

$$= \sum_{k=1}^{\infty}\frac{(i\xi)^k}{k!}\kappa_k$$

where

$$\kappa_k = (k-1)!\,2^{k-1}\textstyle\sum_{j=0}^{\infty} a_j^{\,k}. \tag{8}$$

From the semi-invariants, the moments of y can be found as in Reference 3.

The formula (8) for the semi-invariants can be put in a convenient form not involving the a_j explicitly. From the well known expansion[4]

$$\rho(t, t') = \textstyle\sum_0^{\infty}\lambda_j\varphi_j(t)\varphi_j(t')$$

and the orthonormal properties of the φ's, one finds

$$\kappa_k = \frac{(k-1)!\,2^{k-1}}{T^k}\int_{-T/2}^{T/2}\rho^{(k)}(t, t)\,dt, \tag{9}$$

where the iterated kernel $\rho^{(k)}(t, t')$ is defined by

$$\rho^{(1)}(t, t') = \rho(t, t'),$$

$$\rho^{(n)}(t, t') = \int_{-T/2}^{T/2}\rho(t, x)\rho^{(n-1)}(x, t')\,dx, \qquad n = 2, 3, \cdots.$$

The determination of the higher order iterated kernels generally becomes difficult in practice.

The expression (9) is of the form conjectured by Rice[1] on the basis of computing the first four semi-invariants of y. The formula (7) was given by Kac and Siegert[2] and (9) was noted by Arthur[5] in a special case in connection with the analysis of a frequency discriminator.

The probability density function for y is obtained as the Fourier transform of $C(\xi)$,

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\xi y}\,d\xi}{\prod_{j=0}^{\infty}(1 - 2i\xi a_j)^{1/2}} \tag{10}$$

and the cumulative distribution function can be written as

$$F(y) = 1 - \int_{y}^{\infty} f(x)\,dx. \tag{11}$$

Much of the remainder of this paper will be concerned with evaluating (10) and (11) for specific noises.

IV. COMPUTATIONAL FORM FOR $f(y)$

The evaluation of the integral (10) presents many difficulties even with modern computing machinery. From the physical origins of the problem under discussion, it is clear that for small values of T, $f(y)$ must be a rather broad function (non-localized), whereas for large values of T it must approach a δ-function centered at the point $y = \rho(0, 0)$ when the noise is assumed ergodic. The behavior of (10) therefore depends in detail on the manner in which the a_j approach zero with increasing j.

One seemingly attractive approach to the problem is to truncate the sum in (5) at $i = M$ and correspondingly obtain a product with j running from 0 to M in the denominator of the integral in (10). Procedures are described in the literature[6, 7] for computing the distribution of a finite quadratic form in Gaussian variables. Estimates of the error due to truncation can also be obtained rather readily. Unfortunately, the best such estimates obtained by the author showed that for small values of β or τ, M must be taken quite large (50 or 60) to obtain answers guaranteed accurate to two decimal places. Furthermore, the convergence of the computational schemes described[6, 7] turned out to be very slow. The

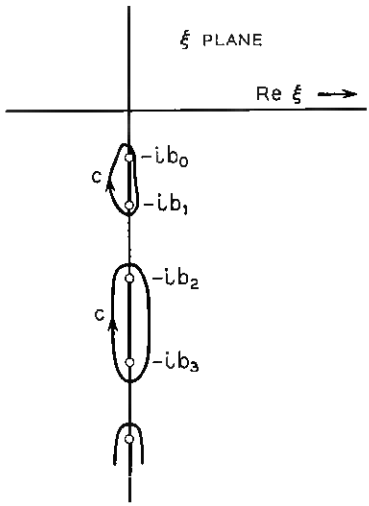

Fig. 12 — Cut lines and contour in complex ξ plane.

following alternative approach which can also be applied to finite sums of the form (5) was used to obtain the curves presented here.

The function $\Pi(1 - 2i\xi a_j)^{1/2}$ in the denominator of (10) has branch points at $-ib_j$, where

$$b_j = \frac{1}{2a_j} = \frac{T}{2\lambda_j}, \qquad j = 0, 1, 2, \ldots. \quad (12)$$

The b's are real positive quantities, for the λ's are eigenvalues of a real symmetric positive definite kernel and must be real positive numbers. Line segment cut-lines are inserted in the complex ξ-plane from $-ib_{2j}$ to $-ib_{2j+1}$, $j = 0, 1, 2, \ldots$ as shown in Fig. 12. When $y < 0$, the value of (10) is zero as can be seen by closing the contour in the upper half plane. When $y \geqq 0$, the contour of integration in (10) is displaced from the axis of reals to the contour, C, shown in Fig. 12. This displacement of contour is easily justified if $\Pi(1 - 2i\xi a_j)^{1/2}$ is of exponential order less than unity, a condition which will be fulfilled in the examples to be treated. The change of variable $\zeta = i\xi$ rotates the contour of Fig. 12 by 90° in the positive direction. If one now collapses the closed contour curves about the cut-lines and takes proper care of the convention already set forth for the square root sign, there results,

$$f(y) = \sum_{k=0}^{\infty}(-1)^k I_k$$

where

$$I_k = \frac{1}{\pi}\int_{b_{2k}}^{b_{2k+1}} \frac{e^{-yt}\,dt}{\sqrt{|\,D(t)\,|}}, \qquad k = 0, 1, 2, \ldots$$

and where

$$D(t) = \prod_{j=0}^{\infty}\left(1 - \frac{t}{b_j}\right). \quad (13)$$

$D(t)$ is closely related to the Freeholm determinant (Reference 4, Chapter 11) of ρ.

In the application to be treated below,

$$D(t) = H(z) \quad (14)$$

where

$$z = g(t) \quad (15)$$

is a non-negative monotone increasing real function of t for $t \geqq b_0$. Denote its inverse by $t = h(z)$. Let

$$z_k = g(b_k)$$
$$c_k = \tfrac{1}{2}(z_{2k+1} - z_{2k})$$
$$d_k = \tfrac{1}{2}(z_{2k+1} + z_{2k})$$

for $k = 0, 1, 2, \ldots$ and let

$$z = c_k \cos \pi x + d_k .$$

Then straightforward substitution yields

$$I_k = \int_0^1 \frac{e^{-yh(z)}h'(z)\,dx}{\sqrt{\left|\dfrac{H(z)}{(z - z_{2k})(z_{2k+1} - z)}\right|}}, \qquad k = 0, 1, 2, \ldots, \tag{16}$$

$$f(y) = \sum_{k=0}^{\infty}(-1)^k I_k . \tag{17}$$

Similarly, one obtains

$$F(y) = 1 - \sum_{k=0}^{\infty}(-1)^k J_k \tag{18}$$

with

$$J_k = \int_0^1 \frac{e^{-yh(z)}h'(z)\,dx}{h(z)\sqrt{\left|\dfrac{H(z)}{(z - z_{2k})(z_{2k+1} - z)}\right|}}. \tag{19}$$

Equations (16) to (19) were used to compute the curves discussed in Section II. The denominators of the integrals in (16) and (19) have no zeros in the range of integration. By use of Gauss's method of numerical integration,[8] evaluation of the integral at $x = 0$ and $x = 1$ where the denominator is an indeterminate form was avoided. In the applications made, it can be shown that for sufficiently large k, I_k and J_k decrease monotonely. Since the series (17) and (18) are alternating, an estimate of the error made by terminating the series at a finite value of k can be obtained. In all cases computed, it was never necessary to take k larger than 18, to obtain 1 per cent accuracy in the final result.

V. DETERMINATION OF EIGENVALUES AND $H(z)$*

For stationary processes, the kernel of the integral equation (4) becomes a difference kernel; that is, $\rho(t, t') = \rho(t - t')$ where $\rho(x)$ is a positive definite function. The Fourier transform of ρ, namely

$$w(f) = \int_{-\infty}^{\infty} e^{2\pi i f x}\rho(x)\,dx$$

is non-negative and is the power density spectrum of the processes.

* An alternative method of evaluating $H(z)$ is described in Reference 12.

Analytic solutions to the integral equation (4) are known in this case only for a relatively small class of kernels. Fortunately, this class is one of considerable interest in communication applications. It is the class of ρ whose spectra $w(f)$ are rational functions of f^2; i.e., ratios of polynominals in f^2. Such spectra are obtained by passing white noise through a finite passive physical electrical network with lumped constants. Details of the method of solution are given in References 9 and 10. It must be pointed out that, even in this case, solutions can be carried out practically only for polynomials of small degree.

5.1 *RC Noise*

If white Gaussian noise is applied to a series RC circuit, the voltage across the capacitor has a power density spectrum proportional to

$$w(f) = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2} \tag{20}$$

where $\alpha = 1/\mathrm{RC}$ is the nominal cut-off frequency of the circuit. The covariance function corresponding to (20) is

$$\rho(t) = e^{-\alpha|\tau|}. \tag{21}$$

Solutions to (4) with this kernel are given in detail in both References 9 and 10.

Let

$$\beta = \frac{\alpha T}{2}. \tag{22}$$

Then

$$b_k = \frac{1}{2\beta}[\beta^2 + z_k^2], \qquad k = 0, 1, 2, \ldots, \tag{23}$$

where the z_k are non-negative roots of either of the equations

$$z \tan z = \beta \tag{24}$$

$$z \cot z = -\beta. \tag{25}$$

If the z_k are labelled so that $z_0 < z_1 < z_2 < \ldots$, then it is readily seen that $z_k \sim k(\pi/2)$, so that $b_k \sim k^2(\pi^2/8\beta)$. The convergence exponent (see Reference 11, p. 14) of the sequence b_k is therefore $\frac{1}{2}$. It follows then (Reference 11, 2.6.5, p. 19) that $D(t)$ as given by (13) is an entire function of order $\frac{1}{2}$.

Now the function $(e^{-2\beta}/\beta)[\beta \cos z + z \sin z][\cos z + \beta(\sin z/z)]$, where $z = \sqrt{2\beta t - \beta^2}$, is an entire function of t of order $\frac{1}{2}$. Its only zeros are

at the points $t = b_k$, $k = 0, 1, 2, \ldots$, given by (23), (24) and (25). At $t = 0$, it has the value unity. It is, therefore, equal to $D(t)$ as can be seen from Hadamard's Factorization Theorem (Reference 11, 2.7.1, p. 22).

The quantities necessary to evaluate (16) and (19) are therefore all known for this case:

$$H(z) = \frac{e^{-2\beta}}{\beta}[\beta \cos z - z \sin z]\left[\cos z + \beta \frac{\sin z}{z}\right]$$

$$h(z) = \frac{1}{2\beta}[\beta^2 + z^2]$$

and the z_k are given by the positive roots of (24) and (25).

The first two semi-invariants of y are found to be

$$\kappa_1 = Ey = 1,$$

$$\kappa_2 = E(y-1)^2 = \sigma^2 = \frac{1}{4\beta^2}[4\beta - 1 + e^{-4\beta}].$$

5.2 *RLC Noise*

If white Gaussian noise is applied to a series RLC circuit, the voltage across the resistor has a power density spectrum proportional to

$$w(f) = \frac{2Q}{\omega_0} \frac{\omega^2}{\omega^2 + \left(\frac{Q}{\omega_0}\right)^2 (\omega^2 - \omega_0^2)^2} \tag{26}$$

where $\omega = 2\pi f$, $Q = \omega_0 L/R$ and $\omega_0^2 = 1/LC$. Introducing parameters u and v defined by

$$u^2 + v^2 = \omega_0^2\left[\frac{1}{Q^2} - 2\right]$$

$$uv = \omega_0^2, \qquad \text{Re } u \geqq 0, \qquad \text{Re } v \geqq 0,$$

one finds

$$w(f) = \frac{2(u+v)\omega^2}{(u^2+\omega^2)(v^2+\omega^2)}$$

and

$$\rho(\tau) = \frac{1}{u-v}[ue^{-u|\tau|} - ve^{-v|\tau|}]. \tag{27}$$

In the special case $Q = \frac{1}{2}$,

$$\rho(\tau) = (1 - \omega_0 \mid \tau \mid)e_0^{-\omega_0 |\tau|}. \tag{28}$$

Solution of (4) with (27) or (28) as kernel is relatively straightforward by the methods of References 9 and 10, although quite laborious. Details can be found in Appendix 2.

Suppose Q and ω_0 are positive real quantities. Then the eigenvalues $\lambda_k = T/2b_k$ are given by

$$b_k = \frac{1}{2r}[r^2 + z_k^2], \qquad k = 0, 1, 2, \ldots \tag{29}$$

where the z_k are non-negative roots of either

$$2r \cos z = (z^2 - r^2)\frac{\sin z}{z} + (z^2 + r^2)\frac{\sin \sqrt{z^2 + s^2}}{\sqrt{z^2 + s^2}} \tag{30}$$

or

$$2r \cos z = (z^2 - r^2)\frac{\sin z}{z} - (z^2 + r^2)\frac{\sin \sqrt{z^2 + s^2}}{\sqrt{z^2 + s^2}}. \tag{31}$$

Here

$$r = \frac{\omega_0 T}{2Q}, \qquad s = \omega_0 T. \tag{32}$$

The eigenfunctions belonging to roots of (30) are of the form

$$A_k \cos \tfrac{1}{2}(z_k + \sqrt{z_k^2 + s^2})t + B_k \cos \tfrac{1}{2}(z_k - \sqrt{z_k^2 + s^2})t$$

while those belonging to roots of (31) are of the form

$$C_k \sin \tfrac{1}{2}(z_k + \sqrt{z_k^2 + s^2})t + D_k \sin \tfrac{1}{2}(z_k - \sqrt{z_k^2 + s^2})t.$$

It is interesting to note that when the λ's are ordered in the usual way, the corresponding eigenfunctions do not in general alternate between even and odd functions of t.

The infinite product (13) with the b's given by (29), (30) and (31) can be written in closed form by arguments similar to those used in Section 5.1. From (30) and (31), it is seen that asymptotically successive z_k are separated by $\pi/2$, so that b_k grows like k^2 and one is again dealing with an entire function of order $\frac{1}{2}$.* For the pertinent quantities

* More generally, it can be shown that for rational spectra if $w(f) \sim f^{-2p}$ then $\lambda_n \sim n^{-2p}$. (Private communication to author by A. Beurling.)

of (16) and (19), one finds

$$H(z) = \frac{e^{-2r}}{4r^2}\left[(z^2 - r^2)\frac{\sin z}{z} - 2r\cos z + (z^2 + r^2)\frac{\sin\sqrt{z^2+s^2}}{\sqrt{z^2+s^2}}\right] \quad (34)$$

$$\cdot\left[(z^2 - r^2)\frac{\sin z}{z} - 2r\cos z - (z^2 + r^2)\frac{\sin\sqrt{z^2+s^2}}{\sqrt{z^2+s^2}}\right],$$

$$h(z) = \frac{1}{2r}[z^2 + r^2] \quad (35)$$

with the z_k given as roots of (30) and (31).

The first two semi-invariants of y are

$$u_1 = Ey = 1$$

$$u_2 = \sigma^2 = \frac{1}{2r^2}\left[2r - 1 + e^{-2r} + 2\frac{r^2e^{-2r}}{r^2 - s^2}\sinh^2\sqrt{r^2 - s^2}\right].$$

VI. ACKNOWLEDGEMENTS

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APPENDIX 1

Let $N(t) = a\cos\omega_0 t + b\sin\omega_0 t$, where a and b are independent Gaussian variates with mean zero and variance unity. Then y as defined by (1) is obtained by direct integration as

$$y = \alpha a^2 + \beta b^2$$

where

$$\alpha = \frac{1}{2}\left(1 + \frac{\sin s}{s}\right), \qquad \beta = \frac{1}{2}\left(1 - \frac{\sin s}{s}\right), \qquad s = \omega_0 T.$$

Since y is the sum of independent random χ^2 variables, the density for y can be obtained as the convolution

$$f(y) = \int_0^y \frac{e^{-(x/2\alpha)}}{\sqrt{2\pi\alpha x}}\frac{e^{-((y-x)/2\beta)}}{\sqrt{2\pi\beta(y-x)}}\,dx.$$

The substitution $x = (y/2)(1 + \cos\theta)$ in this integral leads to

$$f(y) = \frac{e^{-(y/4)[(1/\alpha)+(1/\beta)]}}{2\sqrt{\alpha\beta}} \frac{1}{\pi} \int_0^{\pi} e^{(y/4)(\alpha^{-1}-\beta^{-1})\cos\theta}\, d\theta$$

$$= \frac{e^{-(y/4)[(1/\alpha)+(1/\beta)]}}{2\sqrt{\alpha\beta}} \, J_0\left[i\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{y}{4}\right].$$

Finally, if $\sin\varphi = \sin s/s$,

$$f(y) = \sec\varphi e^{-y\sec^2\varphi} J_0(iy \tan\varphi \sec\varphi).$$

APPENDIX 2

The power spectrum corresponding to the covariance (27) can be written as

$$w = -\frac{2(u+v)p^2}{(p^2-u^2)(p^2-v^2)}$$

where $p = i\omega = 2\pi if$. From Reference 9, then, solutions to (4) with the kernel (27) must satisfy the differential equation

$$\left(\frac{d^2}{dt^2} - u^2\right)\left(\frac{d^2}{dt^2} - v^2\right)\varphi(t) = -\frac{2(u+v)}{\lambda}\frac{d^2}{dt^2}\varphi(t)$$

or

$$\left(\frac{d^2}{dt^2} - \alpha^2\right)\left(\frac{d^2}{dt^2} - \beta^2\right)\varphi(t) = 0, \tag{i}$$

where

$$\alpha^2 + \beta^2 = u^2 + v^2 - \frac{2(u+v)}{\lambda},$$

$$\alpha^2\beta^2 = u^2v^2 = \omega_0^4.$$

We choose α and β so that Re $\alpha \geqq 0$, Re $\beta \geqq 0$. If $\alpha \neq \beta$, then φ is a linear combination of the elementary functions $e^{\alpha t}$, $e^{-\alpha t}$, $e^{\beta t}$, $e^{-\beta t}$.

It is easy to verify that if φ is a solution to (4) with a kernel $\rho(t, t') = \rho(|t - t'|)$, then $\varphi(t) + \varphi(-t)$ and $\varphi(t) - \varphi(-t)$ are also solutions. We can, therefore, restrict attention to even and odd solutions of (4). On substituting

$$\varphi(t) = A \cosh \alpha t + B \cosh \beta t$$

into (4), one finds

$$\frac{\lambda}{T} = -\frac{\alpha^2 r}{(u^2-\alpha^2)(v^2-\alpha^2)} = -\frac{\beta^2 r}{(u^2-\beta^2)(v^2-\beta^2)}, \tag{ii}$$

$$A\frac{u\cosh\frac{\alpha T}{2}+\alpha\sinh\frac{\alpha T}{2}}{u^2-\alpha^2}+B\frac{u\cosh\frac{\beta T}{2}+\beta\sinh\frac{\beta T}{2}}{u^2-\beta^2}=0,$$
$$A\frac{v\cosh\frac{\alpha T}{2}+\alpha\sinh\frac{\alpha T}{2}}{v^2-\alpha^2}+B\frac{v\cosh\frac{\beta T}{2}+\beta\sinh\frac{\beta T}{2}}{v^2-\beta^2}=0. \quad \text{(iii)}$$

The determinant of the system (iii) must vanish. A bit of algebra shows this to be equivalent to

$$2r\cosh x+(x^2+r^2)\frac{\sinh x}{x}+(x^2-r^2)\frac{\sinh\sqrt{x^2-s^2}}{\sqrt{x^2-s^2}}=0\,, \quad \text{(iv)}$$

where $x = (\alpha + \beta)(T/2)$. It is not difficult to show that for positive ω_0 and Q, this equation has roots only if α and β, and hence x, are pure imaginary. Writing $x = iz$, (iv) become (30) and (ii) yields (29).

The substitution of

$$\varphi(t) = C\sinh\alpha t + D\sinh\beta t$$

into (4) again yields (ii) and equations analogous to (iii) with sinh and cosh interchanged. A similar analysis then gives (31).

If $\alpha = \beta$, then from (i), φ must be of the form $A\cosh\alpha t + Bt\sinh\alpha t$ or $C\sinh\alpha t + Dt\cosh\alpha t$. Substitution of these forms into (4) yields equations which cannot be satisfied for positive ω_0 and Q except by the trivial solution $A = B = C = D = 0$.

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The Measurement of Power Spectra from the Point of View of Communications Engineering — Part I

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The measurement of power spectra is a problem of steadily increasing importance which appears to some to be primarily a problem in statistical estimation. Others may see it as a problem of instrumentation, recording and analysis which vitally involves the ideas of transmission theory. Actually, ideas and techniques from both fields are needed. When they are combined, they provide a basis for developing the insight necessary (i) to plan both the acquisition of adequate data and sound procedures for its reduction to meaningful estimates and (ii) to interpret these estimates correctly and usefully. This account attempts to provide and relate the necessary ideas and techniques in reasonable detail. Part II of this article will appear in the March issue of THE JOURNAL.

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I. INTRODUCTION

Communications systems and data-processing systems are generally required to handle a large variety of signals in the presence of noise. The design of these systems depends to a large extent upon the statistical properties of both the signals and the noise. In most cases, the noises may be represented, or approximated, as stationary Gaussian random processes with zero averages, so that all of their relevant statistical properties will be contained by the autocovariance function or the power spectrum. In many cases, the signals may also be represented, or approximated, as stationary Gaussian random processes with zero averages.

Noises, signals, or other ensembles of functions (given continuously or at intervals) which are approximately stationary but not Gaussian are often also usefully studied in terms of autocovariance functions or power spectra. Although the average and the spectrum are no longer the *only* relevant statistical properties, they are usually the most useful ones. Thus, we shall do well to keep as much of our treatment generally applicable as possible.

In almost every case, the autocovariance function or power spectrum of either the noise or the signal will be of interest and importance.

To determine the autocovariance function or power spectrum of an (approximately) stationary random process, we are often reduced to the necessity of measurement and computation. Exact determination

would require a perfectly-measured, infinitely-long piece of a random function (or a collection of pieces of infinite total length), and would require infinitely detailed computations. Both of these requirements are, of course, impractical. Approximate determination, on the other hand, raises the questions of how much data of a given accuracy will be required, what computational approach should be used, and how much reliance may be placed upon the results. Practically useful answers to these questions may be found by combining results from transmission theory and the theory of statistical estimation. These answers prove to be relatively simple. The only major difficulty in their practical application is the extensiveness of the data required for highly precise estimates. This requirement is an inherent, irrevocable characteristic of such random processes.

In this account we shall treat only the measurement of spectra of individual noises or signals. The measurement and utilization of the cross-spectra of pairs of series is also important, but is beyond our present scope. Questions of distribution and anticipated variability of cross-spectral estimates, and of certain estimates derived from them, have recently been cleared up by Goodman.[1]

It is natural to feel that the measurement of power spectra is simple, and that no problems deserving extended discussion arise. After all, are there not commercial "wave analyzers" of many sorts; have not Fourier series served for many years to analyze the frequencies of many signals, (musical instruments, human voices, etc.)? Why should there be a serious problem?

There are two reasons why elementary methods fail us rather frequently. On the one hand, the signal may not be available in indefinitely long time stretches. Either the conditions of observation, experimental or otherwise, or the difficulties of careful recording may make it impractical to have so much data that we can afford to analyze carelessly. (The examples of Sections 26 to 28, involving spectra of radar tracking, noise in very short-lived devices, and irregularities in the earth's rotation, respectively, all illustrate this point). Even if observation and recording can be afforded, the cost of computation often forces careful analysis.

On the other hand, the *random* nature of much noise, and some signals, in which the relative amplitudes and phases of different frequencies are not stably related (in contrast to voices and musical notes), introduces much more difficulty with sampling fluctuations and provides much more significant appearing, thus much more misleading statistical artefacts than experience with simpler signals would lead investigators

to expect. (In postwar oceanography, for example, high mechanical ingenuity was expended in the construction of simple and effective wave analysers to produce detailed spectra of ocean waves. The results were quite misleading, because the frequency resolution obtained was too high for the limited length of records used, and almost the entire appearance of the resulting spectra was an illusion due to the particular fluctuations of the particular record. The use of broader filters has since led to meaningful results which could be related to physically satisfying theories.) All too often, the practical study of spectra requires care.

Effective measurement of power spectra requires understanding of a number of considerations and action guided by all of them. Explaining each individual consideration is necessary, but it is equally necessary to explain how they fit together. The general structure of this description of spectral measurement is the following: an introduction to the concepts (Sections 1–3), brief accounts of individual considerations (Sections 4–19), accounts of how these considerations are assembled in analysis (Sections 20–21), and planning for measurement (Sections 22–28, which include discussion of examples), and Sections in Part II giving the details supporting the earlier sections.

We have attempted to provide, somewhere, most of the facts and attitudes that are needed in the practical analysis of (single) power spectra.

Readers interested in either completing their present knowledge or in gaining a brief overview of the subject may wish to proceed next to Sections 20ff, whence they can be referred to specific sections of interest. For some, reading of Sections 1–3 may be a helpful preliminary for Sections 20ff. For others, who want to build more solidly as they go, reading straight through, perhaps with considerable cross-reference to Part II, may be best.

A function of time $X(t)$ generated by a random (or stochastic) process is one of an ensemble of random functions which might be generated by the process. The value of the function at any particular point in time is thus a random variable with a probability distribution induced by the ensemble. Furthermore, the values of the function at any particular set of points, say $t = t_1, t_2, \cdots, t_n$, have an n-dimensional joint probability distribution also induced by the ensemble. Such probability distributions have an important bearing on the design of any communication system or data-processing system which must handle an output from such a random process, be this output "signal" or "noise".

We shall often, but not always, assume that the random process is *Gaussian*. This means that, for every n, $t_1, t_2, \cdots, t_n$, the joint probability distribution of

$$X(t_1), X(t_2), \cdots, X(t_n),$$

is an n-dimensional Gaussian or normal distribution. Each such distribution is completely determined by the ensemble *averages*

$$\bar{X}(t_i) = \text{ave } \{X(t_i)\},$$

and by the *covariances*

$$\begin{aligned} C_{ij} &= \text{cov } \{X(t_i), X(t_j)\} \\ &= \text{ave } \{[X(t_i) - \bar{X}(t_i)]\cdot[X(t_j) - \bar{X}(t_j)]\} . \end{aligned}$$

As a matter of convenience in development we will assume that the averages $\bar{X}(t_i)$ are zero. The covariances then reduce to

$$C_{ij} = \text{ave } \{X(t_i)\cdot X(t_j)\}.$$

Throughout, we will assume that the random process is *stationary* (that is, temporally homogeneous) in the sense that it is unaffected by translations of the origin for time. The covariances C_{ij} now depend only on the time separation $t_i - t_j$ so that

$$C_{ij} = C(t_i - t_j).$$

Thus, the noise is completely specified by a single function of a single variable. In particular, $C(0)$ is the *variance* (for zero average, the average square) of $X(t)$.

