The Bell System Technical Journal

*Vol. XX* *October, 1941* *No. 4*

# The Reliability of Holding Time Measurements

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## I—The Problem to be Solved

ONE of the fundamental quantities in traffic engineering is the average duration of subscribers' calls. This figure in seconds multiplied by the average number of calls expected over a given route in an hour, and divided by 3600, gives the traffic load submitted in average simultaneous calls—or "the average" as it is commonly called. Tables and curves are widely available which may then be consulted to find the number of paths to be provided so that no more than a desired small percentage of the calls presented will find all paths busy.

The direct measurement of call lengths with a stop watch occurs to one as being the simplest means for obtaining a sample of holding times. It is seldom used, however, due to the relative slowness with which a large number of observations are accumulated coupled with the not inconsiderable expense of the small army of observers required, each looking at one call at a time.

A second direct method of obtaining holding time measurements is by recording mechanically or electrically the length of each call passing over a group of switches or trunks during a certain interval of time. Various holding time recorders or "cabinets" following this principle have been used more or less extensively in the Bell System. Their chief disadvantage has lain in requiring considerable time and labor for summarizing the results. Problems of the perfect maintenance of the measuring equipment have also been present.

To make possible the rapid accumulation of holding time data on a considerable number of calls at relatively slight expense, the method of switch or plug counts has been introduced. This consists in scanning mechanically, electrically, photographically or by eye the group of paths at regular intervals, and recording each time the number found busy.[1] Such data give estimates immediately of the average load being carried, and by a

[1] This number will be highly variable, and even on properly engineered groups great concern need not be felt should few, or even no, cases of "all paths busy" appear since such peaks are of short duration and might easily be missed except in a very long series of counts.

relatively simple analysis a measure of the reliability of such an estimate can be obtained. If in addition for the same period a record is kept on a call- or peg-count meter of the number of calls passing over the group, it is possible also to obtain estimates of the average call holding time and the reliability of such an average.

Direct measurement of holding times or switch counts should naturally be made on groups during periods which are presumably typical of those toward which the engineering is ultimately directed. Usually, although not always, this will be the busy or busiest hours of the day during the busy season of the year. In order to decide intelligently how long a period needs to be studied in any given case some knowledge of the persistence of the same holding time universe is necessary. This might be obtained through relatively small holding time samples made in the hours of interest every day for several weeks in the busy season. If spottiness or "lack of control" is not apparent, the problem will be comparatively simplified. If trends are present, however, it will be necessary to investigate their nature (such as whether some one day of the week shows high holding times) and apportion the main sampling procedure in a fashion to give these peculiarities their proper weighting.

It will be of interest to examine in this respect certain limited data at hand taken by the pen register method some years ago on an inter-office trunk group in Newark, New Jersey. The kind of examination made here will serve to indicate the procedure which may be found suitable in some degree for application to other groups whose characteristics are relatively little known.

## II—Preliminary Study of Newark Data

It has long been known that local subscriber call holding times, $t$, follow remarkably closely the simple exponential frequency distribution,

$$f(t)dt = ke^{-kt}dt, \tag{1}$$

where $\frac{1}{k}$ = the average holding time.[2] This was found to be substantially true of the data collected on the inter-office trunk group in Newark as shown in Fig. 1 for 7385 calls observed in 19 hours having loads in the range 15.0–16.0 average simultaneous calls. The fit of the exponential curve having an average equal to the observed average of 2.380 minutes is seen to be quite good. It may be further noted that in the exponential distribution the standard deviation, $\sigma$, equals the mean $\bar{t}$. In practice $\sigma$

[2] A. K. Erlang apparently was the first to notice this holding time distribution, "Nyt Tidsskrift for Matematik" (Denmark), 1909.