If the process were stationary, with zero averages, but were not Gaussian, then knowledge of the covariance as a function of lag, although providing a very large amount of useful information, would not completely specify the process. The results of this paper fall into two categories: (i) those relating to average values of spectral estimates, and (ii) those relating to variability of spectral estimates. The average-value results apply generally under the assumptions of stationarity (and zero averages), and do not depend upon the Gaussian assumption. The variability results are exact under the Gaussian assumption, and are usually rather good approximations otherwise. Thus, our results have practical value for noises and signals which are not closely Gaussian.

Results about variability are naturally used: (i) for planning the approximate extent of measurement effort, (ii) for indicating the presence of changes, during a series of measurements, in the quantities estimated, and (iii) as a means of judging the precision of an over-all estimate. The results given here are mainly for the first planning use. The additional uncertainties in actual variability due to either non-normality of distri-

bution, or to changing of conditions between runs, or to both, are often all too real, but are rarely large enough to affect planning seriously. The same is true of mild nonstationarity. The Gaussian, stationary results can also be applied to the second use, the detection of changes in the true spectrum, but considerable caution is in order. The precision of final over-all values is ordinarily far more wisely judged from the observed consistency of repeated measurements (as by analysis of variance of logarithms of various spectral density estimates at the same nominal frequency) than from any theoretical variability based on a Gaussian assumption.

Communications engineers are more accustomed to work with a single time function of infinite extent than with an ensemble of finite pieces (of such functions). It is perhaps fortunate, therefore, that averages across an ensemble are equivalent (ergodicity) to averages over time along a single function of infinite extent, whenever a process is stationary, Gaussian, has zero averages, and has a continuous power spectrum (no "lines"). (If the process were not stationary the single function approach could not be used in this way.)

Since we seek to make this account as intuitive as possible for communications engineers, we shall define transforms, and make many other computations in terms of averages along a single function (as limits of integrals over centered intervals). In dealing with more specifically statistical issues, however, we shall write "ave" for average value, "var" for variance and "cov" for covariance, and shall do nothing to hinder the interpretation of these operators as acting across the ensemble. (Those who wish can also think of them in single function terms.)

The covariance at lag τ, in single function terms, is given by

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\cdot X(t+\tau)\cdot dt.$$

In ensemble terms, we would write merely

$$C(\tau) = \text{ave}\,\{X(t)\cdot X(t+\tau)\}.$$

The function $C(\tau)$ is frequently called the autocorrelation function, although historical usage in both statistics and the theory of turbulence (Taylor[2]) shows that this name should be applied to the (normalized) ratio $C(\tau)/C(0)$. We shall call $C(\tau)$ the *autocovariance* function. This name is appropriate to our formal definition of $C(\tau)$ because we have assumed that the averages of our process are all zero. Whenever we give up the assumption of zero averages, as we must almost always do when dealing with actual data, we shall use

$$\text{ave } \{X(t)\cdot X(t+\tau)\} = \text{average lagged product,}$$

$$\text{ave } \{[X(t) - \bar{X}]\cdot[X(t+\tau) - \bar{X}]\} = \text{autocovariance function,}$$

where $\bar{X}$ is the common value of ave $\{X(t)\}$ and ave $\{X(t+\tau)\}$, thus preserving accurate usage.

Because of the direct relationship of the joint probability distribution to the autocovariance function, much of the statistical attention given to Gaussian stationary time series (time-sampled random functions) has been expressed in terms of serial-correlation coefficients (corresponding to lag-sampled autocovariance functions).

A stationary Gaussian random process may be regarded (e.g. Rice[3]) as the result of passing *white* Gaussian noise through a fixed linear network with a prescribed transmission function. White Gaussian noise, in turn, may be regarded as the superposition of the outputs of a set of simple harmonic oscillators (continuously infinite in number) with

(a) a continuous distribution in frequency,*

(b) uniform amplitude over the significant frequency range of the transmission system, and

(c) independent and random phases.

This point of view is particularly suited to the techniques employed by communications engineers. It is therefore not surprising that communications engineers have dealt with stationary Gaussian random processes almost entirely in terms of power spectra.

Because the autocovariance function and the power spectrum are Fourier transforms of each other, it would at first appear to be purely a matter of convenience which one is used in any particular situation. Indeed, optimum filter characteristics for the protection of signal against noise in communications systems and in many types of computing devices have, on occasion, been determined by the use of the autocovariance function. In practice, however, the filter designer almost invariably turns to the power spectrum as the final criterion of adequate design and performance.

In practice also, where the autocovariance function or the power spectrum must be determined by measurement and computation, and then interpreted, the choice is now heavily weighted in favor of interpretation of the power spectrum. Although a great deal of theoretical work has been done on the probability distribution of the serial-correlation coefficients for Gaussian stationary time series of finite length, with a view to the estimation of the confidence which may be placed upon practical

* The term "frequency" is used throughout this paper in the communications engineer's sense, viz., cycles per second of a sinusoidal wave. (Exceptional uses in the statistician's sense are explicitly noted.)

results, the criteria which have been developed along this line are so complicated that it is extremely difficult to apply them in practice, where the joint distribution must be considered. On the other hand, the situation with respect to the power spectrum is now very satisfactory for practical purposes. This stems from results obtained by Tukey,[4] and in part independently by Bartlett,[5] about nine years ago, when studies were made of the effects of sampling, of finite length of series, and of choice of computational procedure on the behavior of the estimated power spectrum. Since that time, applications to such diverse fields as ocean waves (Marks and Pierson[6]), aerodynamics (Press and Houbolt[7]), meteorology (Panofsky[8]), and seismology (Wadsworth, Robinson, Bryan, and Hurley[9]), have shown the practical applicability of these results to a wide variety of physical time series.

Shortly after these studies first reached the stage of practical usefulness, the theoretical analysis was reformulated by Blackman, who expressed it from the point of view of transmission theory, for presentation to members of the technical staff of Bell Telephone Laboratories (Out-of-Hours Courses 1950–1951, Communications Development Training Program 1950–1952).

More recent contributions (1950–1957) to the theory of power spectrum estimation have been reviewed by Bartlett and Medhi,[10] by Bartlett,[11] and by Grenander and Rosenblatt.[12]

2. AUTOCOVARIANCE FUNCTIONS AND POWER SPECTRA

First, let us consider the ideal case. The autocovariance function which was defined in the preceding section by

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t) \cdot X(t + \tau) \cdot dt$$

may be reduced to the form

$$C(\tau) = \int_{-\infty}^{\infty} P(f) \cdot e^{i2\pi f\tau}\, df$$

where

$$P(f) = \lim_{T\to\infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} X(t) \cdot e^{-i2\pi f t}\, dt \right|^2$$

(cp. Section B.2). The function of frequency $P(f)$ describes the *power spectrum* of the stationary random process considered. More precisely $P(f)\, df$ represents the contribution to the *variance* from frequencies between f and $(f + df)$. If we think of $X(t)$ as a voltage across (or current

through) a pure resistance of one ohm, the long-time average power dissipated in the resistance will be strictly proportional to the variance of $X(t)$. This important special case is the excuse for the adjective "power". The pure statistician might prefer to refer to the covariance spectrum or to the second moment spectrum rather than to the power spectrum. For precision, we shall often refer to $P(f)$ as the *spectral density* or *power spectral density*. When no confusion is likely, we may call $P(f)$ merely the *power spectrum*.

The relation exhibiting the autocovariance function as the Fourier transform of the power spectrum may be inverted to express the power spectrum as the Fourier transform of the autocovariance function. Thus, we have

$$P(f) = \int_{-\infty}^{\infty} C(\tau) \cdot e^{-i2\pi f\tau} \, d\tau \, .$$

The autocovariance function $C(\tau)$ and the power spectrum $P(f)$ are, formally at least, even functions of their respective arguments. Hence, the relation between them may be expressed more simply as two-sided cosine transforms, viz.

$$C(\tau) = \int_{-\infty}^{\infty} P(f) \cdot \cos 2\pi f\tau \cdot df,$$

and

$$P(f) = \int_{-\infty}^{\infty} C(\tau) \cdot \cos 2\pi f\tau \cdot d\tau;$$

or perhaps even more simply, as one-sided cosine transforms, viz.

$$C(\tau) = 2 \int_{0}^{\infty} P(f) \cdot \cos 2\pi f\tau \cdot df$$

and

$$P(f) = 2 \int_{0}^{\infty} C(\tau) \cdot \cos 2\pi f\tau \cdot d\tau.$$

Results are usually more conveniently developed in terms of the two-sided forms than in terms of the one-sided forms. In Sections A.3 and B.4 for example, the use of the two-sided forms with exponential kernels will be found to simplify considerably the expression of the operation of convolution between functions of lag or of frequency. In Section B.6, the use of the two-sided forms with exponential kernels avoids some complicated manipulations of trigonometric identities in the early stages of the development.

It should be particularly noted that

$$\text{ave } \{X(t) \cdot X(t + \tau)\} = \int_0^\infty 2P(f) \cdot \cos 2\pi f\tau \cdot df$$

and that (setting $\tau = 0$)

$$\text{var } \{X(t)\} = \int_0^\infty 2P(f) \cdot df.$$

Thus, it is evident that our definition of the power spectrum differs from the usual one which associates the power spectrum only with positive frequencies. References to the power spectrum in practice are usually in terms of a density $2P(f)$ over $0 \leqq f < \infty$ only.

3. THE PRACTICAL SITUATION

In practice we can obtain only a limited number of pieces of $X(t)$ of finite length. Each piece may be regarded as a sample drawn from a population or universe of pieces of $X(t)$ of the same length. The reduction of the data will therefore yield no more than estimates of the autocovariance function and of the power spectrum — estimates which are subject to sampling variations and to biases in the usual statistical sense. This situation is further complicated in those cases in which we can measure, or desire to use, only values of $X(t)$ at uniformly spaced values of t within each piece of $X(t)$; in other words, those cases in which we are dealing with classical time series (discrete time) rather than with time functions (continuous time).

The theoretical study of sampling variability and bias is much simpler in the case of the estimates of the power spectrum than in the case of the estimates of the autocovariance function (or of serial-correlation coefficients). This reflects the fact that, as we consider longer and longer records, and two narrow frequency bands with an arbitrarily small but fixed separation, we may find estimates of the power in these frequency bands which both (i) become arbitrarily precise, and (ii) become arbitrarily nearly (statistically) independent. The existence of such estimates is another particular consequence of the Gaussian character, as expressible in terms of "random and independent phases", of the random process from which we have one or more samples.

Use of the power spectrum has an additional advantage over use of the autocovariance function. In almost all practical situations, the data analyzed does not represent the actual output of the random process. In such cases the data will have been modified, appreciably if not radically, by the transmission characteristics of the devices employed to se-

cure the data. This modification of the data may in fact be intentional, as we shall see when we come to the discussion of "prewhitening" in Section 15. In any case, the estimates will have to be corrected for the effects of this modification of the data. For estimates of the power spectrum, the correction procedure is a simple division of a frequency function by another frequency function. For estimates of the autocovariance function, however, the correction procedure will require a Fourier transformation, division of the resulting frequency function by another frequency function, and an inverse Fourier transformation. This whole sequence of operations on the autocovariance function is the only practical procedure for the inversion of the convolution (see Appendix A.3) which is the effect to be corrected for. (Details are given at the end of Section B.3.)

As we shall see, the measurements and computational operations may involve the use of either analog or digital computation and handling of either continuous "signals" or discrete data. (Whatever be its relation to some communication or data-handling system, we shall call continuous-time signals or noise which we are analyzing "signals", while discrete-time signals or noise, or discrete-time samples thereof, will be called data.) In actual practice, and for well-defined reasons of instrumentation and computation engineering, only a few of the many possible combinations are used.

Spectrum analysis by analog computation is almost always applied to continuous "signals", and makes use of filtering rather than going through autocovariance or mean lagged products. Digital computation must be carried out on discrete data, perhaps time-sampled from a continuous "signal", and preferably uses an indirect route via mean lagged products rather than trying to isolate individual frequency bands directly. In either case, each data value must enter several computations, and it is rarely economic to carry these computations out directly in real time, especially since there will not usually be enough such analysis on a regular basis to saturate the working capacity of the analog or digital computer used. Consequently, recording, either of "signals" or of data or of both, is almost inevitable.

Thus, five stages will be important in nearly every case:

(1) sensing (pick-up, conversion, etc.)
(2) transmission (to recorder or, possibly, to computer)
(3) recording (including play-back, and, perhaps, time-sampling)
(4) computation (formulas, computing circuit performance, etc.)
(5) interpretation.

In every one of these stages, quality of performance (noise level, distortion, etc.) will be of importance.

The present account concentrates on the computational and interpretational stages, but indicates, from time to time, those considerations in the other stages which are peculiar to power spectrum analysis.

We have been unable to find a wholly satisfactory arrangement for the material we wish to present. In order to facilitate a relatively easy once-over, these introductory sections now continue into a condensed account, from which proofs, some reasons, and many helpful remarks have been postponed to the Appendix and sections in Part II. Readers interested in a survey may find it adequate to read only the condensed account. Others may find it best to skim this condensed account first, to read Appendix A next, and then to study similarly numbered sections of Part II and the condensed account.

The continuous record of finite length will be treated first (Sections 4–11); the modifications required for the discrete equally spaced record are covered next (Sections 12–21), and the opening account concludes with a discussion of the planning and analysis of measurement programs (Sections 22–28).

Appendix A (Sections A.1 to A.6) treats fundamental Fourier techniques, and the transform-pairs most closely associated with diffraction, in both the continuous and equi-spaced cases.

Each section of Part II relates to the similarly numbered section of the main body, and contains details of derivations, further reasons, and additional helpful remarks.

Definitions of the technical terms, arranged alphabetically for reference, are included at the end of Part I. Similar definitions of the notation will be given at the end of Part II.

Continuous Records of Finite Length

4. FUNDAMENTALS

Given a continuous record of finite length, it is clear that we cannot estimate the autocovariance function $C(\tau)$ for arbitrarily long lags. Surely, no estimate can be made for lags longer than the record. Furthermore, as we will find in due course, it is usually not desirable to use lags longer than a moderate fraction (perhaps 5 or 10 per cent) of the length of the record. Thus, in place of

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\cdot X(t+\tau)\cdot dt$$

for all values of τ, we will have at our disposal

$$C_{00}(\tau) = \frac{1}{T_n - |\tau|} \int_{-(T_n-|\tau|)/2}^{(T_n-|\tau|)/2} X\left(t-\frac{\tau}{2}\right)\cdot X\left(t+\frac{\tau}{2}\right)\cdot dt = C_{00}(-\tau)$$

only for $| \tau | \leqq T_m < T_n$, where T_n is the length of the record, and T_m is the maximum lag which we desire to use. We will call $C_{00}(\tau)$ the *apparent autocovariance function*, since (on account of ergodicity) its average value is $C(\tau)$ for $| \tau | \leqq T_m$.

The class of estimates for the power spectrum with which we are chiefly concerned will be derived from a *modified apparent autocovariance function* by Fourier transformation. While the modified apparent autocovariance functions, which are obtained by multiplying the apparent autocovariance function by suitable even functions of τ, are often far from being respectable estimates of the true autocovariance function, their transforms are very respectable estimates of *smoothed* values of the true spectral density.

Let $D_i(\tau)$ be a prescribed even function of τ, subject to the restrictions $D_i(0) = 1$, and $D_i(\tau) = 0$ for $| \tau | > T_m$, (where $i = 0, 1, 2, 3, 4$, depending upon the shape of $D_i(\tau)$ for $| \tau | < T_m$), and let the corresponding modified apparent autocovariance function be defined by

$$C_i(\tau) = D_i(\tau) \cdot C_{00}(\tau).$$

We may regard $D_i(\tau)$ as a window of variable transmission which modifies the values of $C_{00}(\tau)$ differently for different lags. It is therefore natural to call $D_i(\tau)$ a *lag window*.

For any lag window which meets the conditions stated above, $C_i(\tau)$ is calculable from the data. Further, it is clear that $C_i(\tau) = 0$ for $| \tau | > T_m$ although $C_{00}(\tau)$ was not defined there. Because $C_i(\tau)$ is defined for all values of τ, it has a perfectly definite Fourier transform $P_i(f)$, which should satisfy the symbolic relation,

$$P_i(f) = Q_i(f) * P_{00}(f)$$

where $Q_i(f)$ is the Fourier transform of $D_i(\tau)$, the asterisk indicates convolution (see Appendix A.3 for discussion), and $P_{00}(f)$ is the Fourier transform of $C_{00}(\tau)$. However, $P_{00}(f)$ is not determinate because $C_{00}(\tau)$ is not specified for $| \tau | > T_m$ (and its definition cannot be directly extended beyond $| \tau | = T_n$). Nevertheless, since

$$\text{ave } \{C_i(\tau)\} = D_i(\tau) \cdot C(\tau)$$

where $C(\tau)$ is the true autocovariance function, it follows that

$$\text{ave } \{P_i(f)\} = Q_i(f) * P(f)$$

where $P(f)$ is the true power spectrum, that is, the Fourier transform of $C(\tau)$. The average may be thought of as either across the ensemble, or along time. (The latter type of averaging would correspond to replacing

$X(t)$ by $X(t - \lambda)$, thus changing the stretch of $X(t)$ which is observed, and then averaging over λ.) The corresponding explicit relation, viz.

$$\text{ave } \{P_i(f_1)\} = \int_{-\infty}^{\infty} Q_i(f_1 - f) \cdot P(f) \cdot df$$

exhibits the average value of $P_i(f_1)$ as a smoothing (average-over-frequency) of the true power spectrum density $P(f)$ over frequencies "near" f_1 with weights proportional to $Q_i(f_1 - f)$. In a manner of speaking $P_i(f_1)$ is the collected impression of the true power spectrum $P(f)$ obtained through a window of variable transmission $Q_i(f_1 - f)$. It is therefore natural to call $Q_i(f)$ the *spectral window* corresponding to the *lag window* $D_i(\tau)$.

The form just given for ave $\{P_i(f_1)\}$ is natural for our two-sided definition of power spectra, but, in order to view the result from the standpoint of transmission theory for real-valued signals, it is convenient to express the result in a form appropriate to a one-sided definition of power spectra. Taking advantage of the fact that $Q_i(f)$ and $P(f)$ are even functions, we may write

$$\text{ave } \{2P_i(f_1)\} = \int_0^{\infty} H_i(f; f_1) \cdot 2P(f) \cdot df$$

where

$$H_i(f; f_1) = Q_i(f + f_1) + Q_i(f - f_1)$$

and where we recall that $2P(f)\ df$ is the amount of power between f and $(f + df)$ in the one-sided true power spectrum. Similarly, $2P_i(f)\ df$ is the amount of power between f and $(f + df)$ in the one-sided estimated power spectrum. The function $H_i(f; f_1)$ has one of the necessary properties of a physically realizable power transfer function inasmuch as it is an even function of f as well as of f_1. In general, however, it does not have the property of being non-negative at all frequencies f. Nevertheless, it is a convenient function to use in the analysis of the variability of the estimated power spectrum. It will be convenient to regard the average value of the smoothed power density estimate ave $\{2P_i(f_1)\}$ as the result of passing the true power spectrum, through a "network" with *power transfer function* $H_i(f; f_1)$.

We see that our procedures will lead us to estimates whose average values are a smoothing (average-over-frequency) of the true power spectral density $P(f)$ over frequencies "near" f_1, and not to estimates of $P(f_1)$ itself. The problem of choosing the shape of the lag window $D_i(\tau)$ so that its Fourier transform $Q_i(f)$ will be concentrated near $f = 0$ is

almost identical to the problem of choosing an intensity distribution along an antenna so that most of the radiation from the antenna will fall in a narrow beam. From this analogy we will use such terms as *main lobe* and *side lobes* for the principal maximum and subsidiary extrema of $Q_i(f)$. (Indeed, any attempt to confine the power transfer function to too narrow a frequency band — too narrow in comparison with the reciprocal of the longest lag used — would be analogous to an attempt to construct a practical hyperdirective antenna.)

It is not surprising that we are led to estimate a smoothed power spectrum. With only a finite length of $X(t)$ available, we should not expect to be able to identify frequencies exactly, and are, indeed, unable to do so. (The presence of neighboring frequencies with random phases will have effects similar to those of noise in preventing such identification.)

5. TWO PARTICULAR WINDOW PAIRS

In order to specify a particular family of estimates within the class of estimates defined in the preceding section, we have only to specify $D_i(\tau)$ or $Q_i(f)$. We would like to concentrate the main lobe of $Q_i(f)$ near $f = 0$, keeping the side lobes as low as feasible. In order to concentrate the main lobe we have to make $D_i(\tau)$ flat and rather blocky. In order to reduce the side lobes, however, we have to make $D_i(\tau)$ smooth and gently changing. Since $D_i(\tau)$ must vanish for $|\tau| > T_m$ we must compromise. So far, cut-and-try inquiry has been more powerful in finding good compromises than has any particular theory.

A simple and convenient compromise is represented by the lag window (whose use is called "hanning")

$$\begin{aligned} D_2(\tau) &= \frac{1}{2}\left(1 + \cos\frac{\pi\tau}{T_m}\right) \qquad \text{for} \qquad |\tau| < T_m \\ &= 0 \qquad\qquad\qquad\qquad\quad \text{for} \qquad |\tau| > T_m . \end{aligned}$$

(Window pairs 0 and 1 are discussed in Section B.5.) An alternative compromise is represented by the lag window (whose use is called "hamming")

$$\begin{aligned} D_3(\tau) &= 0.54 + 0.46\cos\frac{\pi\tau}{T_m} \qquad \text{for} \qquad |\tau| < T_m \\ &= 0 \qquad\qquad\qquad\qquad\quad \text{for} \qquad |\tau| > T_m . \end{aligned}$$

These lag windows and the corresponding spectral windows are illustrated in Fig. 1. Notice that the main lobes are four times as wide as the

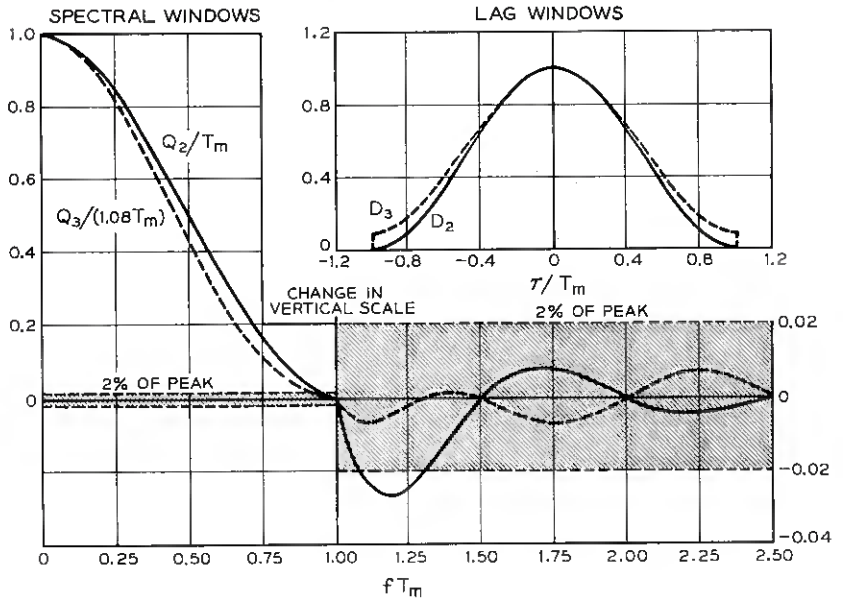

Fig. 1 — Lag windows D_2 and D_3. Spectral windows Q_2 and Q_3.

side lobes (excepting the split side lobes nearest the main lobes), and that the (normal) side lobe width is $1/(2T_m)$.

The general nature of the spectral windows in these two pairs is the same: a main lobe, side lobes at most 1 per cent or 2 per cent of the height of the main lobe. There are differences, which are sometimes relevant, but these may not be obvious. The two most important of these differences are:

(a) The highest side lobe for the "hamming" (spectral) window is about $\frac{1}{3}$ the height of the highest side lobe for the "hanning" window,

(b) The heights of the side lobes for the "hanning" window fall off more rapidly than do those for the "hamming" window.

One difference favors one pair, and one the other.

These and several other window pairs are discussed in Section B.5.

6. COVARIABILITY OF ESTIMATES — BASIC RESULT

It is shown in Section B.6 that, strictly only under Gaussian circumstances, the covariance of any two power density estimates of the sort we have been considering is given to a good degree of approximation by

$$\text{cov } \{2P_i(f_1), 2P_j(f_2)\} \approx \int_0^\infty H_i(f; f_1) \cdot H_j(f; f_2) \cdot 2\Gamma(f) \cdot df$$

where the *power-variance spectrum* $\Gamma(f)$ depends only on the true power spectrum $P(f)$ and the *effective record length* T_n', as described below. Thus, we may regard the covariance of the two power density estimates as the result of passing the power variance spectrum $\Gamma(f)$ through two networks in tandem, one with power transfer function $H_i(f; f_1)$, the other with power transfer function $H_j(f; f_2)$. In other words, we may regard the covariance (of the estimates of the power spectrum) as the power remaining from the *power-variance spectrum* $\Gamma(f)$ after passing through the *two* windows $H_i(f; f_1)$ and $H_j(f; f_2)$ associated with the estimates themselves. *If the windows do not overlap, the estimates do not covary* (at least not in terms of second moments).

In particular, of course,

$$\begin{aligned} \text{var } \{2P_i(f_1)\} &= \text{cov } \{2P_i(f_1), 2P_i(f_1)\} \\ &\approx \int_0^\infty H_i(f; f_1)^2 \cdot 2\Gamma(f) \cdot df \end{aligned}$$

to which we can give a similar interpretation.

These results would become exact if we were to replace $C_{00}(\tau)$ by

$$\tilde{C}_{00}(\tau) = \frac{1}{T_n - T_m} \int_{-(T_n - T_m)/2}^{(T_n - T_m)/2} X\left(t - \frac{\tau}{2}\right) \cdot X\left(t + \frac{\tau}{2}\right) \cdot dt,$$

where $|\tau| \leqq T_m < T_n$. In $C_{00}(\tau)$ we averaged $X(t - (\tau/2)) \cdot X(t + (\tau/2))$ over an interval of t of length $T_n - |\tau|$, varying with τ. In $\tilde{C}_{00}(\tau)$ we would be averaging $X(t - (\tau/2)) \cdot X(t + (\tau/2))$ over an interval of t of length $T_n - T_m$ independent of τ. We could actually do this because $|t \pm (\tau/2)| \leqq T_n/2$ for $|\tau| \leqq T_m$. However, for values of $|\tau|$ less than T_m, $\tilde{C}_{00}(\tau)$ would not make use of some values of $X(t - (\tau/2)) \cdot X(t + (\tau/2))$ which are used in $C_{00}(\tau)$. Thus, $\tilde{C}_{00}(\tau)$ would be wasteful. It seems best, therefore, to use $C_{00}(\tau)$ for computation, but to approximate its variability by the variability corresponding to a $C_{00}'(\tau)$ which could not be calculated from the actual values. This "approximate" hypothetical $C_{00}'(\tau)$ involves a fixed range of integration T_n' part way between $T_n - T_m$ and T_n. The situation is illustrated in Fig. 2, where the ranges of integration are shown for the actually "feasible" $\tilde{C}_{00}(\tau)$, for the $C_{00}(\tau)$ which "wastes not", and for the $C_{00}'(\tau)$ which we use to "approximate" $C_{00}(\tau)$. The shaded areas delineate the products which are actually available.

The best choice of an intermediate value depends somewhat upon the $D_i(\tau)$ and $D_j(\tau)$ involved, and is discussed in Section B.6. In practically useful cases we may take

$$T'_n = T_n - \tfrac{1}{3}T_m .$$

The power-variance spectrum is given approximately and closely by

$$\Gamma(f) = 4\int_{-\infty}^{\infty} P(f+f')\cdot P(f-f')\cdot\left(\frac{\sin \omega' T'_n}{\omega' T'_n}\right)^2 df' \qquad (\omega' = 2\pi f').$$

If we have p pieces of *total length* T_n, and if, in computing our estimate of $C(\tau)$ for each τ, we combine all available lagged products

$$X(t - (\tau/2))\cdot X(t + (\tau/2))$$

without regard to which piece they came from, then we may use this formula for $\Gamma(f)$ with

$$T'_n = T_n - \frac{p}{3}T_m$$

as a satisfactory approximation for the *effective total length.*

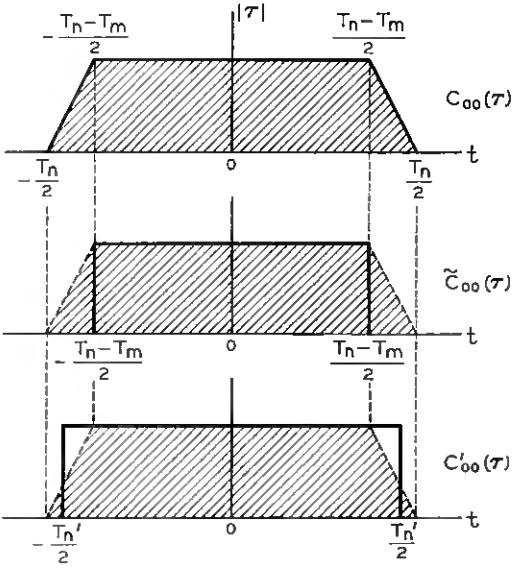

Fig. 2 — Range of integration over t as a function of τ in $\tilde{C}_{00}(\tau)$, $C_{00}(\tau)$, and $C'_{00}(\tau)$.

7. COVARIABILITY OF ESTIMATES — APPROXIMATE FORMS

In assessing the covariability of estimates of the smoothed power spectrum, the relative magnitudes of three distances along the frequency axis are important:

(a) the distance $1/T'_n$, the reciprocal of the effective length of record,

(b) the least distance over which $P(f)$ changes by an important amount for f near f_1, and

(c) the least distance over which $H_i(f; f_1)$ changes by an important amount for f near f_1 (this is of the order of $1/T_m$ and is usually much larger than $1/T'_n$).

If $P(f)$ changes slowly enough to make (b) larger than (a), we may use the approximation

$$\Gamma(f) \approx \frac{2}{T'_n} [P(f)]^2$$

whence, approximately,

$$\text{cov } \{P_i(f_1), P_j(f_2)\} \approx \frac{1}{T'_n} \int_0^\infty P_{i1}(f) \cdot P_{j2}(f) \cdot df$$

where

$$P_{i1}(f) = H_i(f; f_1)P(f)$$
$$P_{j2}(f) = H_j(f; f_2)P(f).$$

In the same terms we have

$$\text{ave } \{P_i(f_1)\} = \int_0^\infty P_{i1}(f) \cdot df$$

and

$$\text{ave } \{P_j(f_2)\} = \int_0^\infty P_{j2}(f) \cdot df.$$

The relation of covariances to averages thus established may be reasonably interpreted as meaning that any cancellations occurring in the average values also occur in the covariances and variances. To the accuracy of this approximation, then, we appear to be using the data rather efficiently.

If, on the other hand, the true spectrum, $P(f)$, consists of a single sharp peak at $f = f_0$, we may use the approximation, derived in Section B.7, namely

$$\text{cov } \{P_i(f_1), P_j(f_2)\} \approx \left[\int_0^\infty P_{i1}(f) \cdot df\right] \cdot \left[\int_0^\infty P_{j2}(f) \cdot df\right]$$
$$\approx \text{ave } \{P_i(f_1)\} \cdot \text{ave } \{P_j(f_2)\},$$

a result which is not influenced by T'_n (so long as T'_n does not become large enough for $1/T'_n$ to become comparable with the width of the peak).

8. VARIABILITY — EQUIVALENT WIDTHS

If $P(f)$ changes slowly in comparison with $1/T'_n$, then, since

$$\text{var } \{P_i(f_1)\} = \text{cov } \{P_i(f_1), P_i(f_1)\},$$

we may write down the dimensionless variability of $P_i(f_1)$ itself as

$$\frac{\text{var } \{P_i(f_1)\}}{[\text{ave } \{P_i(f_1)\}]^2} = \frac{1}{T'_n W_e},$$

where

$$W_e = \frac{\left[\int_0^\infty P_{i1}(f) \cdot df\right]^2}{\int_0^\infty [P_{i1}(f)]^2 \cdot df}$$

is naturally called the *equivalent width* of $P_{i1}(f) = H_i(f; f_1) \cdot P(f)$.

The longer the record, and the wider the equivalent width, the more stable the estimate. (Increasing the width also of course makes the estimate refer to an average power density over a wider frequency interval.)

If, on the other hand, $P(f)$ consists of a sharp peak, then, by the concluding remarks of the preceding section

$$\frac{\text{var } \{P_i(f_1)\}}{[\text{ave } \{P_i(f_1)\}]^2} = 1.$$

The equivalent widths of some simple cases are as follows:

1. If $P_{i1}(f)$ is a rectangle of width W which does not cross $f = 0$, then $W_e = W$.

2. If $P_{i1}(f)$ is a triangle of base W which does not cross $f = 0$, vertex anywhere over the base, then $W_e = 0.75\ W$.

3. If $P_{i1}(f)$ is proportional to

$$\frac{\sin \dfrac{\omega + \omega_1}{W}}{\dfrac{\omega + \omega_1}{W}} + \frac{\sin \dfrac{\omega - \omega_1}{W}}{\dfrac{\omega - \omega_1}{W}}$$

i.e. has the shape of $H_0(f; f_1)$, where $W = W_{\text{main}} = 2W_{\text{side}}$ (these being the widths of main and side lobes, respectively), and if $f_1 \geqq 1/T_m$ then $W_e = 0.5\ W = 0.5\ W_{\text{main}} = W_{\text{side}}$.

4. If $P_{i1}(f)$ has the shape of $H_2(f; f_1)$, i.e. is proportional to a hanning

(0.25, 0.5, 0.25) window, and if $f_1 \geqq 1/T_m$, then $W_e = 0.67\ W_{\text{main}} = 2.67\ W_{\text{side}}$.

5. If $P_{i1}(f)$ has the shape of $H_3(f; f_1)$, i.e. is proportional to a hamming (0.23, 0.54, 0.23) window, and if $f_1 \geqq 1/T_m$, then $W_e = 0.63\ W_{\text{main}} = 2.52\ W_{\text{side}}$.

These cases are illustrated in Fig. 3, a single sketch sufficing for the last two. Note that W_e is close to $\frac{2}{3}W_{\text{main}}$ for practical windows, if $f_1 \geqq 1/T_m$.

For our standard window pairs, hanning or hamming, the width of the normal side lobes is $1/(2T_m)$ and, consequently, $W_e \sim 1.30/T_m$, if $f_1 \geqq 1/T_m$.

These last three equivalent widths decrease somewhat as f_1 becomes small, and the values given should be halved for $f_1 = 0$.

If $P(f)$ varies linearly across $H_i(f; f_1)$, then a calculation discussed in Section B.8 shows that W_e will tend to fall in the range from $1.15/T_m$ to $1.23/T_m$. (A rather peaked case gives $0.94/T_m$.) When we allow for the fact that we are likely to be concerned with processes which are not quite Gaussian, whose variances of estimate are consequently likely to be somewhat larger than for the Gaussian case, a change corresponding to the use of a decreased equivalent width in the formula, the choice

$$W_e \approx \frac{1}{T_m}$$

which introduces a small factor of safety (not more than 1.3) seems de-

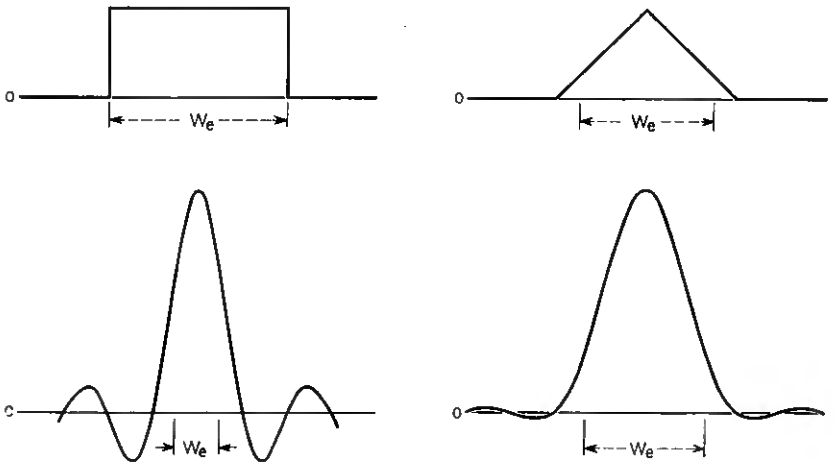

Fig. 3 — Equivalent widths of some spectral windows.

sirable for planning purposes. Consequently, we shall plan according to

$$\frac{\text{var } \{P_i(f_1)\}}{[\text{ave } \{P_i(f_1)\}]^2} \approx \frac{T_m}{T_n'}.$$

If we plan to hold the RMS deviation of each of our estimates below one-third of its average value, we must, accordingly, keep T_m/T_n' below $\frac{1}{9}$. Thus, as noted above, we shall ordinarily keep T_m to a small fraction of T_n.

In making more detailed studies of the variability of spectral estimates, further approximation will be convenient. It is important to note several reasons why we need not be too precise in making such approximations. First, as noted earlier, the variability results depend on the noise being exactly Gaussian. Real noises (and especially real signals) need not be exactly Gaussian. Thus, even exact results in Gaussian theory would be approximations in practice. Second, the chief purposes of studying variability are first to choose, once for all, effective methods of analysis, and then, in each situation, to determine about how much data will be required for the desired or given accuracy. Again, approximate results will be adequate. Third, it would not be safe to use the advance estimates of variability as firm, guaranteed, measures of the stability of the actual computed results in a practical situation, since other sources of variability may well contribute to the deviation of a particular spectral density estimate from its long run value. (Non-constancy of total power level, even with distribution-over-frequency remaining constant, and failures of stationarity are two simple examples.) We must rely on observed changes from trial to trial as basically the safest measure of the lack of stability of our spectral density estimates.

Thus, the purposes of variability theory are well served if its results are approximate — deviations of actual variability from theoretical variability of ± 5 per cent, ± 10 per cent or even ± 20 per cent will be quite satisfactory. Judged by this standard, the variability theory based on (i) the Gaussian assumption and (ii) treating the distribution of the spectral density estimates as if they followed so-called "chi-square" distributions, as we shall do in the next section, will usually be very satisfactory.

9. CHI-SQUARE — EQUIVALENT DEGREES OF FREEDOM

If $y_1, y_2, \cdots, y_k$ are independently distributed according to a standard normal distribution, that is, according to a Gaussian distribution with average zero and unit variance (and, consequently, unit standard

deviation), then

$$\chi_k^2 = y_1^2 + y_2^2 + \cdots + y_k^2,$$

which is obviously positive, follows, by definition, a chi-square distribution with k degrees of freedom. The coefficient of variation of χ_k^2, the ratio of RMS deviation to average value, is $(2/k)^{1/2}$, so that, as k increases, χ_k^2 becomes *relatively* less variable. This statement also applies to any multiple of χ_k^2.

A convenient description of the stability of any positive or nearly-positive estimate is its *equivalent number of degrees of freedom*, the number of degrees of freedom of that χ_k^2 some multiple of which it resembles (in average and variance unless otherwise specified). We can find such a k from

$$k = \frac{2(\text{average})^2}{\text{variance}} = \frac{2}{(\text{coefficient of variation})^2}.$$

Interpretation is aided by Tables I and II. These tables are possible because the distribution of the ratio of any multiple of χ_k^2 to the average value (of that multiple) depends only on k. Thus, if $k = 4$, individual

TABLE I

Distribution of quantities which are distributed as fixed multiple of chi-square. Ratios of individual value to its average value exceeded with given probabilities.

Degrees of freedom	Exceeded by 90% of all values	Exceeded by 50 % of all values	Exceeded by 10% of all values
1	0.016	0.46	2.71
2	0.10	0.70	2.30
3	0.19	0.79	2.08
4	0.26	0.84	1.94
5	0.32	0.87	1.85
10	0.49	0.93	1.60
20	0.62	0.96	1.42
30	0.69	0.98	1.34
40	0.73	0.98	1.30
50	0.75	0.99	1.26
100	0.82	0.99	1.18
200	0.873	1.00	1.139
500	0.920	1.00	1.081
1000	0.943	1.00	1.057

Examples: (1) If the long run average is 10 volts²/cps, then among estimates with 10 degrees of freedom, 10 per cent would fall below 4.9 volts²/cps, and 50 per cent would fall above 9.3 volts²/cps.

(2) If a single observed estimate, with 5 degrees of freedom, is observed to be 2 volts²/cps, then we have 80 per cent confidence that the true long-run value lies between 2/1.85 = 1.08 volts²/cps and 2/0.32 = 6.25 volts²/cps.

TABLE II — BEHAVIOR OF χ_k^2 ON DECIBEL SCALE

Fraction of distribution	Spread* of interval† in db‡	k required for interval† of spread			
		10 db	5 db	2 db	1 db
40%	$6/\sqrt{k-1}$	1	3	11	42
60%	$10/\sqrt{k-1}$	2	5	28	105
80%	$16^*/\sqrt{k-1}$	4	11	63	250
90%	$20/\sqrt{k-1}$	5	18	104	410
96%	$25/\sqrt{k-1}$	8	27	161	640
98%	$29/\sqrt{k-1}$	10	34	207	820

* Accurate to nearest integer in numerator for $k \geqq 4$, except for 80 per cent, where 16 should be replaced by 15 for $k \leqq 11$. Based on Tukey and Winsor.[13] (Spread is the difference between the upper boundary expressed in db, and the lower boundary expressed in db.)

† All intervals are symmetric in the probability sense, half of the non-included probability falling above and half below the interval.

‡ Since we are dealing with measures of variance, analogous to power, 10 db = (factor of 10), and (number of db) = (10 $\log_{10}$ ratio of variances).

values of any particular multiple of χ_4^2 will, in the long run, fall below 0.26 times their average value in 10 per cent of all cases (will be 5.8 db or more below average in 10 per cent of all cases). Similarly, individual values will, in the long run, fall below 0.84 times their average value (be 0.7 db or more below average) in 50 per cent of all cases, and in 90 per cent of all cases will fall below 1.94 times their average value (be 2.9 db or less above average). Thus, in the long run, 80 per cent of all values will fall in an interval of spread (2.9) − (−5.8) = 8.7 db.

Thus, for example, to obtain 4 chances in 5 that a single observed value will lie within ±30 per cent of the true value we require (see Table I) about 40 degrees of freedom, while to obtain 4 chances in 5 that a single observed value will lie in a prescribable interval of length 5 db, we require (see Table II) at least 11 degrees of freedom.

The results of the preceding section indicate that, for an estimate of smoothed spectral density, when $P(f)$ is smooth, the number of degrees of freedom is given by

$$k = 2T_n' W_e = \frac{W_e}{\Delta f},$$

where the latter form expresses the number of degrees of freedom as the number of *elementary frequency bands*, each of width

$$\Delta f = \frac{1}{2T_n'},$$

contained in the equivalent width W_e.

For design purposes, the relation of the last section (including the small safety factor) indicates that

$$k = \frac{2}{[\text{var } \{P_i(f_1)\}]/[\text{ave } \{P_i(f_1)\}]^2} \approx \frac{2T_n'}{T_m}$$

when $P(f)$ varies slowly. (This will usually be the case if (*i*) $k > 3$ or 4, say, and (*ii*) $P_{ii}(f)$ is a moderately smooth single hump. For, under these circumstances, $P_{ii}(f)$ will not change rapidly in a frequency interval $1/T_n'$ and the same property can then be inferred for $P(f)$ itself.)

When, on the other hand, $P(f)$ consists of a single sharp peak, we find, using the last result of Section 7, that $k \approx 2$, so long as $1/T_n'$ is not small enough to be comparable with the width of the peak. At first glance, this result may appear a little surprising, but when we notice that a single spectral line corresponds either (*a*) to frequency $+f_0$ *and* to frequency $-f_0$, or (b) to $\cos \omega_0 t$ *and* to $\sin \omega_0 t$, or (c) to amplitude *and* to phase, it appears quite natural that a sharp line carries *two* degrees of freedom and not merely one.

We may summarize the semi-quantitative study of the stability of estimates of the smoothed power spectrum as follows:

(a) It is not necessary to judge stability with very high accuracy.

(b) It is convenient to measure stability by analogy with the number of degrees of freedom associated with a multiple of a chi-square variate.

(c) The equivalent number of degrees of freedom can be regarded as the number of elementary bands of width Δf in the equivalent width W_e of the *filtered* spectrum

$$2P_{ii}(f) = H_i(f; f_1) \cdot 2P(f) \qquad (f \geqq 0)$$

if the result is not too small (say > 3 or 4) and $P_{ii}(f)$ is moderately smooth.

(*d*) If the filtered spectrum approaches a single sharp peak, the equivalent number of degrees of freedom for the corresponding estimate approaches two.

In interpreting the concept of equivalent number of degrees of freedom, it may be helpful to imagine the continuous density of the *filtered* spectrum replaced by a discrete set of ordinates, one per elementary frequency band. If these ordinates are $p_0, p_1, p_2, \cdots$, the natural approximation to the number of degrees of freedom is

$$k = \frac{(p_0 + p_1 + p_2 + \cdots)^2}{p_0^2 + p_1^2 + p_2^2 + \cdots}$$

as illustrated in Fig. 4. This approximation will usually be satisfactory

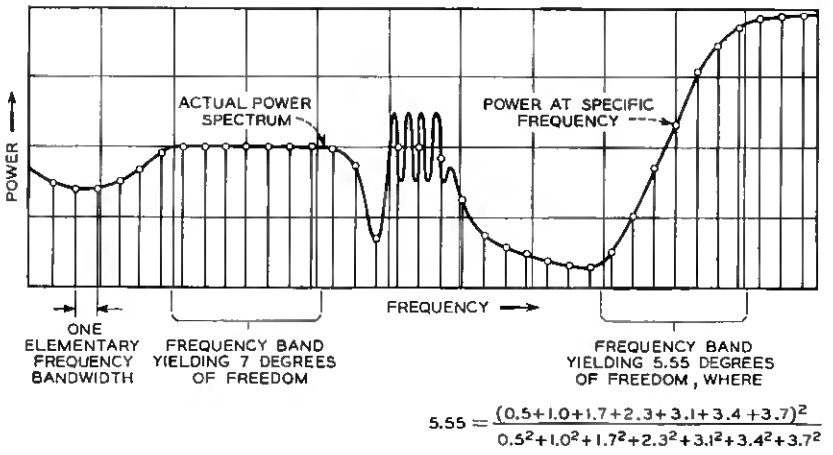

Fig. 4 — Equivalent degrees of freedom.

as long as the effect on k of moving each ordinate around within its elementary frequency band can be neglected. (In more extreme cases, an approximation based on two ordinates per pair of elementary frequency bands is more precise.)

10. DIRECT ANALOG COMPUTATION — GRADED DATA WINDOWS

We have been dealing thus far with continuous time, and the communications engineer will naturally ask, "Why introduce autocovariance functions and all that, why not measure the spectrum by filtering, rectifying, and smoothing?". The only fair answer is "By all means, do so if you can obtain, and maintain, the necessary accuracy economically!" Let us apply our results to such a measurement technique.

Let $X(t)$ be the noise or signal whose power spectrum $P(f)$ we wish to study. Let us pass it through a filter of transfer function $Y(f)$, and designate the result by $X_{\text{out}}(t)$. Its power spectrum, $P_{\text{out}}(f)$, will be given by

$$P_{\text{out}}(f) = |\, Y(f) \,|^2 \cdot P(f)$$

and if a section of $X_{\text{out}}(t)$ of length T_n is applied to an ideal quadratic rectifier and smoothed by a smoothing circuit of infinite time constant, the result will be

$$\int_0^{T_n} [X_{\text{out}}(t)]^2 \, dt.$$

The average value of this result divided by T_n is

$$\int_0^\infty 2P_{out}(f)\, df,$$

and the number of equivalent degrees of freedom is the number of elementary frequency bands, of bandwidth $1/(2T_n)$, contained by the equivalent width of $|\, Y(f)\,|^2 \cdot P(f)$. This last function is of the form

(power transmission function)(original power spectrum)

just as before. We see that the ideal process of filtering, rectifying, and smoothing the actual input has produced the same accuracy as the ideal process of calculating, modifying, and transforming the apparent autocovariance, provided that $|\, Y(f)\,|^2 = H_s(f; f_1)$ for a suitable choice of T_m, f, and f_1. This is what we ought to have expected, since we believe that either method extracts nearly all the information about the spectrum which the data provides.

A few practical considerations deserve mention. They center around the actual switching sitations which can arise, especially when we have only a finite sample of the original noise. In Fig. 5, the watt-second meter includes quadratic rectification and integration functions which we think of as ideal. (It may be very important to allow for the fact that the "ground" position of switch A is not quite at the same potential as the zero of the input noise, but we shall neglect this effect for the moment.)

Some four sorts of operation can arise according to the times at which switch B is operated. The watt-second meter may be connected either at the beginning of the running period T or after some interval of time (to allow initial transients to become negligible), and may be disconnected either at the end of the running period T or after some interval of time (to allow the meter to reach a maximum). These four modes of operation are illustrated in Fig. 6.

In Mode I, providing the initial waiting period is long enough to allow transients to become negligible, the filter output is essentially stationary, and the earlier discussion in this section applies.

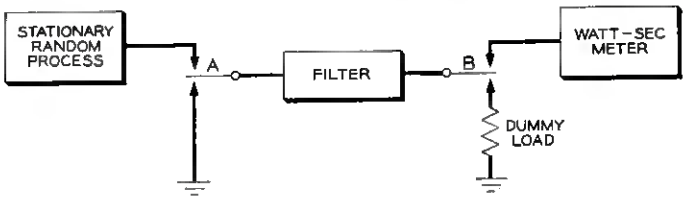

Fig. 5 — Schematic analog circuit.

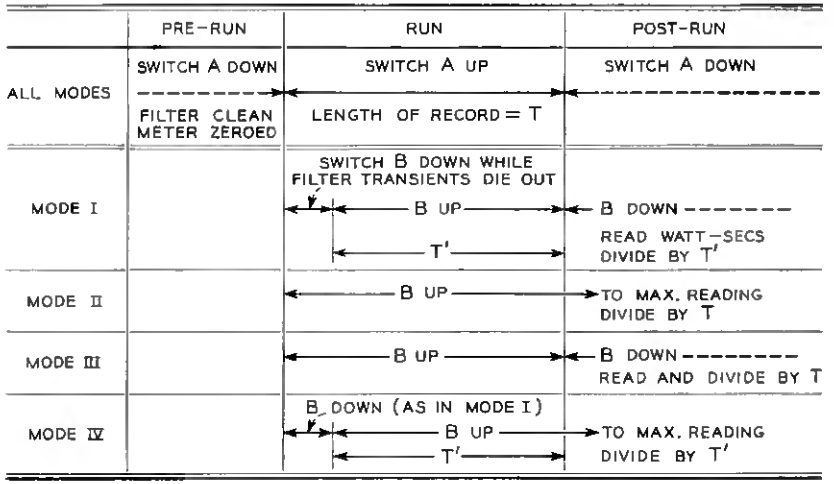

Fig. 6 — Time histories of operation for different modes.

In Mode II, all of the energy output is recorded on the meter, but the reading is divided only by the length of the input data. This mode is amenable to exact and complete analysis which is given in some detail in Section B.10. The results differ from those of Mode I in that the transform of the boxcar function of length T (running period) is convolved twice into the spectral window. (Convolution is defined and discussed in Appendix A.3.) If T is not large, the effects may be somewhat uncomfortable in that the spectral window becomes wider and more ragged.

Mode III, discussed briefly in Section B.10, differs from Mode II by an additional convolution whose effect again disappears as $T \to \infty$.

Mode IV resembles Mode I in that the noise input is passed through the filter until transient effects have become negligible, when the meter is switched on at the filter output. It differs from Mode I in that the meter is read *after* a final waiting period. This seems to offer no advantages over Mode I, and will not be discussed further.

The contrast between Mode I and Mode II is another example of what should now be becoming familiar. Mode I has no additional convolution in the spectral window. Mode II provides data economy by making it possible to integrate over the whole length of the available record. We should really like both advantages.

We can, indeed, obtain most of both advantages, but only by replacing the sharp edges of the switched data window by the smoothed outlines of a graded data window. In other words, we need to introduce

$$X_{in}(t) = B(t) \cdot X(t)$$

at the input of the filter, where $B(t)$ vanishes except for $0 < t < T$, and is smooth enough to have its Fourier transform $J(f)$ concentrated near $f = 0$. Details are discussed in Section B.10.