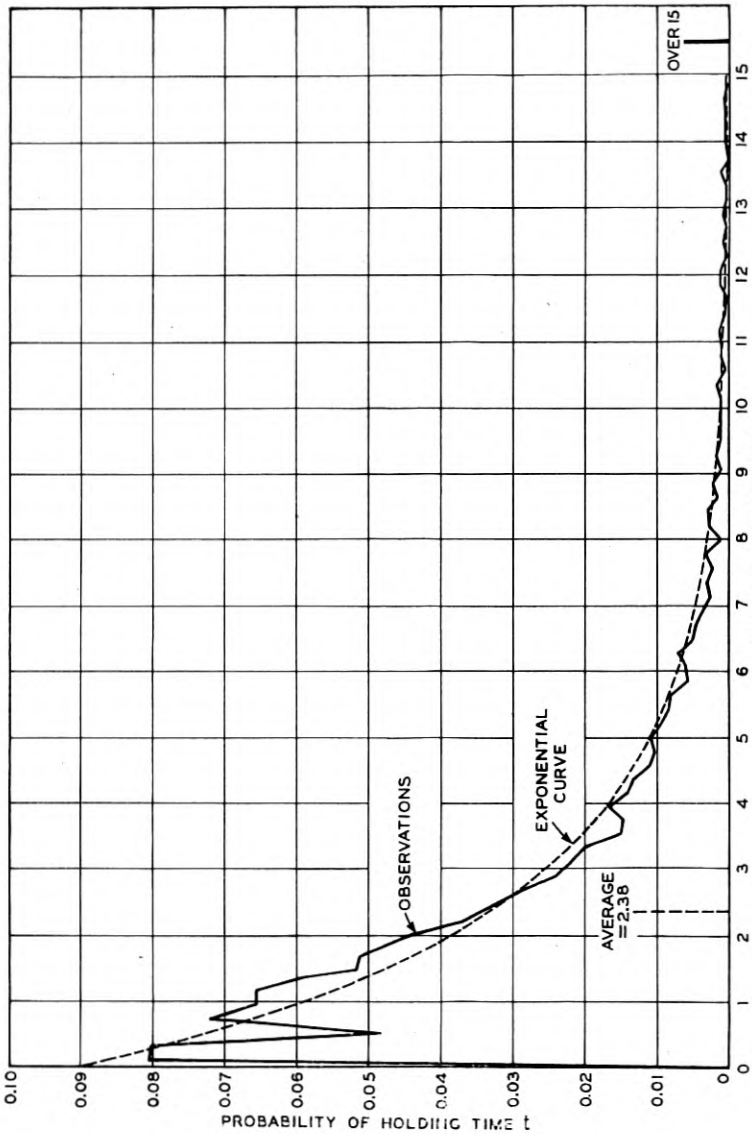



Fig. 1—Distribution of 7385 local holding times, Newark

is usually found to be slightly larger than $\bar{l}$ although not markedly so. We may use this information to test the homogeneity of the holding time universe should all hours of the days be grouped indiscriminately.

TABLE I
HOURLY HOLDING TIME DATA, NEWARK
(Figures in body of table are average holding times in seconds)