Difficulties arising from the fact that the zero of the $X(t)$ input might not be at ground are shown in Section B.10 to behave similarly to those arising from switching transients, namely, no effect in Mode I, possibly uncomfortable in Modes II and III, usually negligible when a well-chosen graded data window is used.

Another device is sometimes used to make maximum use of a finite noise record. The record is merely closed into a continuous loop, and the rectifier-smoother output averaged. It is shown in Section B.10 that here, too, we must use a graded data window $B(t)$.

11. DISTORTION, NOISE, HETERODYNE FILTERING AND PREWHITENING

Another group of very important practical considerations center around the spectrum of the "signal" as it is handled (either instantaneously, or in recorded form). We have spoken of "filtering, rectifying and smoothing" and have treated all these steps as ideal. No attention has been given to the equally vital "gathering" and "transmission and recording" steps. Tacitly, they too have been treated as ideal. Realistically, we must expect a certain amount of distortion (nonlinearity, intermodulation, etc.) and the addition of a certain amount of background noise in all three of the first steps: gathering, transmission and recording, filtration. It often proves to be most important to lessen the ill effects of such distortion and noise addition.

In a perfect system, and with a fixed spectral window, the fluctuations of an estimate are proportional to its average value. If we have a fixed uniform noise level, it will do the least additional damage if all the average values of the estimates are of about the same size, for then no low estimate can "disappear" into the noise.

Intermodulation distortion will have the greatest effect on the signal being transmitted when two strong frequencies combine to produce a modulation product whose frequency falls in a very weak region of the spectrum, for it is in such situations that the fractional distortion of the spectrum reaches its maximum. To minimize possible effects of intermodulation distortion it is again desirable to transmit, record and generally handle signals with a roughly flat spectrum.

To these noise and intermodulation considerations another sort of consideration may be added. Many frequency analyzers use a hetero-

dyne system, bringing the frequency band to be studied to a fixed filter, rather than tuning a filter across a wide frequency band. The power transfer function of the combination of heterodyne modulator and fixed filter, referred to input frequency, will depend only on Δf, the deviation of $|f|$ from $|f_0|$, where f_0 is the nominal frequency of the fixed filter, and will be denoted by $Q_i(\Delta f)$. If demands at different frequencies differ, the shape of $Q_i(\Delta f)$ may have to be a compromise. One sort of demand arises when $P(f)$ varies very rapidly. The net contribution near frequency f_1 to the average value of the spectral density estimate is measured by $H_i(f; f_1) \cdot P(f)$, where, as elsewhere, $H_i(f; f_1) = Q_i(f + f_1) + Q_i(f - f_1)$. If our estimate is to be useful, only f's near f_1 should have a substantial net contribution. If $P(f)$ rises steeply as f leaves f_1, we may have to require a very rapid fall-off in $H_i(f; f_1)$, here practically equal to $Q_i(f - f_1)$, in order to attain this as f leaves f_1. We may thus be forced to compromise properties of $Q_i(\Delta f)$ useful near other frequencies. The simplest way to avoid such problems is to arrange for the $P(f)$ of the "signal" analyzed to be fairly constant, or at most slowly varying.

Thus, for a variety of reasons, we can often gain by introducing "compensation" or "preemphasis" to make more nearly constant the spectrum of the "signal" actually transmitted or recorded, and analyzed. Since the ideal would be to bring the spectrum close to that of white noise, it is natural to refer to this process as *prewhitening*. Such flattening of the spectrum need not be precise, or even closely approximate. We need only to make the rate of change of $P(f)$ with frequency relatively small.

Because of advantages related to the noise and intermodulation distortion introduced in various steps of the sequence, it will be best, other considerations aside, to carry out such prewhitening at as early a point in the measurement-analysis sequence as possible. Sometimes this can even be done in the pick-up or sensing element.

This whole philosophy of prewhitening, which appears quite natural to the communication engineer familiar with preemphasis and other techniques for increased information transfer within a given frequency interval, comes as a great change to the instrumentation engineer, whose clients ordinarily require "faithful" reproduction of an input at the output, by which they mean phase shifts nearly linear with frequency, and a nearly constant amplitude response up to some high frequency. It will be rare indeed, in practical spectrum analysis, that the ideal response for the initial transducer and amplifier will be flat. Instead it should have a characteristic contributing to prewhitening. This characteristic will, of course, have to be measured separately and the corresponding

adjustments to the estimates of the spectral density will have to be made so that these estimates, instead of applying to the "signal" actually analyzed, apply to the original "input signal", but such labor will often be many times repaid.

One further consideration about frequency responses in measurement now enters naturally. In almost every power spectrum problem there is an upper frequency beyond which there is no appreciable interest. In most components used in measurement, transmission, recording, etc., the noise level, and often the level of intermodulation distortion, is roughly a fixed fraction of the peak useful level. If substantial power is present at frequencies so high as to be uninteresting, then the need to keep *total* power below the peak useful level forces us to handle the interesting frequencies at a power level below that which could otherwise be used. The ratio of noise and intermodulation distortion to interesting signal is thus raised — the quality of the analysis and its results degraded. The appropriate remedy is to filter out the uninteresting high frequencies at as early a stage as possible. This is a further reason why a carefully tailored frequency response is an important part of a power spectrum measuring process.

Together with the need for *adequately wide filters* (we can of course use narrower filters when we are prepared to average over homogeneous records of sufficiently long total duration) to provide enough equivalent degrees of freedom, and hence enough stability for the estimates, this tailoring of frequency response is often the crucial part of a power spectrum measuring program. Indeed, there may sometimes be no reasonable way to measure power spectra with an ill-tailored frequency response, even if this response be "flat".

Equally Spaced Records

We come now to treat a modified situation of great practical importance, where the observations are used for analysis only at equally spaced intervals of time — not as a continuous time record. Two new and important features enter: there is aliasing of frequencies, and practical analysis will involve digital rather than analog computation. In general, however, the situation is surprisingly similar to the case of a continuous record, with limitations on data-gathering effort still forcing us to compromise resolution and stability. Advantages of convenient calculation and noise reduction still lead us to prewhitening. Filtering of equi-spaced data must involve transversal filters (see Glossary of Terms for definition) whose transmission properties (in frequency) exhibit a periodic symmetry. This exerts additional pressure toward prewhitening.

Questions regarding computational techniques arise anew because of the nature of digital computation. These include means for reducing the effects of a displaced (perhaps drifting) zero, smoothing by groups to economize arithmetical operations on the whole, and preliminary rough estimation as an aid to planning.

12. ALIASING

We now suppose that $X(t)$ is available, or is to be used, only for uniformly spaced values of t, which we may as well suppose to be

$$t = 0, \Delta t, 2\Delta t, 3\Delta t, \cdots, n\Delta t,$$

so that $C(\tau)$ can only be estimated for

$$|\tau| = 0, \Delta t, 2\Delta t, \cdots, n\Delta t.$$

Now, the equations

$$C(\tau) = \int_0^\infty 2P_A(f) \cdot \cos 2\pi f\tau \cdot df,$$

$$|\tau| = q\Delta t, \qquad q = 0, 1, \cdots, n,$$

if soluble at all, can always be satisfied by a $P_A(f)$ which vanishes for $f > f_N = 1/(2\Delta t)$, although the power spectrum $P(f)$ of the original process (for which the $C(\tau)$ was defined) might actually cover a much wider frequency range. (We shall reserve the notation $P_A(f)$ for such a function, vanishing for $|f| > f_N$.) While frequencies between $f = 0$ and $f = f_N$ are clearly distinct from one another, we face a problem of aliasing, since frequencies above f_N usually contribute some power. Each frequency, no matter how high, is indistinguishable from one in the band from 0 to f_N.

The essential, unavoidable nature of this problem is made clear by Fig. 7 which illustrates how equally spaced time samples from any

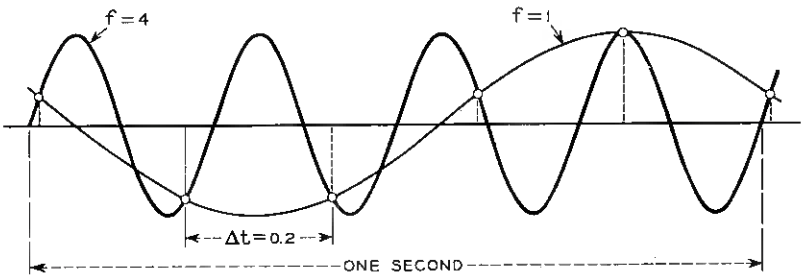

Fig. 7 — Sampling of sinusoidal waves.

cosine wave *could* have come from each of many other cosine waves. (The familiar stroboscope uses a particular expression of this fact in apparently "slowing down" rapidly rotating or oscillating machinery.)

The logical position about $P_A(f)$ depends very much on whether $X(t)$ is thought of as having any real existence for $|t| \neq q\Delta t$.

If $X(t)$ really exists for continuous t, although we have (*i*) failed to observe or record it, or (*ii*) failed to "read" the record, or (*iii*) decided to neglect the available values, then there is a well-defined $P(f)$ corresponding to the process from which each $X(t)$ is a sample, and we must be very careful about the relation between $P(f)$, which is our true concern, and $P_A(f)$, which is clearly all we can strive to estimate directly from the data. It can be shown (see Section B.12) that, in the form appropriate for a one-sided spectrum, if we set

$$2P_a(f) = 2P(f) + 2P(2f_N - f) + 2P(2f_N + f) + 2P(4f_N - f) + 2P(4f_N + f) + \cdots$$

then we may take

$$P_A(f) = \begin{cases} P_a(f), & 0 \leqq |f| \leqq f_N, \\ 0, & \text{otherwise} \end{cases}$$

where $f_N = 1/(2\Delta t)$ is the *folding* (or Nyquist) frequency. We naturally call the frequencies f, $2f_N - f$, $2f_N + f$, $4f_N - f$, $4f_N + f$, and so on, *aliases* of one another, f being the *principal alias*. The *aliased spectrum* $P_a(f)$ is the result of *aliasing* $P(f)$. The *principal part of the aliased spectrum* $P_A(f)$ is the part of $P_a(f)$ which corresponds to principal aliases, positive and negative.

(If $X(t)$ has no natural existence for t's which are not integral multiples of Δt, then $P(f)$ is not uniquely defined, and we are at liberty to choose any normalization we desire. In particular, we may decide to limit $P(f)$ to the interval $|f| \leqq 1/(2\Delta t)$, in which case we will be enforcing $P(f) = P_A(f)$ without any trace of aliasing. We mention this case for logical completeness, but remark that it seems to occur infrequently in practice, whatever the field.)

If the Gaussian noise we are considering has a power spectrum $P(f)$ which extends outside $|f| \leqq 1/(2\Delta t)$, then the Gaussian noise with spectrum $P_A(f)$ is not the same for continuous time. However, if we consider these two noises only for equi-spaced times

$$t = 0, \Delta t, 2\Delta t, \cdots\cdots$$

they are identical. For all first moments vanish and all second moments

coincide, which implies coincidence of the joint distributions of any finite set from $\cdots, X_{-q}, \cdots, X_{-1}, X_0, X_1, \cdots, X_q, \cdots$, and this is our definition of the coincidence of two noises. (If a result concerning such equally spaced values can be established for a Gaussian noise restricted to have $P(f)$ vanish outside $|f| \leqq 1/(2\Delta t)$, it must trivially hold, under the same restriction, when all occurrences of $P(f)$ are changed to $P_A(f)$. It is a consequence of the identification just established that the result, when expressed in terms of $P_A(f)$, must also hold for any Gaussian noise whatever.)

The frequency interval from 0 to f_N contains a certain number of elementary frequency bands in the sense of our treatment of variability. The total length of record is $T_n = n\Delta t$, and if we write $T'_n = n'\Delta t$ for the effective length, then, since

$$\frac{f_N}{\text{elementary frequency bandwidth}} = \frac{\frac{1}{2\Delta t}}{\frac{1}{2T'_n}} = n'$$

there are n' elementary frequency bands between 0 and f_N. As a statistician would have anticipated, we gain one elementary frequency band — one degree of freedom — for each added observation.

It is perhaps natural to base a hope on this result — a hope that taking data more frequently over the same time interval would gain us many degrees of freedom and reduce our difficulties with variability. However, this is not the case (as the expression for the width of an elementary frequency band $1/(2T'_n)$ should have warned us). Taking observations twice as frequently yields twice as many elementary frequency bands, but also doubles the folding frequency f_N and, thus, doubles the frequency interval occupied by principal aliases. The density of elementary frequency bands is not increased one iota. (Clearly, iota was the Greek word for bit!).

It is usual for aliasing to be present and to be of actual or potential importance. This is an inescapable consequence of data taken or read at uniform intervals. (It is not infrequently suggested that there should be a workable scheme of taking discrete data in some definite, but not uniformly spaced pattern, and estimating the power spectrum without aliasing. No such scheme seems so far to have been developed).

13. TRANSFORMATION AND WINDOWS.

Given uniformly spaced values of $X(t)$ — values which we shall now designate $X_0, X_1, \cdots, X_n$, as well as $X(0), X(\Delta t), \cdots, X(n\Delta t)$ — we

expect to calculate "sample autocovariances", modify them, and then Fourier transform the results. There is no possibility of calculating autocovariances for lags other than 0, Δt, $\cdots$, $n\Delta t$, and so we may as well write C_0, C_1, $\cdots$, C_m in place of $C_i(0)$, $C_i(\Delta t)$, $\cdots$, $C_i(m\Delta t)$. If we Fourier transform these $m + 1$ numbers, as obtained or modified, we might obtain a smoothed spectral estimate for any frequency between 0 and $f_N = 1/(2\Delta t)$ that we may wish. It is not surprising, however, that we lose no information (and little explication) if we calculate only $m + 1$ such estimates (one for each C_τ). Nor is it surprising that we regularly take these estimates equally spaced over $0 \leqq f \leqq f_N$, and hence at intervals of $f_N/m = 1/(2m\Delta t)$. As a consequence we have to deal with *finite* Fourier (cosine) *series* transformation (classical harmonic analysis) rather than with *infinite* Fourier *integral* transformation, but the correspondence between multiplication and convolution persists.

The question of modification also requires discussion. In the continuous case we Fourier transformed

$$C_i(\tau) = D_i(\tau) \cdot C_{00}(\tau) = D_i(\tau) \cdot C_0(\tau)$$

where $C_0(\tau)$ coincided with $C_{00}(\tau)$ wherever the latter was defined, and is zero otherwise (cp. Section B.5). The result was, consequently (e.g. see Appendix A.3), the convolution of the Fourier transforms of $D_i(\tau)$ and $C_0(\tau)$. So long as time was continuous and computation was presumably by analog devices, there was a real advantage to modification before transformation. Now that time is discrete and computation presumably digital, the advantage is transferred to first transforming and then convolving. Indeed, because the $D_i(\tau)$, for $i > 1$, are finite sums of cosines, so that their transforms are simply sums of spikes (Dirac delta-functions) at the appropriate spacing, convolution means only smoothing with weights

$$0.25,\ 0.5,\ 0.25 \qquad (i = 2, \text{ hanning})$$

$$0.23,\ 0.54,\ 0.23 \qquad (i = 3, \text{ hamming})$$

and is very simply carried out.

In discussing this program, we gain some generality by using $m + 1$ lags separated by $\Delta\tau = h\Delta t$ for an integer $h > 0$, while our results are no more complicated than if we were to confine ourselves to $h \equiv 1$, which is the practical case. Thus, we first compute the *mean lagged products*

$$C_\tau = \frac{1}{n - \tau h} \sum_{q=0}^{q=n-\tau h} X_q \cdot X_{q+\tau h}$$

for $r = 0, 1, 2, \cdots, m$, where $mh < n$. Note that C_r is heuristically as close as we can come to the apparent autocovariance $C_{00}(\pm r\Delta\tau)$ with the available (equi-spaced) data. Note further that, so far as functions of the C_r are concerned, our effective folding frequency is

$$f_N{}^* = \frac{1}{2\Delta\tau} = \frac{1}{h} f_N .$$

We will usually need to adjust the C_r somewhat to improve very-low-frequency performance, as discussed in Section 19, but this need not concern us for the moment.

Applying a *discrete finite* cosine *series* transform to the sequence C_0, C_1, $\cdots$, C_m, we find

$$V_r = \Delta\tau \cdot \left[C_0 + 2 \sum_{q=1}^{m-1} C_q \cdot \cos \frac{qr\pi}{m} + C_m \cdot \cos r\pi \right].$$

(We may regard this as arising from replacing $C_0(\tau)$ in the expression for $P_0(f)$ as its Fourier integral transform by a finite sequence of spikes (Dirac delta functions) of intensities (areas) proportional to the corresponding values of $C_0(\tau)$.) If we put

$$P_{0A}\left(\frac{r}{2m \cdot \Delta\tau}\right) = V_r$$

then it is shown in Section B.13 that

$$\text{ave } \{P_{0A}(f)\} = \int_{-\infty}^{\infty} Q_0(f - f'; \Delta\tau) \cdot P(f') \cdot df'$$

where

$$Q_0(f; \Delta\tau) = \Delta\tau \cdot \cot \frac{\omega\Delta\tau}{2} \cdot \sin m\omega\Delta\tau.$$

In terms of $Q_0(f)$, which is treated in Section B.5, we have

$$Q_0(f; \Delta\tau) = \sum_{q=-\infty}^{\infty} Q_0\left(f - \frac{q}{\Delta\tau}\right) = Q_{0A}(f).$$

Just as the average value of $P_0(f)$ in the continuous case is the corresponding value of $Q_0(f) * P(f)$, so here the average value of $P_{0A}(f)$ is the corresponding value of $Q_0(f; \Delta t) * P(f)$. Thus, we may consider $P_{0A}(f)$ as estimating the result of "smoothing" $P(f)$ with a window $Q_0(f; \Delta\tau)$ which has repeated major (and concomitant minor) lobes at intervals of $2f_N{}^* = (\Delta\tau)^{-1}$. This is not the most convenient way to consider matters, and in Section B.13 it is shown that there are two equivalent forms for

ave $\{P_{0A}(f)\}$ and, correspondingly, two other, equally appropriate, ways to consider the situation.

These arise from the three-fold identity

$$Q_{0A}(f) * P(f) \equiv Q_0(f) * P_a(f) \equiv Q_{0A}(f) * P_A(f),$$

any member of which represents the average value of $P_{0A}(f)$. Thus, we can also consider $P_{0A}(f)$: (i) as estimating the result of smoothing the infinite, periodic aliased spectrum $P_a(f)$ with the same window as for the continuous case, or (ii) as estimating the result of smoothing the principal part of the aliased spectrum $P_A(f)$ with the aliased window $Q_{0A}(f)$. The latter choice is usually the most helpful of the three possibilities, and is the one we shall adopt.

All this has been discussed for the immediate results of transforming unmodified C_r's. This is only the case $i = 0$ of the identity

$$Q_{iA}(f) * P(f) \equiv Q_i(f) * P_a(f) \equiv Q_{iA}(f) * P_A(f)$$

which holds in general. We should thus usually be concerned with $Q_{iA}(f)$ and with $P_A(f)$.

The case $i = 2$ (hanning) corresponds to the following smoothing after transformation:

$$\begin{aligned} U_0 &= 0.5\ V_0 + 0.5\ V_1\,, \\ U_r &= 0.25\ V_{r-1} + 0.5\ V_r + 0.25\ V_{r+1}\,, \qquad 1 \leqq r \leqq m-1, \\ U_m &= 0.5\ V_{m-1} + 0.5\ V_m\,, \end{aligned}$$

for which $Q_{2A}(f)$ has the form shown in Fig. 8. The curve is for $m = 12$, and the circles are for $m = \infty$, which corresponds exactly to the continuous case. Clearly, for usual values of m, the modification in the lobes due to aliasing is almost surely unimportant.

The frequency separation between adjacent estimates is

$$\frac{1}{2T_m} = \frac{1}{2m\Delta\tau},$$

but the equivalent width of the windows (for $1 \leqq r \leqq m-1$) is about

$$\frac{1.30}{T_m} = \frac{1.30}{m\Delta\tau},$$

just as for the continuous case (see Section 8). For most purposes we may again take the bandwidth corresponding to each estimate as $1/T_m$,

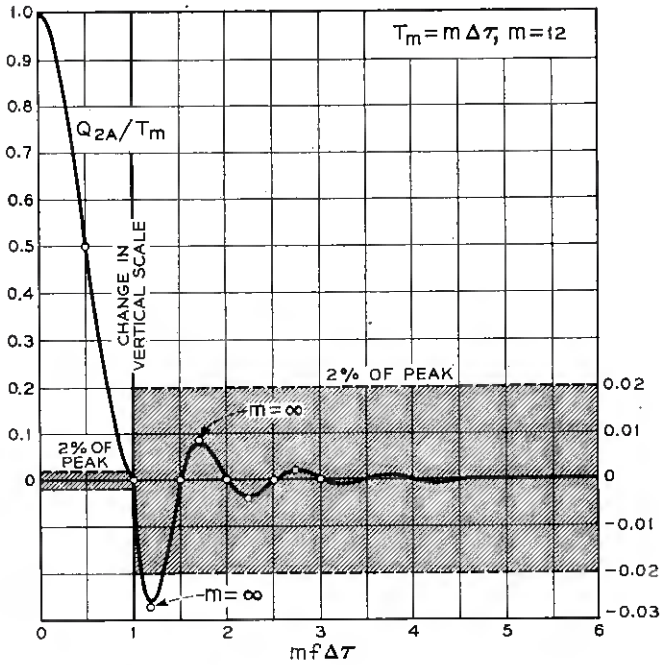

Fig. 8 — Aliased spectral window Q_{2A} for $m = 12$.

so that m satisfies

$$\frac{1}{m} = (\text{bandwidth of estimates}) \cdot (\Delta\tau).$$

If we had neither modified before Fourier transformation, nor smoothed after transformation, we should have faced the uncomfortable minor lobes of $Q_{0A}(f)$ shown in Fig. 9 for $m = 12$ (with circles for $m = \infty$). Generally speaking, all we learned about desirable lag windows for the continuous case carries over with minor modifications, at most. The only serious effect of going to uniformly spaced values is the aliasing (and this may be very serious indeed).

It is well worth noting that the possible spectral windows $Q_{iA}(f)$ are now restricted to be finite Fourier series in $\cos \omega\Delta\tau$, $\cos 2\omega\Delta\tau$, $\cdots$, $\cos m\omega\Delta\tau$, or equivalently, to be polynomials in $\cos \omega\Delta\tau$ of degree m at most.

14. VARIABILITY AND COVARIABILITY

We now extend all our other notation: $H_i(f; f_1)$, $P_i(f_1)$, etc. to corresponding $H_{iA}(f; f_1)$, $P_{iA}(f_1)$, etc. for the uniformly spaced case as

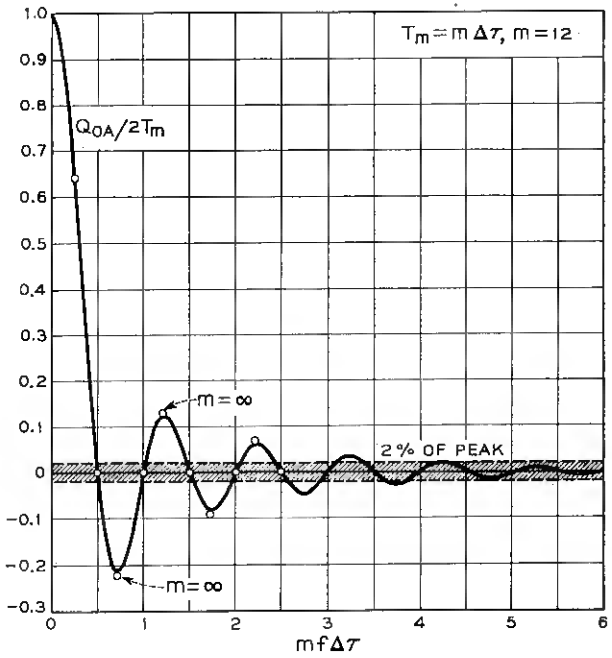

Fig. 9 — Aliased spectral window Q_{0A} for $m = 12$.

specified in Sections B.13 and B.14. It is shown in the latter section that we now have

$$\text{cov } \{2P_{iA}(f_1), 2P_{jA}(f_2)\} = \int_0^\infty H_{iA}(f; f_1) \cdot H_{jA}(f; f_2) \cdot 2\Gamma_{\Delta t}(f) \cdot df$$

where

$$\Gamma_{\Delta t}(f) \approx 4 \int_{-\infty}^{\infty} P_A(f' + f) \cdot P_A(f' - f) \cdot \left(\frac{\sin \omega' T_n'}{\omega' T_n'}\right)^2 \cdot \left(\frac{\sin \omega' \Delta t}{\omega' \Delta t}\right)^{-2} df',$$

($\omega' = 2\pi f'$), with a very slightly different determination of T_n' than before. The only essential change has been the introduction of a new factor, corresponding to aliasing,

$$\left(\frac{\sin \omega' \Delta t}{\omega' \Delta t}\right)^{-2}$$

into the integrand of the power-variance spectrum $\Gamma_{\Delta t}(f)$. For usable values of n, this factor will vary much more slowly than

$$\left(\frac{\sin \omega' T_n'}{\omega' T_n'}\right)^2$$

and can usually be treated as sensibly equal to unity. All the approximate analysis of covariability and variability given for the continuous case now goes through without essential change.

15. PREWHITENING

If the equally spaced data is sampled from a continuously transmitted "signal" or "read" from a continuous recording, then all the points made in Section 11 in favor of early prewhitening are still applicable. If the equally spaced data arises more directly, as by photographing a physical situation, we may not be able to apply prewhitening early. In either case it may still be desirable to prewhiten after the data is obtained at equal intervals, either as a supplement to, or as a partial replacement for, early prewhitening.

The average value of a power density estimate $P_{iA}(f_1)$ is

$$\text{ave } \{P_{iA}(f_1)\} = \int_0^\infty P_{iA1}(f) \cdot df,$$

where

$$P_{iA1}(f) = H_{iA}(f; f_1) \cdot P_A(f).$$

We want this quantity to tell us about the values of $P(f)$ for f near f_1. To do this we must: (i) reduce variability, (ii) ensure that $P_A(f)$ resembles $P(f)$ sufficiently, and (iii) concentrate $P_{iA1}(f)$ near $f = f_1$. We must be concerned with: (i) adequately broad windows, (ii) sufficiently weak aliasing, and (iii) enough sharpness in the effective filter. This sharpness can be obtained in a combination of ways.