| Day | Date | Hour of day | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 9–10 am | 10–11 am | 11–12 am | 12–1 pm | 1–2 pm | 2–3 pm | 3–4 pm | 4–5 pm |
| Monday | 7– 8–18 | 131.0 | 139.0 | 140.5 | | | | | |
| Tuesday | 7– 9–18 | 151.1 | 151.0 | 159.0 | | | | | |
| Wednesday | 7–10–18 | 146.3 | 161.4 | 140.0 | | | | | |
| Thursday | 7–11–18 | 138.5 | 133.7 | 151.4 | | | | | |
| Friday | 7–12–18 | — | 146.0 | 138.1 | | | | | |
| Saturday | 7–13–18 | 123.7 | 139.5 | 130.5 | | | | | |
| M. | 7–15–18 | 135.1 | 152.6 | 147.0 | | | | | |
| T. | 7–16–18 | 134.5 | 138.4 | — | | | | | |
| W. | 7–17–18 | 138.4 | 151.3 | 159.8 | | | | | |
| Th. | 7–18–18 | 148.2 | 148.0 | — | | | | | |
| F. | 7–19–18 | 147.1 | 136.4 | — | | | | | |
| S. | 7–20–18 | 146.9 | 131.8 | — | | | | | |
| M. | 7–22–18 | 145.2 | 146.7 | 148.7 | | | | | |
| T. | 7–23–18 | 154.5 | 145.3 | 143.7 | | | | | |
| W. | 7–24–18 | 132.5 | 137.7 | 157.5 | | | | | |
| Th. | 7–25–18 | — | 149.0 | — | | | | | |
| F. | 7–26–18 | 138.0 | 157.9 | 174.4 | | | | | |
| S. | 7–27–18 | 128.5 | 142.0 | 150.1 | | | | | |
| M. | 7–29–18 | 132.3 | 141.5 | — | | 166.7 | | | |
| T. | 7–30–18 | 151.3 | 143.4 | 139.5 | | | | | |
| W. | 7–31–18 | 142.1 | 129.4 | 144.1 | | | | | |
| Th. | 8– 1–18 | 141.1 | 134.4 | 154.1 | | | | | |
| F. | 8– 2–18 | 161.6 | 150.5 | 150.3 | | | | | |
| S. | 8– 3–18 | 148.7 | 147.7 | 134.5 | | | | | |
| M. | 8– 5–18 | 139.4 | 131.0 | 142.9 | | | | | |
| T. | 8– 6–18 | 158.0 | 141.4 | 158.5 | | | | | |
| T. | 8–13–18 | 162.8 | 141.6 | — | | | | | |
| W. | 8–14–18 | 136.0 | 150.8 | — | | | | | |
| Th. | 8–15–18 | 153.0 | 141.8 | 139.0 | | | | | |
| F. | 8–16–18 | 141.0 | 160.1 | 151.6 | | | | | |
| Th. | 9– 5–18 | 144.7 | 158.9 | — | | | | | |
| F. | 9– 6–18 | — | 139.5 | — | | | | | |
| W. | 9–25–18 | — | — | 144.3 | | | | 157.6 | |
| Th. | 9–26–18 | 152.1 | 143.1 | 134.2 | | | | | |
| F. | 9–27–18 | 139.7 | 160.5 | 149.9 | | | | | |
| M. | 9–30–18 | — | 132.8 | 128.9 | | 138.7 | | | 152.6 |
| T. | 10– 1–18 | 129.7 | 137.5 | 150.0 | | 158.3 | | | 166.0 |
| W. | 10– 2–18 | 138.5 | 135.2 | 132.5 | | | 161.0 | | |
| Th. | 10– 3–18 | 142.0 | 143.0 | 152.0 | 174.3 | 153.0 | 165.0 | | |
| F. | 10– 4–18 | 128.4 | 136.9 | 145.7 | | 150.5 | | | |
| S. | 10– 5–18 | 137.0 | 138.4 | 138.5 | | | | | |
| M. | 10– 7–18 | — | — | — | | 150.0 | 139.2 | 137.7 | |
| T. | 10– 8–18 | 131.0 | 136.4 | 145.1 | | | 150.0 | 145.1 | 152.2 |
| W. | 10– 9–18 | 138.3 | 144.4 | 142.0 | | 174.8 | | 145.4 | 151.6 |
| Th. | 10–11–18 | 135.4 | 149.8 | — | | | | | |
| *Summary:* | | | | | | | | | |
| No. Hours | | 39 | 43 | 33 | 1 | 7 | 4 | 4 | 4 |
| Average | | 141.63 | 143.67 | 146.01 | 174.3 | 156.3 | 153.8 | 146.4 | 155.6 |

In Table I and on Fig. 2 are shown the holding time averages for 135 hours observed at various times of the day over a period of 3 months. At first glance these appear to fall in two rather distinct groups, those before noon and those after noon. If the 115 hours before noon be considered as defining a homogeneous group, could those holding time averages found in the afternoon be reasonably considered as coming from the same universe? We first find the average holding time of the 115 forenoon hours to be 143.5 seconds. Since these hours averaged about $n = 390$ calls each, the standard deviation of the means $\sigma_{\bar{t}}$ should, by theory, be closely

$$\sigma_{\bar{t}} = \frac{\sigma}{\sqrt{n}} = \frac{\bar{t}}{\sqrt{390}} = 7.29.$$