Note that we asked for $P_{iA1}(f)$, which measures the net contribution to the average value, to be localized. We did not merely ask that $H_{iA}(f; f_1)$ should be localized. For, if

$$P_A(f_2) >>> P_A(f_1),$$

although

$$H_{iA}(f_2 ; f_1) << H_{iA}(f_1 ; f_1),$$

it is still possible for

$$P_{iA1}(f_2) = H_{iA}(f_2 ; f_1) \cdot P_A(f_2)$$

to outweigh

$$P_{iA1}(f_1) = H_{iA}(f_1 ; f_1) \cdot P_A(f_1),$$

so that our estimate tells us more about $P(f)$ near $f = f_2$ than it does about $P(f)$ near $f = f_1$. To avoid such unfortunate situations either we

must choose our window pair in a very particular manner (so as to make $H_{iA}(f_2\,;\,f_1)$ exceptionally small) or we must avoid $P_A(f_2) >>> P_A(f_1)$. Both courses are possible and sometimes necessary. Usually, the second course is simpler.

Following the second course is simple in principle. Given actual values X_q, we apply a selected linear procedure to obtain new values $\tilde{X}_q$ and analyze these. The aliased spectrum $\tilde{P}_A(f)$ of the $\tilde{X}_q$ differs from the aliased spectrum $P_A(f)$ of the X_q by a known multiplicative function of frequency. (See Section B.15 for details.) Thus, (i) we may convert estimates of $\tilde{P}_A(f)$ into estimates of $P_A(f)$, and (ii) we may choose the linear procedure to make the aliased spectrum $\tilde{P}_A(f)$ of the $\tilde{X}_q$ reasonably flat.

The simplest linear procedures are probably the formation of moving linear combinations and the construction of autoregressive series. A simple example of a moving linear combination is

$$\tilde{X}_q = X_q - \alpha X_{q-1} - \beta X_{q-2} - \gamma X_{q-3}$$

for which the relation between the spectra is

$$\frac{\tilde{P}_A(f)}{P_A(f)} = \frac{\tilde{P}(f)}{P(f)} = |\,1 - \alpha e^{-i\omega\Delta t} - \beta e^{-i2\omega\Delta t} - \gamma e^{-i3\omega\Delta t}\,|^2$$

$$= \text{a cubic in } \cos\,\omega\Delta t.$$

A suitable moving linear combination will generate any desired non-negative polynomial in $\cos\,\omega\Delta t$.

A simple example of an autoregressive combination is

$$\tilde{X}_q = X_q + \lambda\tilde{X}_{q-1} + \mu\tilde{X}_{q-2} + \nu\tilde{X}_{q-3}$$

for which the relation (reciprocal to that just considered) between the spectra is

$$\frac{\tilde{P}_A(f)}{P_A(f)} = \frac{\tilde{P}(f)}{P(f)} = \{|\,1 - \lambda e^{-i\omega\Delta t} - \mu e^{-i2\omega\Delta t} - \nu e^{-i3\omega\Delta t}\,|^2\}^{-1}$$

$$= (\text{a cubic in } \cos\,\omega\Delta t)^{-1}.$$

A suitable autoregressive combination will, when indefinitely continued, generate the reciprocal of any desired non-negative polynomial in $\cos\,\omega\Delta t$.

By combining a suitable moving linear combination with suitable autoregression, as for instance in

$$\tilde{X}_q = X_q - \alpha X_{q-1} + \lambda\tilde{X}_{q-1}\,,$$

which may also be written

$$\tilde{X}_q - \lambda \tilde{X}_{q-1} = X_q - \alpha X_{q-1} ,$$

for which

$$\frac{\tilde{P}_A(f)}{P_A(f)} = \frac{\tilde{P}(f)}{P(f)} = \left| \frac{1 - \alpha e^{-i\omega\Delta t}}{1 - \lambda e^{-i\omega\Delta t}} \right|^2$$

$$= \frac{1 + \alpha^2 - 2\alpha \cos \omega\Delta t}{1 + \lambda^2 - 2\lambda \cos \omega\Delta t}$$

$$= \text{a rational function of } \cos \omega\Delta t,$$

we can modify $P_A(f)$ by multiplication by an arbitrary non-negative rational function of $\cos \omega\Delta t$.

Freedom to multiply by any (simple) non-negative rational function of $\cos \omega\Delta t$ is very substantial freedom. If we have a rough idea (see Section 18) of the behavior of $P_A(f)$, and if this behaviour is moderately smooth, though perhaps quite steep in places, we can usually do a very good job of flattening the spectrum by prewhitening after obtaining discrete (digital) values. Unless still bothered with steep slopes, we will usually then find that hanning, with its (0.25, 0.50, 0.25) weights and lower outer lobes is slightly preferable to hamming, with its (0.23, 0.54, 0.23) weights and reduced first minor lobes.

The main purpose of prewhitening *after* data has been obtained in digital form at equally spaced intervals is to avoid difficulty with the minor lobes of our spectral windows. We may regard the whole process of prewhitening, analysis with standard spectral windows, and, finally, compensation of estimate, as a means of constructing a set of specially shaped spectral windows, one for each center frequency, specially adapted to the data we are processing. This point of view is illustrated in Fig. 10. The uppermost curve shows the power transfer function of a hypothetical prewhitening filter, one which enhances mid-frequencies in comparison with those lower and higher. The next line shows two standard spectral windows, with symmetrical side lobes. The third line shows the effective spectral windows when prewhitening is followed by standard analysis, as given by the product of prewhitening power transfer function and spectral window. In either case, the side lobe toward mid-frequencies is higher than the corresponding side lobe on the opposite side, which is lower than for the standard. The lowest curve shows alternative spectra for time series which might reasonably be processed by the combination of prewhitening and standard analysis shown (since the prewhitened spectra would change only slowly). In every case, the side lobes of the

special spectral windows are automatically so related to these spectra, as to balance and reduce the amount of leakage through them, as given by the product of special spectral window side lobe and original spectral density.

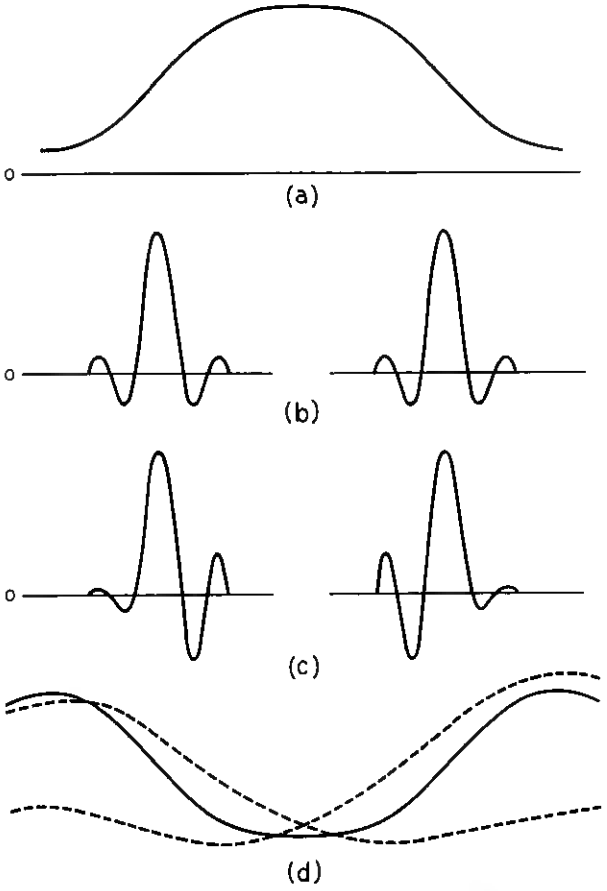

Fig. 10 — Illustration of prewhitening; (a) prewhitening power transfer function, (b) standard spectral windows, (c) effective spectral windows, and (d) typical input spectra to which (a) might be applied.

Easing of requirements for accuracy (number of significant figures, etc.) during computation are ordinarily quite secondary, though pleasant, advantages of prewhitening during digital calculation.

16. REJECTION FILTERING AND SEPARATION

If the difficulties in handling $P(f)$ are due, wholly or in part, to one or more quite narrow and very high peaks ("lines" or "narrow bands")

then we cannot expect either to afford, or to be able to estimate, the great number of accurately chosen constants which would be required to obtain a rational function whose reciprocal has a shape very close to the given narrow peak. We must adopt a slightly different approach, and plan to make at least two analyses of the data — one to estimate the behavior at the peak, and another to estimate the behavior away from the peak.

In order to separate the bulk of the information in the data from the variation associated with the sharp peak which may be troubling us, we may apply to the data a moving linear combination (possibly combined with autoregression) whose power transfer function (the factor by which the spectrum is altered) has one or more zeroes near the peak. The resulting sequence will be largely free of contribution from the peak and hence will be suitable for further prewhitening (if required) and analysis. (This operation can often, of course, be combined with further prewhitening so far as actual calculation goes. It will of course be necessary to compensate for the effects of this transformation at frequencies away from the peak, when preparing the final spectrum estimates for interpretation.)

There remains the estimation of the power in the peak, and possibly some inquiry into its width. A number of approaches are possible:

(1) We may analyze the original data as well as the data with the peak rejected, obtaining an estimate at the peak and possibly confirmatory estimates far from the peak.

(2) We may subtract a suitable multiple of the modified data from the original data so as to retain the peak and partially reduce other frequencies; and then analyze the difference.

(3) We may apply a band-pass filter to isolate frequencies at and near the peak, and then analyze the result.

Any of these techniques may be applicable in suitable circumstances.

Other related procedures are sometimes more natural than the use of moving linear combinations. Rejection of zero frequency, for example, is more naturally, and computationally more easily, accomplished by subtraction of the mean of all the data from each X_q than by the subtraction of a *moving* linear combination from each.

Rejection filtration has been applied in oceanography by Groves,[14] Seiwell,[15] Seiwell and Wadsworth,[16] to the elimination of various well-defined tides from records. It almost always has to be used to eliminate possible peaks at zero frequency (see Section 19 below).

In electronic measurements we may also anticipate its possible use in measurements: (i) close to a substantial harmonic of 60 cycles per second (such as 120 cps or 1380 cps), or (ii) near some strong "carrier".

17. SMOOTHING BY GROUPS

The cost of digital power spectrum analysis, once initial investments in programming, etc. have been made, and assuming records to have already been made and "read", is likely to be associated with the number of multiplications involved in computing the mean lagged products (in original or modified form). If there are n observations, and m lags are used, then there will be roughly nm multiplications.

Ways of reducing this number substantially are naturally of interest. Most of these must depend for their efficacy on our interest in something less than the whole spectrum. We have already discussed (in passing) a situation which would naturally arise only when we are interested only in the lower part of the aliased spectrum. This is the use of lags which are multiples of $\Delta\tau = h\Delta t$ with $h > 1$. The use of lags up to $m\Delta\tau = hm\Delta t$ allows us to explore the spectrum down to frequencies almost of the order $1/hm\Delta t$, which, had we used all multiples of Δt up to hm, would have required $hm + 1$ values of C_τ (or of its modifications) instead of $m + 1$. The price of doing this is the aliasing of the spectrum with folding frequency $1/(2\Delta\tau) = (1/h)\,(1/(2\Delta t))$, which is h times as much aliasing as if all multiples of Δt up to hm had been used, yielding a folding frequency of $1/(2\Delta t)$.

If such intensive aliasing is bearable, this procedure with $\Delta\tau > \Delta t$ is simple, even though it is not necessarily economical. Indeed, if so much aliasing were permissible, we need only have "read" every hth data value. In many situations, however, especially where Δt has been taken as large as aliasing will permit, such further aliasing is unbearable. If we are to look at the low frequency part of the aliased spectrum $P_A(f)$with computational economy, another course will have to be found.

Our use of linear schemes in prewhitening shows us a possible course. Let us begin by applying a linear scheme to the given values X_q, which attenuates all high frequencies. Then we can face further aliasing, and proceed apace.

If simplicity is controlling, then we take

$$\tilde{X}_q = X_q + X_{q-1} + \cdots + X_{q-k+1} \qquad (k \text{ terms})$$

for which the relation between the spectra (the power transfer function of the smoothing) is

$$\frac{\tilde{P}(f)}{P(f)} = \left| 1 + e^{-i\omega\Delta t} + \cdots + e^{-i(k-1)\omega\Delta t} \right|^2$$

$$= \left[\frac{\sin \dfrac{k\omega\Delta t}{2}}{\sin \dfrac{\omega\Delta t}{2}} \right]^2 .$$

This will give us zeroes at frequencies which are multiples of $1/k\Delta t$, and we can avoid folding the first two side lobes of this function onto the main lobe and still take a folding frequency as small as $2/k\Delta t$. Such a choice will fold the second, fifth, sixth, etc. side lobes onto the first side lobe, and it will fold the third, fourth, seventh, eighth, etc. side lobes onto the main lobe. We obtain such a folding frequency by retaining only one in every $k/4$ of the $\tilde{X}_q$'s. These *decimated** $\tilde{X}_q$'s may, in particular, be obtained by summing the $\tilde{X}_q$'s in non-overlapping blocks of $k/4$, and then summing these block sums in all possible (overlapping) sets of four successive blocks. (This requires $(k + 8)/k$ additions per original value.) The estimated spectrum below $1/k\Delta t$ has to be multiplied by

$$\left[\frac{\sin\dfrac{\omega\Delta t}{2}}{\sin\dfrac{k\omega\Delta t}{2}}\right]^2,$$

and only aliases which are usually negligible will have been superposed on the principal aliases. About one kth of the original principal spectrum will be available for analysis.

The stability obtained by this process can be easily compared with that obtained by using all X_q and taking $\Delta\tau = k\Delta t/4$. In each case, the width of the elementary frequency bands is approximately $1/2T'_n$ where T'_n has slightly different, but not substantially different values. The process just described yields nearly the same stability as $\Delta\tau = k\Delta t/4$, and usually involves much less computation, besides avoiding serious aliasing. It will almost always be preferred to using $\Delta\tau = h\Delta t$ with $h > 1$.

Other schemes of smoothing by groups are discussed in Section B.17.

18. PILOT ESTIMATION

The prewhitening procedure demands a rough knowledge of the spectrum for its effective use. Sometimes this rough knowledge can be obtained from theoretical considerations, or from past experience, but in many cases it must be obtained from a preliminary (pilot) analysis of the data. Such pilot analyses should be as simple and cheap as possible. We now discuss a pilot analysis giving very rough results quite easily.

Table III exemplifies a form of calculation which is easily carried out either entirely by hand, or with a desk calculator. The symbols "δ" and "σ" refer to differences and sums of consecutive numbers in non-overlapping pairs. Taking the numbers in non-overlapping pairs is not neces-

* Although this word should refer strictly to the deletion of only every 10th item, we shall apply it to the retention of only every jth item, for whatever j may be relevant.

TABLE III — COMPUTATION OF PILOT ESTIMATES

q	X_q	δX_q	$(\delta X_q)^2$	σX_q	$\delta\sigma X_q$	$(\delta\sigma X_q)^2$	$\sigma^2 X_q$	$\delta\sigma^2 X_q$	$(\delta\sigma^2 X_q)^2$
1	3								
2	4	1	1	7					
3	−1								
4	−2	−1	1	−3	−10	100	4		
5	2								
6	7	5	25	9					
7	5								
8	−1	−6	36	4	−5	25	13	9	81
9	−3								
10	2	5	25	−1					
11	5								
12	4	−1	1	9	10	100	8		
13	7								
14	3	−4	16	10					
15	4								
16	−1	−5	25	3	−7	49	13	5	25
17	−4								
18	2	6	36	−2					
19	4								
20	0	−4	16	4	6	36	2		
21	1								
22	−1	−2	4	0					
23	1								
24	2	1	1	3	3	9	3	1	1
25	4								
26	3	−1	1	7					
27	0								
28	−4	−4	16	−4	−11	121	3		
29	−1								
30	−2	−1	1	−3					
31	−2								
32*	−2	0	0	−4	−1	1	−7	−10	100
Totals			205			441			207

CONTINUATION OF TABLE III TO THE RIGHT (COMPRESSED)

q	$\sigma^3 X_q$	$\delta\sigma^3 X_q$	$(\delta\sigma^3 X_q)^2$	$\sigma^4 X_q$	$\delta\sigma^4 X_q$	$(\delta\sigma^4 X_q)^2$	$\sigma^5 X_q$	$(\sigma^5 X_q)^2$
8	17							
16	21	4	16	38				
24	5							
32	−4	−9	81	1	−37	1369	39	1521
			97			1369		1521

(* Note: $32 = 2^5$.)

sary, but saves much calculation at little cost in accuracy. (In this table sums and differences are entered in the lower of the two lines to which they correspond.)

The final sums of squares are roughly proportional to the power in successive octaves coming down from the folding frequency. They differ by only a constant factor, equal to the number 2^ℓ of values X_q used,

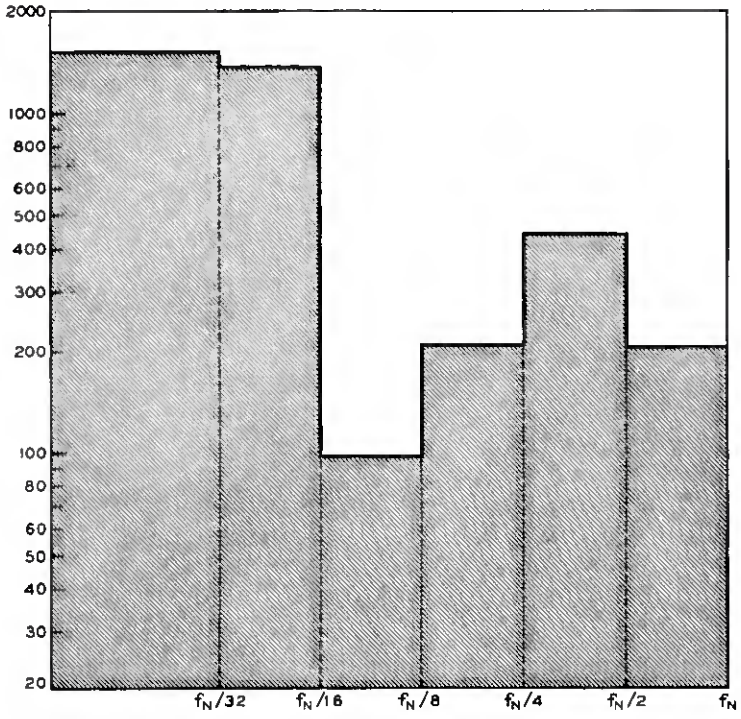

Fig. 11 — Pilot-estimated power spectrum.

from the mean squares of a nested analysis of variance. For many purposes they can be used as they come.

For the example of Table III we obtain sums of squares of 205, 441, 207, 97, 1369, and 1521. These are plotted in Fig. 11 for the successive octaves f_N to $f_N/2$, $f_N/2$ to $f_N/4$, $f_N/4$ to $f_N/8$, $f_N/8$ to $f_N/16$, $f_N/16$ to $f_N/32$, and the remaining range $f_N/32$ to 0. We see that the spectrum is roughly flat.

When medium or large stored-program digital computers are available, and the data is already available in machine-processable form (so-called diamond copy), it will often pay to use less elementary pilot calculations. Possible alternatives are discussed in Section B.18.

19. VERY LOW FREQUENCIES

The change from continuous "signals" processed in analog equipment to equally spaced "data" processed digitally has another important practical effect. Analog equipment, unless special care is taken, does not respond all the way down to zero frequency, and this automatically filters

out the very lowest frequencies. This fact allowed us, in dealing with continuous records, to treat the "signals" being processed as if they had zero means. In dealing digitally with equally spaced data, all frequencies down to zero are transmitted, *unless* we take special precautions. Consequently, we must give serious attention to the very lowest frequencies.

(We must now distinguish between power (in the sense of a line) at zero frequency and power density at zero frequency. The power spectrum of a stationary random process with zero means may have finite power density at zero frequency without having finite power there. However, finite power at zero frequency may be introduced into the data in measurement. It would then be desirable to filter out the power at (exactly) zero frequency without affecting the power density at and near zero frequency due to the stationary random process, but this cannot be done perfectly.)

The need for such attention becomes clear when we consider the effect of "small" displacements of the average. Suppose that most of the observations (say about 999 in 1000) lie between -100 to $+100$, with a few falling outside one limit or the other. This would be the case when the standard deviation is about 30, the variance about 900. If the average of the observations were 5 or even 10, we might or might not detect at a glance its failure to be zero.

The total power is the square of the average (dc power) plus the variance. Numerically, perhaps $25 + 900 = 925$ or $100 + 900 = 1000$. All the dc power belongs to the very lowest frequency band, whose width is

$$\Delta f = \frac{1}{2T'_n} .$$

If we have data at one second intervals for a period of 15 minutes, a total of 900 points, we will have a folding frequency of one-half cycle per second, and 900 elementary frequency bands before we reach the folding frequency. Thus up to one tenth of all the power may be concentrated in one 900th of the spectrum, so that the lowest frequency band has a power density up to 90 times that of the average of the 899 others. It is not surprising that precautions need to be taken to deal with such possibilities. (After all, our standard spectral windows have side lobes more than 1 per cent the height of the main lobe.)

Slow trends, which may reasonably be regarded as zero-frequency sine waves, just as constant displacements are regarded as zero frequency cosine waves, are not nearly so likely to involve quite so substantial excesses of power density, but instances of this may and do arise.

Any way of dealing with these effects must essentially remove the

lowest elementary frequency band, or both this band and the next to lowest one. In the process it will also have to eliminate some parts of the next higher elementary bands as well, since we cannot design a filtering procedure entirely free of side lobes. Two classes of ways of doing this are important. Either the X_i's can be linearly altered, as by subtracting the mean of them all from each of them, before the mean lagged products are calculated — calculated from modified data as if they were original data — or additional computations may be made and combined with either the mean lagged products or their cosine series transforms. Thus, for example, the mean of all data may be calculated and the square of this mean subtracted from each and every mean lagged product. The effect of all of these modifications can, however, be summarized as applying the finite cosine series transform to

$$C_r - E_{kr}$$

where k identifies a specific method of modification, rather than to the C_r alone.

In place of

$$\text{ave } \{P_{iA}(f)\} = Q_{iA}(f) * P_A(f),$$

we shall now have

$$\text{ave } \{P_{iAk}(f)\} = Q_{iAk}(f) * P_A(f) = [Q_{iA}(f) - R_{ik}(f)] * P_A(f)$$

where $R_{ik}(f)$ is related to the E_{kr} in the same way that $Q_i(f)$ is related to the C_r.

Details for certain special choices for E_{kr} are given in Section B.19. It is there concluded that, among others, satisfactory choices for practical calculation appear, for the present, to be, for removing possible constants,

$$E_{0r} = (\bar{X})^2 \qquad \text{(independent of } r)$$

and, for removing the effects of both possible constants and possible linear trends,

$$E_{1r} = (\bar{X})^2 + \frac{3}{16}\left(1 - \frac{1}{n^2} - \frac{2r}{n} - \frac{2r^2}{n^2}\right)(\overline{X^+} - \overline{X^-})^2$$

where $\overline{X^+}$ and $\overline{X^-}$ are the means of the right- and left-hand thirds of the X values.

WARNING: It will *almost never be wise* to fail to use some E_{kr} in a digital computation.

Analysis in Practice

The two sections which follow discuss the questioning and planning required whenever a digital analysis of equally spaced data is to be made, and exhibit a sample sequence of calculation formulas which might result from such planning. They are intended to summarize the previous material in its application to analysis. (Application to planning for measurement is treated next after this.)

20. Practical Analysis of an Equally Spaced Record

We may logically and usefully separate the analysis of an equally spaced record into four stages — each stage characterized by a question:

(a) Can the available data provide a meaningful estimated spectrum?

(b) Can the desires of the engineer for resolution and precision be harmonized with what the data can furnish?

(c) What modifications of the data are desirable or required before routine processing?

(d) How should modification and routine processing be carried out?

Failure to adequately consider any one question properly, or failure to apply any one answer, can make the entire analysis worthless.

The data presented will have come about by measuring some physical phenomenon at regular intervals. Thus,

1. the spectrum of the phenomenon
2. the frequency response of the instruments used to make the measurements
3. the probable magnitudes of measuring, and recording or reading errors, and
4. the time separation between adjacent values

are all relevant.

The first stage of consideration is to inquire generally about these quantities, and to determine whether either aliasing (see Section 12) or background noise is so heavy as to make the values almost wholly useless. Thus, if the spectrum is believed to extend up to 10 megacycles with substantial intensity, if the measuring equipment is flat to 1.2 kilocycles and is 60 db down at 5 kilocycles, and if the values are measured every $\frac{1}{200}$ of a second, we may as well stop here and go no further, since the whole available spectrum (up to 100 cycles) will be aliased more than a dozen times over. (The 1.2 kilocycle measurement bandwidth, which will be aliased 12 layers deep, will control rather than the 10 megacycle phenomenon bandwidth.)

If, on the other hand, the equipment was flat to 10 cycles, down about

6 db at 20 cycles, 15 db at 30 cycles, and 60 db at 50 cycles, we would not expect any irremovable aliasing difficulties, and would expect to be able to estimate the spectrum up to some moderate frequency — up to, say, 20 cycles, 30 cycles, or 40 cycles, depending upon how much background noise was present. (The energy above 100 cycles would not be recorded.)

In the next stage we should inquire into

1. the frequency resolution required
2. the fractional accuracy of estimation required, and
3. the total duration of data available, and the number of pieces into which it falls.

Items 1 and 3 can be combined and converted into the approximate number of elementary frequency bands (number of degrees of freedom — see Section 9 which is based on Sections 6 to 8) possibly available for each of the proposed estimates. This number can then be compared with the number of degrees of freedom required (also see Section 9) to give the desired fractional accuracy. If these are consistent, or if the desired accuracy, or the desired resolution, or both can be modified to make them consistent, then there is a good chance that the data can be persuaded to yield the desired results, and further inquiry is indicated. If not, we should stop here.

Explicit relations among duration, resolution, and fractional accuracy, the latter expressed in terms of 90 per cent interval (cp. Tables I and II), are given in Section B.23. These lead to an approximate 90 per cent spread, expressed in db (decibels), of

$$\frac{14}{\sqrt{(\text{total duration in secs})(\text{resolution in cps}) - \frac{1}{2} - \frac{1}{8}\,(\text{number of pieces})}}$$

a result which may often be conveniently used in such an inquiry.

At the beginning of the third stage, information should be sought as to

1. over what range of frequencies the spectrum is desired, and
2. whether any lines or high and narrow peaks are to be expected, and at what frequencies.

Guided by this information, it should be possible to decide whether either

a. smoothing by groups (as in Section 17) to reduce computation without loss of low-frequency information, or

b. rejection filtration (as in Section 16) to suppress well-established lines or high and narrow peaks,

or both, are desirable. If desirable, they are then carried out before or during the next step.

Unless advance information about the spectrum is exceedingly good,

a pilot analysis (see Section 18) to establish the rough form of the spectrum will now be very much worthwhile. The result (or the very good advance information, if available) will now make it possible to choose a reasonable prewhitening procedure (or, possibly, to choose not to prewhiten). Once suitable prewhitening (see Sections 11 and 15) has been chosen, and either carried out or planned for, the third stage is complete.

Finally, the information on resolution and accuracy combine to specify the width of spectral window desired, and hence (see Section 13) the number of lags for which mean lagged products should be calculated. When these are in hand, they are modified and transformed (or, perhaps more simply, transformed and convolved — see Section 13), adjusted to screen out very low frequencies, and the resulting power density estimates are corrected for the prewhitening, and for grouping and/or rejection filtration (if any) used. The final estimates are best plotted on a logarithmic power scale, since their accuracy will be roughly constant on this scale. Crude confidence limits can then be calculated from the number of degrees of freedom (see Section 9) which would be present in the individual estimates if: (i) the process were Gaussian, and (ii) the prewhitened spectrum were flat. (The factor of safety of Section 8 will ordinarily be adequate.)

21. SAMPLE COMPUTING FORMULAS

We cannot prescribe one set of computing formulas for general use, since there are rational reasons for different choices. All we can do is illustrate a procedure which may work fairly well in many cases. (And our example is not likely to be the only one with such properties. If the reader understands, by comparison with adjacent sections, just why we do what we do, he can compare other procedures with this example in a meaningful way. He will have to understand much of what is said in order to do this.)

If X_t, $t = 0, 1, \cdots, n$ are the given observations, which we will treat as if at unit spacing, it is likely that $P_A(f)$ decreases substantially as f goes from 0.0 to $0.5 = f_N$. (If it does not, then aliasing is likely to have been serious, and satisfactory analysis at this spacing may be impossible.) Prewhitening by

$$\tilde{X}_t = X_t - 0.6\, X_{t-1}$$

which multiplies $P_A(f)$ by

$$1.36 - 1.20 \cos 2\pi f,$$

a factor increasing from 0.16 to 2.56, *may* be a wise prewhitening. (The index t will now start at 1, and not at zero.)

We calculate next

$$C'_r = \frac{1}{n-r}\sum_{1}^{n-r} \tilde{X}_t \tilde{X}_{t+r} - \left(\frac{1}{n}\sum_{1}^{n} \tilde{X}_t\right)^2,$$

namely mean lagged products with an adjustment for the mean. (Further adjustment for a linear trend might have been necessary. See Section 19.) Let us suppose that we do this for $r = 0, 1, 2, \cdots, 24 = m$. (Some other choice may have been appropriate.)

Next we calculate the finite cosine series transform

$$V_r = \left[C'_0 + 2\sum_{q=1}^{m-1} C'_q \cdot \cos\frac{qr\pi}{m} + C'_m \cdot \cos r\pi\right]$$

and the results of hanning (see Sections 5 and 13)

$$\begin{aligned} U_0 &= \tfrac{1}{2}(V_0 + V_1) \\ U_r &= \tfrac{1}{4}V_{r-1} + \tfrac{1}{2}V_r + \tfrac{1}{4}V_{r+1}, \qquad 1 \leqq r \leqq m-1, \\ U_m &= \tfrac{1}{2}V_{m-1} + \tfrac{1}{2}V_m . \end{aligned}$$

These can then be corrected for both prewhitening and the correction for the mean by forming (see Section B.21)

$$\begin{aligned} &\frac{n}{n-m}\,\frac{1}{1.36 - 1.20\cos\dfrac{2\pi}{6m}}\,U_0, \\ &\frac{1}{1.36 - 1.20\cos\dfrac{2r\pi}{2m}}\,U_r, \qquad 1 \leqq r \leqq m-1, \\ &\frac{1}{1.36 - 1.20\cos\left(1 - \dfrac{1}{6m}\right)2\pi}\,U_m, \end{aligned}$$

as smoothed estimates of the power density. Estimates with subscript 0 will apply in the range just above zero frequency, those with subscript r near a frequency of $r/(2m)$ cycle per observation, and those with subscript m in the range just below a frequency of 0.5 cycle per observation.

In interpreting these estimates four cautions are important:

(a) aliasing of frequencies (see Section 12) may have taken place,

(b) the estimates are smoothed with a crudely isosceles triangular weighting function (see Sections 5 and 13) of full width $4/(2m)$,

(c) no estimate will be more stable than chi-square on $(2n)/m$ degrees of freedom and, wherever the spectrum is not smooth, the stability of the estimates will be appreciably less (see Section 9),

(d) adjacent estimates will not have independent sampling errors, though those not adjacent are at least very close to being uncorrelated.

The units involved are such that the smoothed one-sided, aliased power density on $0.0 \leqq f \leqq 0.5$ is approximated by twice the estimates. The pieces into which the variance would be divided, each coming from a frequency band of width $1/(2m)$ cycles per observation, are estimated by $1/(2m)$ times the corrected estimates.

Planning for Measurement

Up to this point, with the exception of part of Section 11, our discussion has been concerned (i) with what happens when certain operations are performed, and hence (ii) with how we should make the best of what we already have.

The third aspect — planning the measurements or observations to meet requirements — has not been adequately treated. (Both statisticians and engineers concerned with measurement will agree that this is the most vital aspect of all, but will, unfortunately, also have to admit that, all too often, "salvage" work will be required because this third aspect was omitted, and the observations made unwisely.)

In discussing "What data shall we take?", "How shall we measure it?", the same considerations will recur as in discussing "How shall we analyze it?", but (i) they will be looked at from quite different aspects and (ii) they will be even more important. Now, by planning in advance of data-gathering, we may be able either to replace useless or difficult-to-analyze measurements by usable ones, or to avoid making measurements which could never provide the desired information.

The first basic decision has to do with the type of recording and analysis to be used. Three types are in use today:

(1) *Spaced:* Analog use of intermittent recorders (photography of situations or of dials, etc.) or digital recording at equally spaced intervals (electronic reading of dials, photography of counters, etc.).

(2) *Mixed:* Continuous recording (on film, calibrated paper rolls, etc.) with the intention of analyzing equally spaced values to be "read" from these continuous records.

(3) *Continuous:* Continuous recording (FM recording on magnetic tape, etc.) with the intention of making an analog analysis.

The choice among these types will depend on their particular advantages and disadvantages, and on the availability of equipment, both for recording and analysis. In almost every case, however, the detailed problems will be surprisingly similar.

22. CHOICE OF FREQUENCY RESPONSE

In each instance there will be a problem of the response of the observing and transmitting or recording elements to high frequencies. When less quantitative studies are made, it is usual to worry whether the high-frequency response is large enough to "follow" the phenomena precisely. To be sure, if recording is only at intervals, and the needle is so blurred as not to be read, the high-frequency response may indeed be reduced by filtering. Such filtering is too likely to be regarded as unfortunate rather than helpful. Effort tends always to be applied for "faithful" recording. This is appropriate for recording specific *individual* time histories for *visual* study, but is often *most inappropriate* for recording *sample* time histories for *statistical* study with the aid of sensitive *filters* (analog or digital). (When the recording is continuous, be it on film, oscillograph paper, or magnetic tape, the "writing" means has a limited frequency response, and this will usually help to keep the record from blurring.)

When the analysis is to be made on equally spaced data, whether the recording be continuous or equi-spaced, there is a real problem of aliasing. And there is need for a basic choice of a frequency cutoff, usually in terms of two frequencies such that (i) the experiment is only concerned with frequencies up to the lower one, and (ii) frequencies beyond the upper one will not be recorded. The need for such a choice in a continuous system may not appear to be so acute, since only problems of noise or non-linear distortion are involved (see Section 11). Yet in practice, it will almost always be made — indirectly — by the choice of a writing speed (which implies a frequency cutoff for a continuous recorder). Economic pressures to reduce both the volume of record, and the extent of measurement and computation, act to lower the frequency cutoff, while desires to follow the spectrum to higher frequencies act to raise it. The proper choice comes from balancing these pressures.

Sometimes in mixed systems, when continuously recorded data is to be subjected to equi-spaced analysis, an attempt is made to compromise matters by recording with a high cutoff, and then asking that the measurements of this record be "eye averages" over periods long enough for the record to show considerable variation. Such compromises do not seem to work nearly as well in practice as their proponents suppose. Re-

placing the "averages" by the results of "reading to the line" at equi-spaced points often seems to give better results, even though a smaller, but unknown amount of aliasing is thus replaced by a larger, known amount. Putting the filtering into the observing and writing equipment, rather than into the (human) measurer and transcriber, will usually do even better — better by a large margin.

If one can be confident of the upper limit, beyond which the power spectrum will not be needed, it is usually best to record with a related frequency cutoff, thus reducing noise complications, aliasing difficulties, and the necessary bulk of the record.

Conversely, however, points must be recorded or measured frequently enough (or a high-enough writing speed used) so that aliasing (or loss of high-frequency response) is not serious. (For a given maximum usable frequency, the sharper the cutoff, the less stringent this requirement.)

To summarize, the problems surrounding aliasing should lead to the choice of a frequency cutoff which is usefully described by two frequencies (which may reasonably be in the ratio of 1 to 2):

(a) a lower frequency, which is the highest at which important power spectrum estimates will be made, and

(b) a higher frequency, at and above which no serious amount of recording is done.

Both of these need to be chosen before settling finally on observing and recording equipment. If equi-spaced data is produced, the folding frequency may be as low as half-way between these two frequencies.

A prime essential to keep in mind is that all measurement, transmission, *and analysis* systems are essentially band-limited. It is always inadvisable to try to cover too many octaves of log frequency while using exactly the same techniques.

23. DURATION OF DATA REQUIRED

Instead of trying to compromise resolution and stability within the limitations of available data, we may now consider the costs and advantages of getting still more data, or, perhaps, somewhat less data. We face a three-way compromise among effort, resolution, and stability (precision) of estimate.

Effort has to be measured in various ways, but the duration of initial record will almost certainly have to be considered as one measure. It is shown in Section B.23, where both precise definitions of the quantities, and a corresponding formula for the necessary numbers of pieces of a given length will also be found, that

$$(\text{total duration in seconds}) = \frac{\frac{1}{2} + \frac{200}{(90\%\text{ range in db})^2} + \frac{(\text{pieces})}{3}}{(\text{resolution in cps})}.$$

If, for example, a resolution of 0.1 cps is to be obtained from 6 pieces of record and is to furnish stability of ± 2 db for (on the average) $\frac{9}{10}$ of the individual estimates, then the necessary duration will be

$$\frac{\frac{1}{2} + \frac{200}{4^2} + \frac{6}{3}}{0.1} = 150 \text{ seconds.}$$

This applies equally to analog processing of continuous records or to digital processing of spaced records, so long as we apply the best methods which we know to a shape of spectrum which is not exceptionally difficult to handle.

24. AMOUNT OF DIGITAL DATA-HANDLING REQUIRED

If spaced data are to be digitally processed, both the number of data points to be used and the number of multiplications involved are of interest.

If we can easily build in the desirable frequency cutoff, and have to resolve a number of equally spaced bands spaced evenly from zero frequency to some maximum frequency, then we will require about

$$\left[\frac{3}{2} + \frac{600}{(90\%\text{ range in db})^2} + (\text{pieces})\right] (\text{number of bands resolved})$$

data points and, roughly about

$$\left(\frac{9}{2} + \frac{1800}{(90\%\text{ range in db})^2} + 3\ (\text{pieces})\right) (\text{number of bands resolved})^2$$

multiplications.

These last two results often give only preliminary indications. Aliasing difficulties will increase these numbers. The possibility of smoothing by groups will decrease them. Details and possible modifications of the proposed system of data gathering and analysis need to be studied carefully before final estimates of the number of data points and the rough number of multiplications are finally settled upon.

25. QUALITY OF MEASUREMENT AND HANDLING

In every case, careful consideration should be given to the quality of measurement and data handling required (in terms of the dangers of,

e.g.: time-varying frequency response, introduced noise, intermodulation distortion, etc.). An extensive catalog would be out of place here, since the problems are basically those of instrumentation engineering. But a few reminders may indicate the diversity of problems which might arise.

A camera may be "clamped" to some object to record the relative orientation of that object and something visible to the camera. The mounting of the camera is never perfectly rigid, and vibrations will occur ordinarily at frequencies far above the data-taking rate. Whatever the frequency, these vibrations will introduce "noise" into the record. At least an order-of-magnitude calculation of the effects of likely vibration is needed.

Storage of a signal on magnetic tape will be a part of many measurement-analysis systems. Because only rough spectra are wanted, AM (amplitude modulation) recording may be planned. If the fact that AM recording and playback is subject to considerable fluctuation in over-all gain (db's, not tenths db) is neglected, measurement planning may be quite misleading.

In a complex analysis, where several spectra and cross-spectra (whose analysis we have not specifically discussed) are involved, it might be planned to plot the estimates of each spectrum and cross-spectrum against frequency, draw smooth curves, and compute derived quantities from values read from these curves. Such a process has led to great difficulties in certain actual situations, because of the "noise" introduced by such visual smoothing which appears to have distinctive but unknown properties. Such a graphical step may appear to be good engineering, but it cannot be high quality data handling. Its use may nullify the careful selection of other data processes, some of which are delicately balanced.

Graphical analysis should ordinarily be reserved for:

(a) display of whatever spectrum or function of spectra is really a final output,

(b) description of the actual effects of computational procedures, and

(c) trouble-shooting.

20. EXAMPLE A

Suppose first that the spectrum of some aspect of the angular tracking performance of a new radar is to be obtained; that angular tracking can only be studied by photographing the target with a camera clamped to the antenna; that frequencies near 0.27 cps are of special interest; that the spectrum of tracking performance at higher frequencies is relatively flat up to 10 cps and then falls rapidly enough to be negligible beyond 40

cps; that estimates at all frequencies up to 25 cps are desired; and that stability to ± 1 db is derived. What are the requirements?

The total amount of tracking required is fixed by the resolution requirement near 0.27 cps, which we may suppose to be either 0.05 cps or 0.02 cps. These lead, respectively to durations of

$$\left(\frac{1}{2} + 50 + \frac{p}{3}\right)\frac{1}{0.05} > 1000 \text{ seconds}$$

and

$$\left(\frac{1}{2} + 50 + \frac{p}{3}\right)\frac{1}{0.02} > 2500 \text{ seconds.}$$

Single stretches of either 16 or 40 minutes continuous tracking are almost certain to be out of the question. The length of piece available would depend on the aspect of tracking performance studied, but a fair figure for this illustration might be 200 seconds. Going to Section B.23 for the necessary formula, we find

$$(\text{number of pieces}) = \frac{\frac{1}{2} + 50}{(200)(0.05) - \frac{1}{3}} = \frac{50.5}{9.67} = 5.2$$

or

$$(\text{number of pieces}) = \frac{\frac{1}{2} + 50}{(200)(0.02) - \frac{1}{3}} = \frac{50.5}{3.67} = 13.7.$$

From a purely experimental point of view, these amounts of data are moderately hard to substantially hard to obtain, but we may suppose them available as far as radar and target availability are concerned.

We come next to data taking and availability problems. We must study the spectrum up to 25 cps. Since the spectrum is negligible only above 40 cps, our folding frequency must be at least 32.5 cps, which would fold 40 cps exactly back to 25 cps. Hence we need at least 65 frames a second. Consideration of available frame rates bring us to 64 frames a second as probably reasonable. This is 12,800 frames in each 200 second piece, a total film reading load of between 50 and 150 thousand frames. This will require some hundreds of man-days of film reading, but may perhaps be faced.

To calculate *directly* the rough number of multiplications involved, we

may begin by assuming that we are going to require the 0.05 or 0.02 cps resolution all the way from 0 to 25 cps. Were this the case, then we would require to resolve from

$$\frac{25}{0.05} = 500$$

to

$$\frac{25}{0.02} = 1{,}250$$

frequency bands. The corresponding numbers of multiplications range from

$$[4.5 + 450 + 3 \text{ (pieces)}]\ (500)^2 \approx 120 \text{ million}$$

to

$$[4.5 + 450 + 3 \text{ (pieces)}]\ (1{,}250)^2 \approx 750 \text{ million.}$$

The running time of an IBM 650 calculator on such a problem is about 10 hours per million multiplications, so that between

$$1{,}200 \text{ hours} = 30 \text{ shift-weeks}$$

and

$$7{,}500 \text{ hours} = 188 \text{ shift-weeks}$$

would be required. Clearly these machine times are out of line, and attention should be given to ways of reducing this aspect of effort.

An application of smoothing by groups seems most likely to be effective, especially since the high resolution is only wanted near the low frequency of 0.27 cps. Let us suppose that, in view of the supposed rather flat spectrum out to 10 cps, the engineers concerned will be content with two spectrum analyses, one with 0.5 cps resolution extending all the way to 25 cps, and the other with 0.02 cps resolution extending only to 1 cps. What effect will this have on the computational load?

Notice first that it will have no effect on the radar-and-target operating and film-reading loads. These were fixed by the resolution-precision requirements, and by the combination of this with the upper limit of the actual spectrum affecting the camera. Replanning details of the analysis will save nothing on either of these.

The broad-frequency low-resolution analysis will resolve about

$$\frac{25}{0.5} = 50 \text{ bands}$$

and require roughly

$$[4.5 + 450 + 3(14)]\cdot(50)^2 = 1.24 \text{ million multiplications}$$

(since we shall need 14 pieces to obtain the required precision at a resolution of 0.02 cps). This would require about 12.4 hours machine time, a quite reasonable amount.

The preparation of data for the low-frequency high-resolution analysis — if we follow the suggestion of Section 17, requires less than 1.5 additions per original frame, since each datum contributes to four means. This is at most 0.2 million additions and can probably be combined with the next step so as not to involve substantial machine time.

The conduct of the low-frequency high-resolution analysis will resolve about

$$\frac{1.0}{0.02} = 50 \text{ bands}$$

and will require about another 12.4 hours of machine time.

Thus we have reduced machine time to about 25–30 hours, in pleasant contrast with the remaining requirements of some hundreds of hours of film reading and 14 test runs of 200 seconds each. The balance is approximately restored.

Our apparently blind use of the multiplications-required formula has concealed one important point. Our calculation of the time required for the high-frequency low-resolution analysis tacitly assumed that we have processed no more of the data than is required to meet the actual resolution-precision requirement.

The loosening of resolution from 0.02 cps to 0.5 cps in this part of the analysis has reduced by a factor 25 the amount of data which must be processed to meet the ± 1 db (90 per cent) requirement. Hence the two hours machine time is predicated on processing only $\frac{1}{25}$th of the available data. If only about $\frac{1}{25}$ of the data is to be processed for the high frequency analysis, then it will be desirable to take the most typical 8 or 10 seconds from each piece. The losses due to end effects will be somewhat greater, it is true, but the advantages of increased coverage of the effects of unplanned variation, consequent on using parts of all 14 runs, far outweigh such considerations.

It would be possible to use only one run for the high-frequency analysis, a possibility which emphasizes the fact that $\frac{13}{14}$ths of the film reading is done to obtain the raw material for averaging, for filtering out high frequencies. If the hundreds of man-days of film reading look out of line, *and if* the line from the radar to the target is known not to change

rapidly (with respect to an inertial frame of reference), then we are driven to consider whether the "clamping" of the camera to the antenna could modified in such a way as to provide a frequency cutoff between antenna position and camera position. What would be desired would be a reliable mechanical filter with a cutoff at 1 or 2 cps, and substantial, reproducible transmission up to, say, 0.5 cps. If such a mount could be taken down from the shelf, then it would suffice to make (a) one 200-second run with a stiff mount and 64 frames per second, and, say, (b) thirteen 200-second runs with a mount of such designed softness, and, say 4 frames per second. The total number of frames for reading would now be 12,800 for run (a) and 800 for each run (b), a total of about 23,000 frames. This might require about a man-month to read, a saving of several man-months. Unfortunately, such a sharply-tuned low-pass mount would not be likely to be on the shelf.

27. EXAMPLE B

As a second example, suppose a new solid-state device develops a noise voltage with a power spectrum roughly proportional to $1/f^2$ when under test under most extreme circumstances — circumstances so extreme that its average life is 30 to 50 milliseconds, and suppose that the detailed behaviour of this spectrum is believed likely to provide a clue to the proper theoretical treatment of some of the properties of this device. Suppose further that, while it was believed that the shape of the spectrum of the noise from different examples of this device was the same, the voltage levels of different devices were quite different. It might be reasonable to ask for spectral measurements to ± 0.25 db resolving 1 cps and covering from 1 cps to 500 cps. Direct measurements are likely to be most difficult, for the power between 499 and 500 cps is about $\frac{1}{120,000}$th the power between 1 and 2 cps, a difference of 51 db in level. Our recording and processing equipment is not likely to have the dynamic range required for direct analysis.

Clearly we should prewhiten our noise as early in the measurement and analysis system as we reasonably can. Fortunately, prewhitening here is

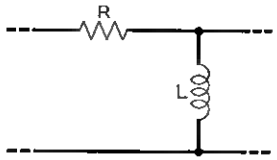

Fig. 12 — RL voltage divider.

operationally simple. A RL voltage divider, as indicated in Fig. 12 will introduce an attenuation of voltage, if the load impedance is high, amounting to

$$\left| \frac{R + j\omega L}{j\omega L} \right|^2 = 1 + \frac{R^2}{\omega^2 L^2}.$$

If the original spectrum were

$$P(f) = \frac{A}{f^2} = \frac{4\pi^2 A}{\omega^2},$$

then the prewhitened spectrum would be

$$\frac{\frac{4\pi^2 A}{\omega^2}}{1 + \frac{R^2}{\omega^2 L^2}} = \frac{4\pi^2 A L^2}{R^2 + \omega^2 L^2}$$

which will be initially constant, and then decrease 6 db per octave, with a corner at $\omega_c = R/L, f_c = R/2\pi L$. As a first step in measuring a spectrum out, to say, $f = 2R/2\pi L$, at which frequency the prewhitened spectrum would be down about 7 db, such a change would be useful. The range of frequencies which could be usefully studied would not be appreciably reduced by such a change, even though the low frequency power level would be greatly reduced by the prewhitening network, since the low-frequency power level would not be seriously reduced below the former power level at the corner frequency. If one could have been studied, the other can be studied.

28. EXAMPLE C

The irregularities in the earth's rotation have been studied by Brouwer,[17] who reduced the available observations (times of occultation and meridian passage) by averaging over individual years. He states "occultations so reduced in recent years have been demonstrated to yield annual means essentially free from systematic errors if the observations are well distributed over the year. . . . The δ's may themselves be the accumulations of numerous smaller random changes with average intervals much smaller than a year. The astronomical evidence throws no further light on this, though perhaps something may be gained by an analysis of residuals in the moon's mean longitude taken by lunations."[18] These comments suggest that astronomical data can supply values once a year, possibly no more frequently, and may be able to supply values

about 13 times a year (once per lunation), certainly no more frequently. Let us accept the first possibility as a basis for an example. (This is the best example we know of a situation where equally spaced data cannot, in principle, be had at a finer spacing.)

The information most nearly directly supplied by the astronomical observations is Δt, the difference between ephemeris time and mean solar time. Brouwer discusses two statistical models for its structure, both of which are most easily described in terms of the behavior of the second differences of the observations. In the first, the true second differences are constant over periods of varying length. In the second model, the true second differences are independently and randomly distributed. In either case, observational errors, independent from observation-period to observation-period also contribute to the observed Δt's.

If we were to plan an observational program to decide between these hypotheses by spectral analysis we need first to specify the alternative spectra. The first model seems never to have been made as precise statistically as the second. Brouwer's fitted curves correspond to constancy over periods of from 4 to 15 years. We should like to get a general idea of the possible spectra corresponding to this model without making the model too specific. Consider first a situation in which, except for the effects of second differences of experimental errors, the observations are constant in blocks of five, and where the values assigned to different blocks are independent. The successive average lagged products (starting with lag zero) are proportional to 5, 4, 3, 2, 1, 0, 0, 0, . . . and it follows that the power density is proportional to

$$1 + \frac{8}{5}\cos \pi f/f_N + \frac{6}{5}\cos 2\pi f/f_N + \frac{4}{5}\cos 3\pi f/f_N + \frac{2}{5}\cos 4\pi f/f_N .$$

Calculation shows that this is high near zero frequency, falling rapidly until, beyond about $f/f_N = 0.3$, it consists of ripples with an average height of less than $\frac{1}{25}$th the low frequency peak. If, instead of "constant by fives", the specific model were "constant by eights" or "constant by tens", still with independence between blocks, this peaking would be more pronounced and confined to still lower frequencies. If the lengths of the blocks were to vary at random, according to some distribution, still with independence of value, the spectrum would be the corresponding average of such spectra for fixed block lengths. The spectrum to be expected *for second differences* of annual average observations then should consist of a sum of two components:

(1) a "true" component peaked at low frequencies, falling rapidly by, say, $f/f_N = 0.2$ or 0.25, and continuing to $f/f_N = 1.00$ with an average height perhaps 1 per cent or 2 per cent of the low-frequency value,

(2) an "observational error" component, corresponding to independent errors in the annual averages, and hence proportional to $(1 - \cos(\pi f/f_N))^2$.

In case the second model should apply, the first component would be replaced by one with a flat density.

Fig. 13 shows the shapes of the three possible components. The natural way to try to distinguish between the two models by spectral analysis is to compare the spectral density in the middle range, say $f/f_N = 0.25$ to 0.5 with that in a lower range, say below $f/f_N = 0.25$. According to Model I, the low-range density should be substantially higher than the middle-range density, the latter consisting of the effects of observational error (whose strength can be well estimated at the upper end of the spectrum). According to Model II, the middle-range density should be slightly to somewhat greater than the low-range density, the increment representing effects of observational error.