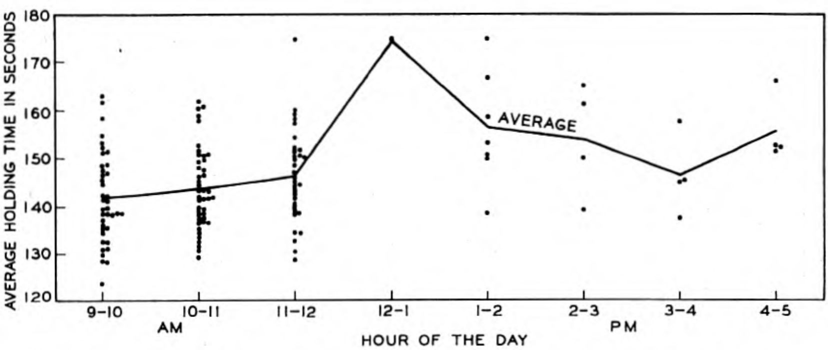



Fig. 2—Day to day holding time averages by hours of the day, 135 hours, Newark

The standard deviation observed is 9.26, some 27% higher, which, however, agrees with the observation made in the previous paragraph. On the hypothesis that the universe of 115 early hours has the parameters of $\bar{t} = 143.5$ and $\sigma = 7.29$, we see that the observations for each of these three clock hours could readily have occurred. The deviations of their averages from 143.5 are 1.9, .17 and 2.5 seconds, respectively, and according to theory the corresponding standard errors in these averages are $1.168 \left(= \frac{7.29}{\sqrt{39}}\right)$, $1.111 \left(= \frac{7.29}{\sqrt{43}}\right)$, and $1.270 \left(= \frac{7.29}{\sqrt{33}}\right)$. All the deviations are well within two times the standard error of the assumed mean of the holding time universe. The remaining 20 observations from noon on, however, average 154.6 seconds, and if they could reasonably have come from the hypothesized universe, this figure should not differ from 143.5 by more than, say, three times the standard error $1.630 \left(= \frac{7.29}{\sqrt{20}}\right)$. Actually the difference is more than six times the standard error, strongly indicating a significant difference between the forenoon and afternoon holding times. We conclude that between 9 a.m. and noon the holding times are satisfactorily controlled but that we should not attempt to include observations on afternoon hours with them. Since the heaviest loads here occurred generally in the morning we should confine our direct measurements or switch counts to these hours for determining the engineering holding time.

It may occasionally be well to investigate the possibility that certain days of the week have, on the average, longer holding times than other days. If the Newark 9–12 a.m. data are plotted by days of the week as in Fig. 3 we see that the averages for each day fluctuate considerably as shown by the heavy dots. In testing these points the simple average of the $\sigma$'s for

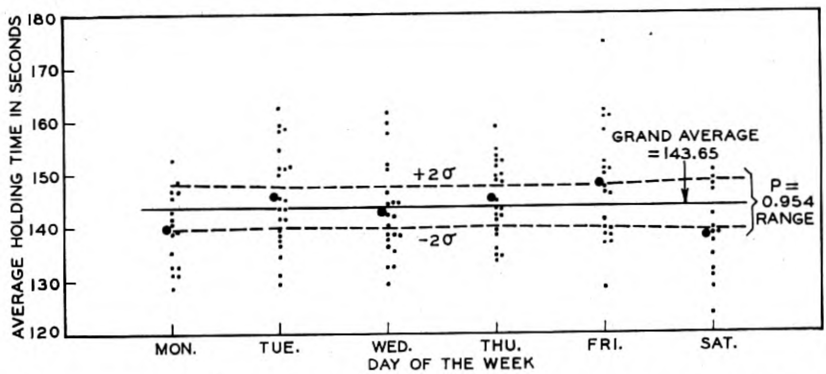



Fig. 3—Variations in average holding times, 115 9–12 a.m. hours, by days of the week, Newark

each day's hours is taken as an estimate of the standard deviation of the homogeneous universe from which all the hours are presumed to be drawn. Then with the weighted arithmetic means of the daily averages as the best estimate available for the mean of the universe, each day's average is tested to see whether it could reasonably have arisen from it. The $\pm 2\sigma$ lines which should include some 95% of the day-averages are shown on Fig. 3. It is seen that two of the six points fall slightly outside these limits indicating a moderately significant difference in the holding time conditions for Friday and Saturday. The sampling procedure to follow in such cases of non-controlled populations is not rigorously definable. However, it is clear that the samples should be drawn from the various groups of controlled elements which probably go to make up the universe, and roughly in proportion to the importance to be assigned to each such group. In our example here we would probably want to draw samples of about equal size from the calls of each week day in the week.