Without more detailed estimates of the relative sizes of the components, it would be difficult to specify exactly how many observations would be required to separate Model I from Model II, but 10 to 20 degrees of freedom in each of the ranges discussed should be quite helpful. This suggests 100 values of annual second differences, corresponding to 102 years of careful astronomy, as likely to be helpful. Since Brouwer gives annual values for 131 years, some 129 annual second differences are

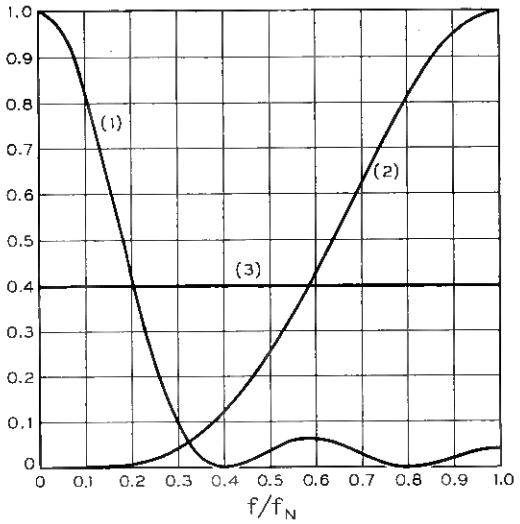

Fig. 13 — Components for two models of earth-rotation irregularities: (1) "true irregularity" component for first model, (2) "observational error" component for either model, (3) "true irregularity" component for second model.

available for trial, and it may be possible to answer the question without waiting for many more years to pass.

It might well suffice to estimate smoothed densities over octaves such as $0.0625 \leqq f/f_N \leqq 0.125$, $0.125 \leqq f/f_N \leqq 0.25$, $0.25 \leqq f/f_N \leqq 0.5$ and $0.5 \leqq f/f_N \leqq 1$. Thus we might consider using the add-and-subtract pilot estimation method for initial exploration. The actual analysis of Brouwer's data is considered further in Section B.28.

Appendix A

Fundamental Fourier Techniques

In this appendix we review briefly certain aspects of Fourier transformation. These aspects may be regarded as dealing mainly with diffraction by slits, rectangular or graded, and by analogs made up of discrete "lines". Convolution and the so-called Dirac functions are specially important as convenient tools. Some parts of the discussion will have no direct bearing on the analysis of procedures for power spectrum estimation, but are intended to familiarize the reader with analytical tools which are used frequently throughout the remainder of this paper, and which may be used to advantage in many other analyses of a similar nature.

A.1 *Fourier Transformation*

There are several formulations of Fourier transformation which differ according to custom, convenience, or taste. The formulation which we will adopt here is the one used by Campbell and Foster.[19] Given a function of time, $G(t)$, its Fourier transform is a function of frequency, and is given by the formula

$$S(f) = \int_{-\infty}^{\infty} G(t) \cdot e^{-i\omega t}\, dt \qquad (\omega = 2\pi f).$$

Conversely, given a function of frequency, $S(f)$, its Fourier transform is a function of time, and is given by the formula

$$G(t) = \int_{-\infty}^{\infty} S(f) \cdot e^{i\omega t}\, df \qquad (\omega = 2\pi f).$$

The term "frequency" is used here, not in the probability or statistical sense, but in the sense of sinusoidal or cisoidal functions of time ($\cos \omega t$, $\sin \omega t$, $e^{i\omega t}$).

Our preference for the Campbell-Foster formulation is based on the

following points, arranged approximately in the order of increasing weight.

1. Frequencies are expressed in cycles per second more naturally and much more frequently than in radians per second. (In our analysis we use ω only as an abbreviation of $2\pi f$, and only if it is typographically convenient.)

2. Except for the sign of the exponent in the kernels, the transformation formulae are symmetrical. The assignment of the signs here is the conventional one in transmission theory.

3. In most of the applications to communications problems, the frequency functions are rational functions of $p = i\omega$, with real coefficients. Hence, the reformulation of the transformation of $S(f)$ to $G(t)$ as

$$G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} S\left(\frac{p}{2\pi i}\right) \cdot e^{pt}\, dp$$

is a natural and convenient step in the calculation of the integral by the method of residues.

4. The transformation formulae correspond to the conventional relations between the *impulse response* (response due to a unit impulse applied at $t = 0$) and the *transfer function* (ratio of steady-state response to excitation, for the complex excitation $e^{i\omega t}$) of a fixed linear transmission network. These network functional relations are commonly regarded as Laplace transformations rather than Fourier transformations. As a matter of fact, however, the circumstances in almost all practical applications are such that there is no essential difference between Laplace transformations and Fourier transformations. Impulse responses are zero for $t < 0$ and vanish exponentially as $t \to \infty$, and transfer functions are analytic on and to the right of the imaginary axis (including the point at infinity) in the complex p-plane. On the very rare occasions when a communications engineer might be interested in the behavior of a network under energetic initial conditions, he has ways of introducing the initial conditions without using Laplace transforms (Guillemin[20]).

It should be noted that, since $G(t)$ must be a real function, the real part of $S(f)$ must be an even function, and the imaginary part of $S(f)$ must be an odd function. The even part of $G(t)$ and the even (real) part of $S(f)$ are cosine-transforms of each other. The odd part of $G(t)$ and the odd (imaginary) part of $S(f)$ are negative sine-transforms of each other. It should be noted also that if $G(t)$ and $S(f)$ constitute a transform-pair, then $G(-t)$ and $S(-f)$ also constitute a transform-pair. Further, $S(-f)$ is equal to $S^*(f)$, the complex conjugate of $S(f)$.

A.2 *Some Transform-Pairs*

We will now turn our attention to some transform-pairs which we will require directly or indirectly in the analysis of procedures for power spectrum estimation. We will use special symbols for some of these transform-pairs. For later reference, these transform-pairs will be collected in Table IV.

The first transform-pair, which is easily worked out, involves a symmetrical rectangular time function (box car of length $2T_m$), viz.

$$\begin{aligned} D_0(t) &= 1, && |t| < T_m, \\ &= \tfrac{1}{2}, && |t| = T_m, \\ &= 0, && |t| > T_m. \end{aligned}$$

TABLE IV

1.	$D_0(t) = 1, \ \|t\| < T_m$ $= \frac{1}{2}, \ \|t\| = T_m$ $= 0, \ \|t\| > T_m$	$Q_0(f) = 2T_m \frac{\sin \omega T_m}{\omega T_m}$ $= 2T_m \operatorname{dif} 2fT_m$
2.	$D_1(t) = 1 - \frac{\|t\|}{T_m}, \ \|t\| \leqq T_m$ $= 0, \quad \|t\| \geqq T_m$	$Q_1(f) = T_m \left(\frac{\sin \pi f T_m}{\pi f T_m}\right)^2$ $= T_m \ (\operatorname{dif} fT_m)^2$
3.	$\delta(t - t_0)$	$e^{-i\omega t_0}$
4.	$\cos \omega_0 t$	$\frac{1}{2}[\delta(f + f_0) + \delta(f - f_0)]$
5.	$\nabla_m(t; \Delta t) = \frac{\Delta t}{2}\delta(t + m\Delta t) + \Delta t \cdot \sum_{q=-m+1}^{q=m-1} \delta(t - q\Delta t) + \frac{\Delta t}{2}\delta(t - m\Delta t)$	$Q_0(f; \Delta t) = \Delta t \cdot \cot \frac{\omega \Delta t}{2} \cdot \sin m\omega\Delta t$ $= 2(m \cdot \Delta t) \cos (\pi f \cdot \Delta t) \frac{\operatorname{dif} 2f(m \cdot \Delta t)}{\operatorname{dif} f \cdot \Delta t}$
6.	$\nabla(t; \Delta t) = \Delta t \cdot \sum_{q=-\infty}^{q=\infty} \delta(t - q\Delta t)$	$A\left(f; \frac{1}{\Delta t}\right) = \sum_{q=-\infty}^{q=\infty} \delta\left(f - \frac{q}{\Delta t}\right)$
7.	$A(t; \Delta t)$	$\nabla\left(f; \frac{1}{\Delta t}\right)$

The corresponding frequency function is

$$Q_0(f) = 2T_m \frac{\sin \omega T_m}{\omega T_m} = 2T_m \cdot \text{dif } 2fT_m.$$

(The values assigned to $D_0(t)$ at the end points $|t| = T_m$ are those resulting from the transformation of $Q_0(f)$ to $D_0(t)$. Of course the values assigned at these two points do not influence the result of the transformation of $D_0(t)$ to $Q_0(f)$). Except for scale factors, this frequency function is the function dif $u = \sin \pi u / \pi u$ which recurs constantly in this subject. It is often convenient to regard it as the diffraction pattern (in frequency) due to passage through a rectangular slot (in time). The behaviour of dif $2fT_m$ is shown in Fig. 14.

The second transform-pair, which is almost as readily worked out as the first, involves a symmetrical triangular time function, viz.

$$D_1(t) = 1 - \frac{|t|}{T_m}, \qquad |t| \leqq T_m,$$

$$= 0, \qquad |t| \geqq T_m.$$

The corresponding frequency function is

$$Q_1(f) = T_m \left(\frac{\sin \pi f T_m}{\pi f T_m} \right)^2 = T_m (\text{dif } fT_m)^2.$$

Except for scale factors, this frequency function behaves as shown in Fig. 14.

The third transform-pair involves a so-called Dirac *function* as the time function. The Dirac function is not a function in the strict mathematical sense. It is called a "measure" by L. Schwartz.[21] For our purposes, it will only be necessary to identify $\delta(t - t_0) \cdot dt$ formally with $dh(t - t_0)$ where $h(t - t_0)$ is Heaviside's unit-step function, viz.,

$$h(t - t_0) = 0, \qquad t < t_0$$

$$= 1, \qquad t > t_0$$

and to interpret all integrals as Stieltjes integrals. Hence if the time function (to use the term loosely) is

$$G(t) = \delta(t - t_0)$$

then, the corresponding frequency function is

$$S(f) = e^{-i\omega t_0}.$$

It should be noted that while $\delta(t - t_0)$ is easily formally transformed

into a frequency function, the latter is not so readily transformed into the original time function.

The fourth transform-pair involves a symmetrical pair of Dirac functions as the frequency function. Thus, the time function

$$G(t) = \cos \omega_0 t \qquad (\omega_0 = 2\pi f_0)$$

corresponds to the frequency function

$$S(f) = \tfrac{1}{2}[\delta(f + f_0) + \delta(f - f_0)].$$

If the reader is disturbed over the fact that we are evidently going to base our analysis, at least initially, on the use of Dirac functions, he should note that Dirac functions are always paired with functions which are used widely and freely in transmission theory although they are not realistic in a physical sense. Functions of time, such as $\cos \omega_0 t$, which represent an infinitely long past and future history of activity, are not a bit more realistic in a physical sense than are "infinitely sharp" lines in the frequency spectrum. Similarly, functions of frequency, such as $\exp(-i\omega t_0)$, whose absolute values do not vanish as $f \to \infty$, are not a bit more realistic than impulsive "functions" of time. Nevertheless, as we

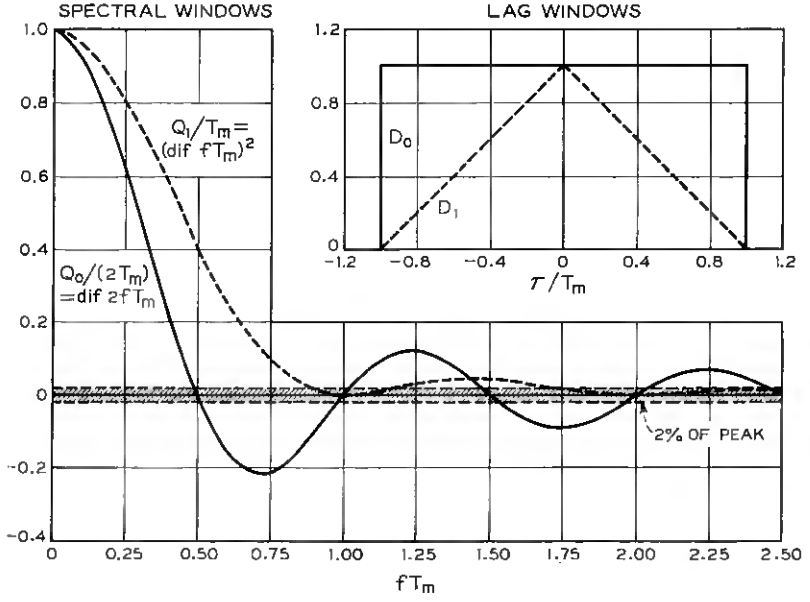

Fig. 14 — Lag windows D_0 and D_1. Spectral windows Q_0 and Q_1.

will see later on, these unrealistic pairs may be used as convenient bases for a wide variety of realistic pairs. They thus serve a very useful purpose.

The fifth transform-pair involves a *finite Dirac comb* as the time function, viz.

$$\nabla_m(t;\Delta t) = \frac{\Delta t}{2}\delta(t+m\Delta t) + \Delta t\cdot\sum_{q=m+1}^{q=m-1}\delta(t-q\Delta t) + \frac{\Delta t}{2}\delta(t-m\Delta t).$$

This is clearly a discrete approximation to $D_0(t)$ for $T_m = m\cdot\Delta t$. The corresponding frequency function, which is easily worked out with the help of the third transform-pair (summing the exponential terms before introducing trigonometric equivalents), is

$$Q_0(f;\Delta t) = \Delta t\cot\frac{\omega\Delta t}{2}\cdot\sin m\omega\Delta t = 2(m\cdot\Delta t)\cos(\pi f\cdot\Delta t)\frac{\mathrm{dif}\,2f(m\cdot\Delta t)}{\mathrm{dif}\,f\cdot\Delta t}.$$

Except for a scale factor, the initial behaviour of this frequency function is illustrated in Fig. 9. Clearly, since $\cos 0 = \mathrm{dif}\ 0 = 1$, the limit of $Q_0(f;\Delta t)$, when $\Delta t \to 0$ with $m\cdot\Delta t = T_m$ held constant, is $Q_0(f)$. This corresponds to the formal convergence of $\nabla_m(t;\Delta t)$ to $D_0(t)$.

We have defined this finite Dirac comb with a half-sized Dirac function at each end because the corresponding frequency function has smaller side lobes, relative to the main lobe, than for the finite Dirac comb with a whole Dirac function at each end. This is easily seen from the fact that the effect of adding a further half-sized Dirac function at each end of $\nabla_m(t;\Delta t)$ is to add $\Delta t\cdot\cos m\omega\Delta t$ to $Q_0(f;\Delta t)$.

The frequency function $Q_0(f;\Delta t)$ is periodic, with a period of $1/\Delta t$ cps. It is symmetrical about every integral multiple of $1/(2\Delta t)$ cps. Thus, it has an absolutely maximum value of $2m\cdot\Delta t$ at the integral multiples of $1/\Delta t$ cps. It is zero at the integral multiples of $1/(2m\Delta t)$ cps which are not integral multiples of $1/\Delta t$ cps. For large values of m and small values of $\omega\Delta t$, it behaves approximately like $Q_0(f)$.

The sixth transform-pair involves an *infinite Dirac comb* in time, and, as it turns out, also an infinite Dirac comb in frequency. The time function is the formal limit of $\nabla_m(t;\Delta t)$ as $m \to \infty$, namely,

$$\nabla(t;\Delta t) = \Delta t\cdot\sum_{q=-\infty}^{q=\infty}\delta(t-q\Delta t).$$

The corresponding frequency function is

$$A\left(f;\frac{1}{\Delta t}\right) = \sum_{q=-\infty}^{q=\infty}\delta\left(f-\frac{q}{\Delta t}\right) = \Delta t\cdot\nabla\left(f;\frac{1}{\Delta t}\right).$$

This may be surmised from the fact that

$$\int_{-1/(2\Delta t)}^{1/(2\Delta t)} Q_0(f;\Delta t)\,df = 1 \qquad \text{for any } m$$

while

$$\lim_{m\to\infty}\int_{-\epsilon}^{\epsilon} Q_0(f;dt)\,df = \frac{2}{\pi}\lim_{m\to\infty} Si(2\pi m\epsilon\Delta t), \quad \left(\text{where } Si(x) \equiv \int_0^x \frac{\sin y}{y}\,dy\right)$$

$$= 1 \text{ for any } \epsilon \text{ in } 0 < \epsilon < \frac{1}{2\Delta t}.$$

The result may indeed be obtained by applying the fourth transform-pair with $T_m = m\cdot\Delta t$ to the formal Fourier series representation of the infinite comb

$$\nabla(t;\Delta t) = 1 + 2\sum_{q=1}^{q=\infty} \cos\frac{2\pi qt}{\Delta t}.$$

Since

$$\nabla_m(t;\Delta t) = D_0(t)\cdot\nabla(t;\Delta t)$$

we also have, as we shall see in the next section,

$$Q_0(f;\Delta t) = Q_0(f) * A\left(f;\frac{1}{\Delta t}\right)$$

$$= \sum_{q=-\infty}^{\infty} Q_0\left(f - \frac{q}{\Delta t}\right).$$

The seventh transform-pair arises from the sixth by dividing by Δt on both sides.

A.3 *Convolution*

If $G(t) = G_1(t)\cdot G_2(t)$, then the Fourier transform of $G(t)$ may be expressed in terms of those of $G_1(t)$ and $G_2(t)$ as follows.

$$S(f) = \int_{-\infty}^{\infty} G_1(t)\cdot G_2(t)\cdot e^{-i\omega t}\,dt,$$

$$= \int_{-\infty}^{\infty} G_1(t)\cdot\left[\int_{-\infty}^{\infty} S_2(\xi)\cdot e^{i2\pi\xi t}\,d\xi\right]\cdot e^{-i\omega t}\,dt,$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} G_1(t)\cdot e^{-i2\pi(f-\xi)t}\,dt\right]\cdot S_2(\xi)\,d\xi,$$

$$= \int_{-\infty}^{\infty} S_1(f-\xi)\cdot S_2(\xi)\,d\xi.$$

This relation, in which $S_1(f)$ and $S_2(f)$ are interchangeable, is commonly expressed in the symbolic form

$$S(f) = S_1(f) * S_2(f).$$

The implied operation on $S_1(f)$ and $S_2(f)$ is called a *convolution*. In particular, $S(f)$ is said to be the convolution of $S_1(f)$ with $S_2(f)$.

Similarly, if $S(f) = S_1(f) \cdot S_2(f)$, then

$$G(t) = \int_{-\infty}^{\infty} G_1(t - \lambda) \cdot G_2(\lambda)\, d\lambda$$

$$= G_1(t) * G_2(t).$$

Thus, multiplication and convolution constitute an *operational transform-pair*.

(Convolution is often called by a variety of names such as Superposition theorem, Faltungsintegral, Green's theorem, Duhamel's theorem, Borel's theorem, and Boltzmann-Hopkinson theorem.)

It may be noted in the detailed derivation above (putting $f = 0$), that

$$\int_{-\infty}^{\infty} G_1(t) \cdot G_2(t)\, dt = \int_{-\infty}^{\infty} S_1^*(f) \cdot S_2(f) \cdot df$$

where $S_1^*(f)$ is the complex conjugate of $S_1(f)$. This is *Parseval's theorem* of which a very useful special case is

$$\int_{-\infty}^{\infty} [G(t)]^2\, dt = \int_{-\infty}^{\infty} |\, S(f)\, |^2\, df.$$

An example of convolution is supplied by the symmetrical triangular time function in the second transform-pair. This time function is the convolution of two symmetrical rectangular time functions from the first transform-pair, with appropriate scalar adjustments. Another example is the infinite Dirac comb $\nabla(t;\, \Delta t)$, which may be regarded as the convolution of the finite Dirac comb $\nabla_m(t;\, \Delta t)$ with the infinite Dirac comb $A(t;\, 2m\Delta t)$, that is

$$\nabla(t;\, \Delta t) = \nabla_m(t;\, \Delta t) * A(t;\, 2m\Delta t).$$

As the reader may easily verify, this corresponds to

$$A\left(f;\, \frac{1}{\Delta t}\right) = Q_0(f;\, \Delta t) \cdot \nabla\left(f;\, \frac{1}{2m\Delta t}\right).$$

Convolution of time functions occurs in communications systems whenever a signal is transmitted through a fixed linear network. If the input

signal is $G_1(t)$, and if the impulse response of the network is $W(t)$, then the output signal is*

$$G(t) = \int_{-\infty}^{\infty} W(t - \lambda) \cdot G_1(\lambda)\, d\lambda$$

$$= W(t) * G_1(t).$$

The so-called *linear distortion* of the signal due to transmission through the network can be (and occasionally is) examined in terms of the effects of convolution, but the common practice among circuit engineers is to conduct the examination in terms of the corresponding frequency functions. There are good reasons for this common practice. The most important of these reasons are:

1. The relation between the frequency functions is simpler, viz.

$$S(f) = Y(f) \cdot S_1(f)$$

where $Y(f)$ is the transfer function of the network.

2. The effects of *amplitude distortion* of the signal and of *phase distortion* (of the unmodulated signal) may be examined independently. While phase distortion is critical in the transmission of pictures (facsimile), it is relatively unimportant in the transmission of speech or music.

3. The transmission characteristics of fixed linear networks are most easily calculated or measured accurately in terms of frequency rather than time.

4. Fixed linear network design techniques based on frequency functions are today much further developed (simpler, more powerful, and more versatile) than those based on time functions.

Convolution of frequency functions occurs in communications systems whenever a carrier wave is amplitude-modulated by a signal. If the input signal is $G_1(t)$, and if the carrier wave is $\cos \omega_0 t$, then the output signal, with *suppressed carrier*, is

$$G(t) = G_1(t) \cdot \cos \omega_0 t$$

* It may be of some help here to think of λ as "excitation time", and of t as "response time". In the equivalent formulation

$$G(t) = \int_{-\infty}^{\infty} W(\tau) \cdot G_1(t - \tau)\, d\tau$$

we may think of $\tau = t - \lambda$ as the "age" of input data at response time.

At this point attention is called to a device which will be used many times to simplify analysis, which is to use $-\infty$ and $+\infty$ as limits of integration, letting the integrand take care of the effective range of integration. In this case, if $G_1(\lambda) \equiv 0$ for $\lambda < t_0$, and $W(\tau) \equiv 0$ for $\tau < 0$, the effective range of integration would be $t_0 < \lambda < t$ or $0 < \tau < t - t_0$.

and the relation among the corresponding frequency functions is

$$S(f) = S_1(f) * \tfrac{1}{2}\,[\delta(f + f_0) + \delta(f - f_0)]$$
$$= \tfrac{1}{2}\,S_1(f + f_0) + \tfrac{1}{2}\,S_1(f - f_0).$$

The convolution of frequency functions corresponding to the amplitude-modulation of a carrier wave is so naturally visualized simply as shifting the signal spectrum (frequency function) that it is almost never visualized in any other way. It should be observed, however, that this point of view depends critically upon the two-sided specification of the signal spectrum, in amplitude and phase, to give the correct picture of the sidebands, whether the amplitude-modulation scheme under consideration be double-sideband, single sideband, vestigial sideband, or two-phase (as in TV chrominance signals). Further, the two-sided specification of the modulated-carrier spectrum is essential for a correct picture of the demodulation process used to recover the signal.

For present purposes we will be interested in convolution not only as a tool for the synthesis of new transform-pairs but also as an analytical tool. For example, by regarding a time function $G(t)$ as the product of two other time functions $G_1(t)$ and $G_2(t)$ we can make use of the relation $S(f) = S_1(f) * S_2(f)$ to reach insights about $S(f)$ which do not come easily from the explicit form of $S(f)$.

To make convolution a useful analytical tool, we have to visualize it in some convenient way. This may be done in three ways. The relative merits of these three points of view depend upon the circumstances in any particular case.

In the first place, convolution may be visualized as a stretching process. For example, in the equation

$$G(t) = \int_{-\infty}^{\infty} G_1(t - \lambda)\cdot G_2(\lambda)\, d\lambda$$

we visualize $G_2(\lambda)\cdot d\lambda$ as a rectangular element of $G_2(t)$, originally concentrated at $t = \lambda$. This rectangular element is then stretched into the area under the elementary curve $G_1(t - \lambda)\cdot G_2(\lambda)\cdot d\lambda$ regarded as a function of t. This elementary curve has the shape of $G_1(t)$ with origin shifted to $t = \lambda$. The total effect at any particular value of t is then obtained by integration over λ. In this example, we have regarded $G_1(t)$ as the "stretcher" operating on each element of $G_2(t)$. Of course, since convolution is commutative, we may interchange the roles of the two functions.

In the second place, if one of the functions in the convolution consists exclusively of Dirac functions, each Dirac function may be regarded as a "shifter" operating on the other function in the convolution. For ex-

ample,

$$\delta(t - a) * G(t) = \int_{-\infty}^{\infty} \delta(t - a - \lambda) \cdot G(\lambda)\, d\lambda = G(t - a).$$

In the third place, convolution may be visualized as a weighted integration with a moving weight function. For example, in the equation

$$G(t) = \int_{-\infty}^{\infty} G_1(t - \lambda) \cdot G_2(\lambda)\, d\lambda$$

we regard $G(t)$ as the integral of $G_2(\lambda)$ with weight function $G_1(t - \lambda)$. The position of the weight function with respect to the λ scale depends upon the value of t. In the event that the weight function has unit area, $G(t)$ may be regarded as the moving weighted average of $G_2(\lambda)$. (As previously noted, the roles of the two functions may be interchanged.)

As an example of the use of the ideas described above, let us assume that we have a function $G_0(t)$ which is zero outside of the interval $0 < t < T$, and for which the frequency function is $S_0(f)$. Let us generate a periodic function $G(t)$ by convolving $G_0(t)$ with $A(t; T)$. Then, since

$$G(t) = G_0(t) * A(t; T)$$

the frequency function corresponding to $G(t)$ is, from Item 7 of Table IV,

$$S(f) = S_0(f) \cdot \nabla\left(f; \frac{1}{T}\right).$$

As we expect, $S(f)$ consists of "lines" (of infinite height but finite area) at uniform intervals of $1/T$ cps. The complex intensities (areas) of these lines represent the amplitudes and relative phases of the terms in the conventional Fourier series representation of $G(t)$. Thus,

$$\begin{aligned} G(t) &= \int_{-\infty}^{\infty} S(f) \cdot e^{i\omega t}\, df \\ &= \frac{1}{T} \sum_{q=-\infty}^{q=\infty} S_0\left(\frac{q}{T}\right) \cdot e^{(i2\pi q t)/T}. \end{aligned}$$

As a second example, which is in a sense the dual of the first, let us assume that we have a function $G_0(t)$ for which the frequency function $S_0(f)$ is zero outside of the band $-f_0 < f < f_0$. Let us generate a discrete time series $G(t)$ by sampling $G_0(t)$ at uniform intervals of $1/(2f_0)$ seconds. If we regard sampling as a multiplication by (or as amplitude-modula-

tion of) an infinite Dirac comb, then

$$G(t) = G_0(t) \cdot A\left(t; \frac{1}{2f_0}\right).$$

Hence, the frequency function corresponding to $G(t)$ is

$$S(f) = S_0(f) * \nabla(f; 2f_0),$$

or, explicitly,

$$S(f) = 2f_0 \cdot \sum_{q=-\infty}^{q=\infty} S_0(f - 2qf_0).$$

If this frequency function is multiplied by a frequency function $S_1(f)$, where

$$S_1(f) = \frac{1}{2f_0}, \quad |f| < f_0$$
$$= 0, \quad |f| > f_0$$

it will revert to $S_0(f)$. Thus,

$$S_1(f) \cdot S(f) = S_0(f).$$

Hence, if $G_1(t)$ is the time function corresponding to $S_1(f)$, namely,

$$G_1(t) = \frac{\sin \omega_0 t}{\omega_0 t},$$

then

$$G_1(t) * G(t) = G_0(t).$$

Thus, sampling $G_0(t)$ to get the discrete time series $G(t)$, and convolving $G(t)$ with $G_1(t)$, restores $G_0(t)$ exactly. This result reflects the well-known *sampling theorem* in information theory. The effect of sampling $G_0(t)$ at uniform intervals of other than $1/(2f_0)$ seconds is readily visualized.

A.4 *Windows*

If a time function is even (and of course real), the corresponding frequency function is real (and of course even), and conversely. These circumstances will prevail when we deal with autocovariance functions, power spectra, and appropriate weight functions. Under these circumstances, the weight functions will be called *windows*. Such windows will be considered in transform-pairs, and the members of any pair will be distinguished as the *lag window*, and the *spectral window*.

Time windows convolved with periodic functions of time have been used by Guillemin,[22] under the name "scanning functions", to examine the behavior of weighted partial sums of Fourier series. We use them in Sections B.4 and B.10 where we call them *data windows*, and their Fourier transforms (which may be complex) *frequency windows*.

A.5 *Realistic Pairs from Unrealistic Pairs*

Transform-pairs which involve Dirac functions are very easily converted into a wide variety of realistic pairs. As an example, let us consider the sixth pair (infinite Dirac combs) which requires two convolutions for conversion to a realistic pair. If we convolve the time functions of the first and sixth pairs, taking $T_m \ll \Delta t$, we get a time function which represents an infinite train of narrow rectangular pulses of unit height. The corresponding frequency function still consists of Dirac functions but these now do not have a uniform intensity. If we next multiply the time function of this pair by the time function of the first pair, taking $T_m \gg \Delta t$, we get a time function which represents a long but finite train of narrow rectangular pulses. The corresponding frequency function is continuous and consists chiefly of very narrow peaks of finite height approaching zero as $f \to \infty$.

A sinusoidal carrier wave of finite though great length may be represented as the product of the time functions of the first and fourth pairs with $T_m \gg 1/f_0$. The corresponding frequency function is continuous and consists of very narrow peaks at $\pm f_0$, with much lower subsidiary peaks of height approaching zero as $f \to \infty$.

If the time function of the third pair is convolved with the time function

$$\begin{aligned} G(t) &= 0 && t < 0 \\ &= \frac{1}{T} e^{-t/T} && t > 0, \end{aligned}$$

the resultant frequency function is

$$S(f) = \frac{1}{1 + i\omega T} e^{-i\omega t_0}$$

of which the absolute value falls off asymptotically like $1/f$ as $f \to \infty$, however small $T(>0)$ might be.

In line with this discussion, it should be noted that a realistic "white noise" spectrum must be effectively band-limited by an asymptotic fall-off at least as fast as $1/f^2$. Under certain circumstances, however, we may assume that the spectrum is flat to any frequency. Let us suppose that

the spectrum is in fact

$$P(f) = \frac{\dfrac{\sigma^2}{\pi f_c}}{1 + \left(\dfrac{f}{f_c}\right)^2}, \qquad \int_{-\infty}^{\infty} P(f)\, df = \sigma^2$$

where σ^2 is the variance. The autocovariance is

$$C(\tau) = \sigma^2 \cdot e^{-\omega_c |\tau|}.$$

If we transmit this noise through a network with an effective cutoff frequency well below f_c, we may assume for an approximation that

$$P(f) \approx \frac{\sigma^2}{\pi f_c}$$

and, therefore, that

$$C(\tau) \approx \frac{\sigma^2}{\pi f_c}\, \delta(\tau)$$

although such an assumption is unrealistic if carried to indefinitely high frequencies (the input noise would have infinite variance). Hence, if the impulse response of the network is $W(t)$, the autocovariance of the output noise is

$$\begin{aligned} C_{\text{out}}(t_i - t_j) &= \text{ave}\left\{\int_{-\infty}^{\infty} W(\tau_1) X(t_i - \tau_1)\, d\tau_1 \right. \\ &\qquad \left. \cdot \int_{-\infty}^{\infty} W(\tau_2) \cdot X(t_j - \tau_2)\, d\tau_2 \right\} \\ &= \iint_{-\infty}^{\infty} W(\tau_1) W(\tau_2) \cdot C(t_i - t_j - \tau_1 + \tau_2) d\tau_1\, d\tau_2 \\ &\approx \frac{\sigma^2}{\pi f_c} \int_{-\infty}^{\infty} W(\tau_1) \cdot W(\tau_1 - t_i + t_j)\, d\tau_1 . \end{aligned}$$

In particular, the variance of the output noise is

$$C_{\text{out}}(0) \approx \frac{\sigma^2}{\pi f_c} \int_{-\infty}^{\infty} [W(\tau_1)]^2\, d\tau_1$$

which by Parseval's theorem is equivalent to

$$C_{\text{out}}(0) \approx \frac{\sigma^2}{\pi f_c} \int_{-\infty}^{\infty} |\, Y(f)\, |^2\, df$$

where $Y(f)$ is the transfer function of the network. These results are

realistic. (The variance of the output noise is finite and approximately correct).

A.6 *Some Trigonometric Identities*

In this section we develop some trigonometric identities which will be needed later on. We start with the equation

$$\sum_{-a}^{b} \cos(\psi + 2hu) = \frac{\sin(a + b + 1)u}{\sin u} \cos[\psi + (b - a)u]$$

which is easily obtained by substituting

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

in the left-hand member, summing the exponential terms and making some elementary trigonometric substitutions. By substituting $\psi \pm \pi/2$ for ψ we then get

$$\sum_{-a}^{b} \sin(\psi + 2hu) = \frac{\sin(a + b + 1)u}{\sin u} \sin[\psi + (b - a)u].$$

Now, setting $u = \pi f$, and using the function introduced in Section A.2,

$$\frac{\sin pu}{p \sin u} = \frac{(\sin p\pi f)/p\pi f}{(\sin \pi f)/\pi f} = \frac{\operatorname{dif} pf}{\operatorname{dif} f},$$

which, on differentiation, yields

$$\frac{d}{df}\frac{(\operatorname{dif} pf)}{(\operatorname{dif} f)} = \left(\frac{\operatorname{dif} pf}{\operatorname{dif} f}\right)\left(p\frac{\operatorname{dif}' pf}{\operatorname{dif} pf} - \frac{\operatorname{dif}' f}{\operatorname{dif} f}\right).$$

Before we rewrite our summation formulas in terms of such ratios of "dif" functions, we need to appreciate their behavior. For p not very small, (dif pf)/(dif f) behaves much like the numerator for pf small and moderate. The effect of the denominator is to force symmetry around integer multiples of $\frac{1}{2}$, so that the peak at $f = 0$ is repeated at $f = 1, 2, 3, \cdots$, thus making its behavior consistent with aliasing. For $0 \leqq f \leqq \frac{1}{2}$ its other effects are minor, since in this range $(2/\pi) \leqq \operatorname{dif} f \leqq 1$, while the extrema of dif pf have shrunk from $+1$ to $\pm 2/(p\pi)$. For most considerations, therefore, we can approximate this ratio by the numerator.

We now rewrite our summations as means, introducing (dif pf)/(dif f), finding

$$\frac{1}{a + b + 1}\sum_{-a}^{b} \cos(\psi + 2h\pi f) = \frac{\operatorname{dif}(a + b + 1)f}{\operatorname{dif} f} \cos[\psi + (b - a)\pi f]$$

$$\frac{1}{a + b + 1}\sum_{-a}^{b} \sin(\psi + 2h\pi f) = \frac{\operatorname{dif}(a + b + 1)f}{\operatorname{dif} f} \sin[\psi + (b - a)\pi f].$$

Differentiating with respect to f, and multiplying through by

$$-(a + b + 1)/(2\pi),$$

we get

$$\sum_{-a}^{b} h \sin(\psi + 2h\pi f) = \left(\frac{\operatorname{dif}(a + b + 1)f}{\operatorname{dif} f}\right) \cdot \left[\frac{b - a}{2}(a + b + 1) \sin[\psi + (b - a)\pi f] - \left(\frac{(a + b + 1)^2}{2\pi} \frac{\operatorname{dif}'(a + b + 1)f}{\operatorname{dif}(a + b + 1)f} - \frac{a + b + 1}{2\pi} \frac{\operatorname{dif}' f}{\operatorname{dif} f}\right) \cos[\psi + (b - a)\pi f]\right]$$

with a similar formula for

$$\sum_{-a}^{b} h \cos(\psi + 2h\pi f).$$

We shall now use these formulas to obtain results about the average values of certain quadratic functions of chance variables $X_0, X_1, \cdots, X_n$. The average value of any such quadratic function can be represented in terms of a corresponding spectral window $Q(f)$ in the form

$$\int_0^\infty Q(f) \cdot 2P(f)\, df$$

whenever

$$\text{ave}\{X_t X_{t+q}\} = \int_0^\infty \cos 2\pi q f \cdot 2P(f)\, df$$

for all suitable integers t and q, since the quadratic function can be expressed as a sum of multiples of terms of the form $X_t X_{t+q}$. To determine the height, $Q(f_0)$, of the spectral window corresponding to a specific quadratic function, it suffices to consider the special case $2P(f) = \delta(f - f_0)$, for which ave $\{X_t X_{t+q}\} = \cos 2\pi q f_0$, when the average value of the quadratic function for such a special set of X_t is exactly $Q(f_0)$.

If ave $\{X_t X_{t+q}\} = \cos 2\pi q f$, we easily find that

$$\text{ave}\left\{\left(\frac{1}{a + b + 1} \sum_{-a}^{b} X_h\right)\left(\frac{1}{c + d + 1} \sum_{-c}^{d} X_g\right)\right\} = \frac{1}{a + b + 1} \sum_{-a}^{b} \frac{1}{c + d + 1} \sum_{-c}^{d} \cos 2\pi f(g - h)$$

$$= \frac{1}{a+b+1} \sum_{-a}^{b} \frac{\operatorname{dif}(c+d+1)f}{\operatorname{dif} f} \cos(-2\pi fh + (d-c)\pi f)$$

$$= \frac{\operatorname{dif}(a+b+1)f}{\operatorname{dif} f} \frac{\operatorname{dif}(c+d+1)f}{\operatorname{dif} f} \cos(d-c-a+b)\pi f$$

$$\approx \operatorname{dif}(a+b+1)f \operatorname{dif}(c+d+1)f \cos(d-c-a+b)\pi f$$

any of these expressions being the spectral window corresponding to

$$\left(\frac{1}{a+b+1} \sum_{-a}^{b} X_h\right)\left(\frac{1}{c+d+1} \sum_{-c}^{d} X_g\right).$$

Making the same assumption, we find that (where $n = 2\ell + 1$)

$$\operatorname{ave}\left\{\left(\sum_{-\ell}^{\ell} hX_h\right)\left(\sum_{-\ell}^{\ell} gX_g\right)\right\}$$

$$= \sum_{-\ell}^{\ell} \sum_{-\ell}^{\ell} gh \cos 2\pi f(g-h)$$

$$= \sum_{-\ell}^{\ell} g \left(\frac{\operatorname{dif} nf}{\operatorname{dif} f}\right)\left(\frac{n^2}{2\pi} \frac{\operatorname{dif}' nf}{\operatorname{dif} nf} - \frac{n}{2\pi} \frac{\operatorname{dif}' f}{\operatorname{dif} f}\right) \sin 2\pi fg$$

$$= \frac{n^4}{4}\left(\frac{\operatorname{dif} nf}{\operatorname{dif} f}\right)^2 \left(\frac{1}{\pi} \frac{\operatorname{dif}' nf}{\operatorname{dif} nf} - \frac{1}{n\pi} \frac{\operatorname{dif}' f}{\operatorname{dif} f}\right)^2.$$

These expressions therefore represent the spectral windows corresponding to

$$\left(\sum_{-\ell}^{\ell} hX_h\right)^2.$$

Glossary of Terms

Add-and-subtract method

A method of roughly estimating spectra based on successive additions by non-overlapping two's followed by a differencing. (18, B.18 and B.28.)

Alias

In equally spaced data, two frequencies are aliases of one another if sinusoids of the corresponding frequencies cannot be distinguished by their equally spaced values (this occurs when $f_1 = 2kf_N \pm f_2$ for integer k); the *principal aliases* lie in the interval $-f_N \leqq f \leqq f_N$. (See also 14.) (Also *aliased, aliasing*, etc.)

Aliased spectrum

See Spectrum, aliased.

Analysis, pilot

Any of a number of methods of obtaining a rough spectrum, including the add-and-subtract method (18, etc.) the cascade method (B.18), the complete add-and-subtract method (B.18).

Autocorrelation function

The normalized autocovariance function (normalized so that its value for lag zero is unity).

Autocovariance function

The covariance between $X(t)$ and $X(t + \tau)$ as a function of the lag τ. If averages of $X(t)$ and $X(t + \tau)$ are zero, it is equal to the average value of $X(t) \cdot X(t + \tau)$. It can be defined for a whole ensemble, a whole function stretching from $-\infty$ to $+\infty$, or for a finite piece of a function; in the latter case it is called the *apparent autocovariance function* (see 4). Certain related functions are called modified apparent autocovariance functions (also see 4).

Autoregressive series

A time series generated from another time series as the solution of a

linear difference equation. (Usually where previous values of output series enter into determination of current value.)

Average

The arithmetic mean, usually over an ensemble, a population, or some reasonable facsimile thereof.

Band-limited function

Strictly, a function whose Fourier transform vanishes outside some finite interval (and hence is an entire function of exponential type); practically, a function whose Fourier transform is very small outside some finite interval.

Box-car function

A function zero except over a finite interval, in the interior of which it takes a constant value (often $+1$).

Cardinal theorem (of interpolation theory)

A precise statement of the conditions under which values given at a doubly infinite set of equally spaced points can be interpolated (with the aid of the function $(\sin (x - x_i))/(x - x_i)$ to yield a continuous band-limited function. (See B.1.)

Cascade process (of spectral estimation)

A process of spectral estimation in which a single step is repeated again and again, each step yielding both certain estimates and a condensed set of data (ready for input to the next step). (See B.18.)

Chi-square

A quantity distributed (strictly exactly, but practically approximately) as $x_1^2 + x_2^2 + \cdots + x_k^2$ where $x_1 , x_2 , \cdots , x_k$ are independent and Gaussian, and have average zero and variance unity.

Continuous power spectrum

A power spectrum representable by the indefinite integral of a suitable (spectral density) function. (All power spectra of physical systems are continuous.)

Convolution

The operation on one side of a Fourier transformation corresponding to multiplication on the other side. (See A.3 for detailed discussion.)

Cosine transform

A series (see 13) or integral (see 2) transform in which a cosine of the product of the variables is the kernel.

Covariance

A measure of (linear) common variation between two quantities, equal to the average product of deviations from averages. (See 1.)

Cross-spectrum

The expression of the mutual frequency properties of two series analogous to the spectrum of a single series. (Because mutual relations at a single frequency can be in phase, in quadrature, or in any mixture of these, either a single complex-valued cross-spectrum or a pair of real-valued *cross-spectra* are required.) (Also *cross-spectral.*)

Data

As specifically used in this paper, values given at equally spaced intervals of time (often called time series).

Data window

A time function which vanishes outside a given interval and which is regarded as multiplying data or signals defined for a more extended period. (Data windows are usually smooth (graded) to improve the quality of later frequency analysis.)

Degrees of freedom

As applied to chi-square distributions arising from quadratic forms in Gaussian (normal) variables, the number of linearly independent squared terms of equal size into which the form can be divided. In general, a measure of stability equal to twice the square of the average divided by the variance.

Delta-component

A finite contribution to the spectrum at one frequency (B.10 only).

Diffraction function

$$\text{dif } x \equiv \frac{\sin \pi x}{\pi x}.$$

Dirac comb

An array of equally spaced Dirac functions, usually most of which are of equal height.

Dirac function

The limit of functions of unit integral concentrated in smaller and smaller intervals near zero. (See A.2 for fuller discussion.)

Distortion

Failure of output to match input. (Often specified as to kind of failure as *linear*, *amplitude*, *phase*, *non-linear*, etc., cp. A.3.)

Effective record length

Actual length of record available reduced to allow for end effects. (See 6.)

Elementary frequency band

An interval of frequency conveniently thought of as containing "a single degree of freedom", equal to the reciprocal of twice the duration of observation or record. (Since both sines and cosines may occur, it requires *two* elementary frequency bands to contain "an independently observable frequency.")

Ensemble

A family of functions (here functions of either continuous or equi-spaced time) with probabilities assigned to relevant sub-families.

Equivalent number (of degrees of freedom)

See second sentence under *degrees of freedom.*

Equivalent width

The extent of a function regarded as a window as expressed by the ratio of the square of its integral to the integral of its square. (See 8.)

Filtered spectrum

Spectrum of the output from any process which can be regarded as a filter.

Folding frequency

The lowest frequency which "is its own alias", that is, is the limit of both a sequence of frequencies and of the sequence of their aliases, given by the reciprocal of twice the time-spacing between values, also called *Nyquist* frequency.

Fourier transform

Operations making functions out of functions by integration against a kernel of the form exponential function of $\sqrt{-1}$ times frequency times time. Often, including here, defined differently for transforming time functions into frequency functions than for transforming frequency functions into time functions. (See A.1 for details.)

Frequency

A measure of rate of repetition; unless otherwise specified, the number of cycles per second. The *angular frequency* is measured in radians per second, and is, consequently, larger by a factor of 2π.

Frequency window

The Fourier transform of a data window.

Gaussian

A single quantity, or a finite number of quantities distributed according to a probability density representable as e to the power minus a quadratic form. (Also called *normal*, *Maxwellian*, etc.) Also, a function or ensemble, distributed in such a way that all finite sections are Gaussian. (See 1.)

Hamming

The operation of smoothing with weights 0.23, 0.54 and 0.23. (After R. W. Hamming.)

Hanning

The operation of smoothing with weights 0.25, 0.50 and 0.25. (After Julius von Hann.)

Hyperdirective antenna

An antenna or antenna system so energized as to have a more compact directional pattern than naturally corresponds to its extent (as measured in wavelengths).

Impulse response

The time function describing a linear system in terms of the output resulting from an input described by a Dirac function.

Independence (statistical, of estimates)

In general, two quantities are statistically independent if they possess a joint distribution such that (incomplete or complete) knowledge of one does not alter the distribution of the other. Estimates are statistically independent if this property holds for each fixed true situation.

Independent phases

An ensemble has independent phases when it can be approximated by ensembles consisting of finite sums of (phased) cosines (of fixed frequencies) whose phases are mutually independent. Continuous spectrum and independent phases imply Gaussian character. Every Gaussian ensemble has independent phases.

Intermodulation distortion

Non-linear distortion, especially as recognized in the output of a system when two or more frequencies enter the input simultaneously.

Joint probability distribution

Expression of the probability of simultaneous occurrence of values of two or more quantities.

Lag

A difference in time (epoch) of two events or values considered together.

Lag window

A function of lag, vanishing outside a finite interval, and either multiplying or regarded as multiplying the quantities of a family of quantities with differing lags.

Lagged product

The product of two values corresponding to different times. (In a *mean lagged product* the lags are usually all the same.)

Lead

The negative of lag.

Line (in a power spectrum)

Theoretically, and as used in this paper, a finite contribution associated with a single frequency. Physically, not used here, a finite contribution associated with a very narrow spectral region.

Lobe

A bulge, positive or negative, especially in a spectral window. (In most spectral windows, a large central *main lobe* is surrounded on both sides by smaller *side lobes*.)

Mean lagged product

The (arithmetic) mean of products of equally lagged quantities.

Moving linear combination

A transformation expressing the values of an output time series as linear combinations of values of the input series in specified relations of lag (or lead).

Negative frequencies

When sines and cosines are jointly represented by two imaginary exponentials, one has a positive frequency and the other a negative frequency. (Not specifiable for a single time function in real terms.)

Network (linear)

In this account, an otherwise unspecified physical device which converts an input function (of continuous time) *linearly* into an output function (of continuous time).

Noise

In general, an undesired time-function, or component of a function.

Non-normality

Failure to follow a normal or Gaussian distribution.

Normality

The property of following a normal or Gaussian distribution.

Nyquist frequency

The lowest frequency coinciding with one of its own aliases, the reciprocal of twice the time interval between values (same as *folding frequency*).

Octave

An interval of frequencies, the highest of which is double the lowest.

Pilot (*analysis* or *estimation*)

A process yielding rough estimates of spectral density intended mainly as a basis for planning more complete and precise analyses.

Population

A collection of objects (in particular, of numbers or of functions), with probabilities attached to relevant subcollections.

Power transfer function

The function expressing the ratio of output power near a given frequency to the input power near that frequency.

Power-variance spectrum

A function of frequency, in terms of which the variances and covariances of a family of spectral estimates can be expressed in standard form. (See 6 and 14 for details in the continuous and equi-spaced cases, respectively.)

Preemphasis

Emphasis of certain frequencies (in comparison with others), before processing, as an aid to the quality of result.

Prewhitening

Preemphasis designed to make the spectral density more nearly constant (the spectrum more nearly flat).

Principal alias

An alias falling between zero and plus or minus the folding or Nyquist frequency.

Process (*random* or *stochastic*)

An ensemble of functions. (Often composed of functions of time regarded as unfolding or developing.)

Protection ratio

The ratio of transmission at a desired frequency to the transmission at an undesired alias of that frequency.

Recording

Is *spaced* when originally taken at equal intervals, *mixed* when taken continuously and processed at equal intervals, *continuous* when taken and processed on a continuous basis.

Resolution

A measure of the concentration of a spectral estimate expressed in frequency units, here taken (for the important cases) as equal to the width of the major lobe. (See B.23.)

Resolved bands (number of)

The ratio of the Nyquist or folding frequency to the resolution.

Sampling theorem (of information theory)

Nyquist's result that equi-spaced data, with two or more points per cycle of highest frequency, allows reconstruction of band-limited functions. (See *Cardinal theorem.*)

Serial correlation coefficients

Ratios of the autocovariances to the variance of a process, ensemble, etc.

Signal

A time function desired as (potentially) carrying intelligence.

"Signal"

A function of continuous time, which may be either a signal, a noise, or a combination of both. (Contrasted with *data*, a function of discrete time.)

Single function approach

A mode of representing certain ensembles by the translations of a single time function (in single function terms).

Smoothed function

The result of weighted averaging of nearby values of the original function.

Smoothing

In the narrow sense, forming (continuous or discrete) moving linear combinations with unit total weight.

Smoothing and decimation procedure

A procedure which may be regarded as the formation of discrete moving linear combinations, followed by the omission of all but every kth such. (See 17 and B.17.)

Spectrum (also power spectrum)

An expression of the second moments of an ensemble, process, etc. (i) in terms of frequencies, (ii) in such a form as to diagonalize the effects on second moments of time-invariant linear transformations applied to the ensemble or process. (adjective: *spectral*).

Spectrum, aliased

For equally spaced data, the *principal part* of the *aliased spectrum* expresses contributions to the variance in terms of frequencies between zero and the Nyquist or folding frequency, all contributions from frequencies having the same principal alias and sign having been combined by addition. (The *aliased spectrum* repeats the principal part periodically with period $2f_N$. See 14.)

Spectral density

A value of a function (or the entire function) whose integral over any frequency interval represents the contribution to the variance from that frequency interval.

Spectral density estimates

Estimates of spectral density, termed *raw* when obtained from equispaced mean lagged products by cosine series transformation, *refined* when hanned or hammed from raw estimates or obtained by an equivalent process. (See B.13.)

Spectral window

A function of frequency expressing the contribution of the spectral density at each frequency to the average value of an estimate of (smoothed) spectral density.

Stationary (*ensemble* or *random process*)

An ensemble of time functions (or random process) is stationary if any translation of the time origin leaves its statistical properties unaffected.

Superposition theorem

A statement that the output of a linear device is the convolution of its input with its impulse response. (See B.2.)

Temporally homogeneous

Sometimes used in place of *stationary*, especially when speaking of stochastic processes.

Transfer function

The *transfer function* of a network or other linear device is a complex-valued function expressing the amplitude and phase changes suffered by cosinusoidal inputs in becoming outputs. (See A.5.) The square of the absolute value of the transfer function is the *power transfer function*, which expresses the factors by which spectral densities are changed as inputs become outputs. (See 4.)

Transmission

The coefficient with which power at a given frequency contributes to power at the (new) principal alias as a result of the application of a smoothing and decimation procedure.

Transversal filtering

Time domain filtering by forming linear combinations of lagged values, use of moving linear combinations for filtering. (See Kallmann[32] for the origin of this term.)

Trend

A systematic, smooth component of a time function (time series), as, for example, a linear function of time (a *linear trend*).

True

Often used to refer to average values over the ensemble, as contrasted with mean values over the observations.

Universe

A collection of objects (numbers, functions, etc.) with probabilities attached to relevant subcollections.

Variance

A quadratic measure of variability, the average squared deviation from the average.

White noise

An ensemble whose spectral density is sensibly constant from zero frequency through the frequencies of interest (in equi-spaced situations, up to the folding or Nyquist frequency). (The values of equi-spaced white noise at different times are independent.)

Window

A function expressing, as a multiplicative factor, the tendency or possibility of the various values of some function to enter into some calculation or contribute to the average value of some quantity. (See *data*, *lag*, *spectral*, etc. for specific instances.)

Windowless quadratic

A quadratic expression is windowless if its average value vanishes for every stationary ensemble of finite variance (See B.19).

Window pair

Two windows related by a Fourier transformation, as lag and spectral windows or data and frequency windows. (See A.4 and 4.)

Zero-frequency waves (cosine and sine)

The limiting forms of very-low-frequency cosinusoids, namely constants and linear trends. (See 19.)

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