Finally there may be some question as to the busy season, its length and stability. Plotting of the same data for the 9–12 a.m. hours as in Fig. 4A and 4B will help to decide these points. The morning hours' holding time averages for each day of the week are plotted for several weeks during the suspected busy season. Wide changes in the load through the passage

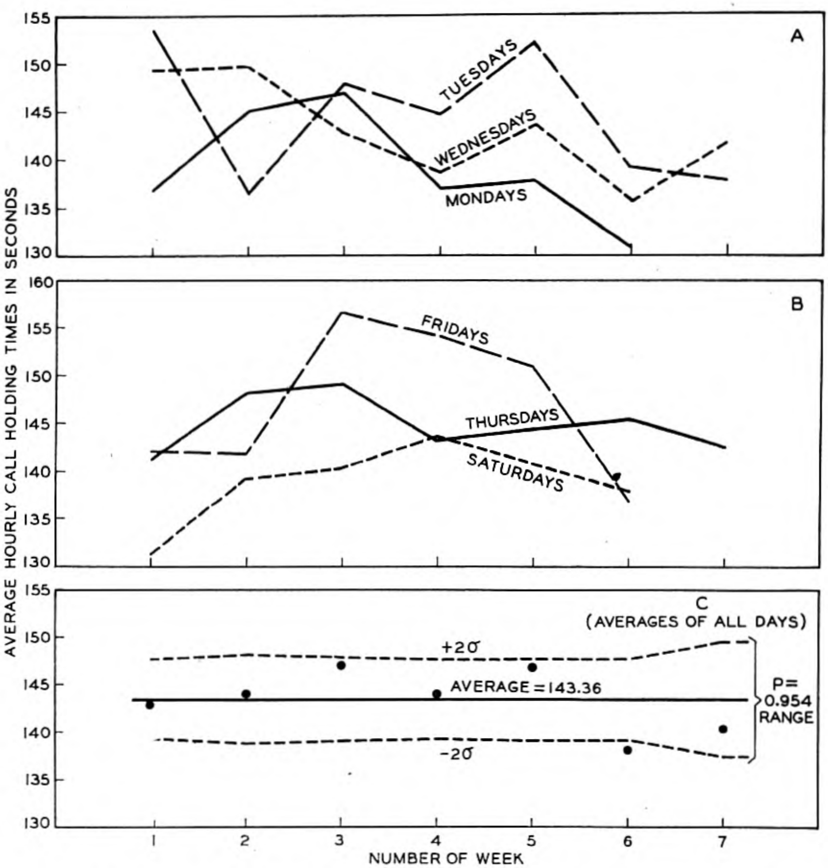



Fig. 4—Seasonal trend in holding time averages, 105 9–12 a.m. hours, Newark

of weeks can be noted by eye. In the case illustrated there appears no consistency of movement. Applying the identical test used in the previous paragraph for the day-of-the-week changes, we find in Fig. 4C that in the first five-week period no movements of significance took place. However, the sixth week, which followed the first five after an interval of about six weeks, showed a significant drop suggesting the approach of a lower level of traffic. The traffic engineer would probably decide to schedule his holding time observations during the weeks numbered one to five inclusive.

Having determined something as to the character of the holding time trends, if any, with hours of the day, with days of the week, and seasons of the year the traffic engineer is in a better position to lay out a program for sampling. He will especially want to apportion the total sample between the hours or days which show significant differences among themselves roughly in proportion to the relative traffics flowing at those levels. The less specific the information on the traffic flow characteristics the more important it will be to spread the observations over a variety of hours, days or weeks.

## III—A Satisfactory Sample of Directly Measured Holding Times

If the standard deviation $\sigma$ of individual call lengths is known, we can estimate the standard error of the average of $n$ measurements as

$$\sigma_{\text{avg}} = \frac{\sigma}{\sqrt{n}}. \tag{2}$$

Since $n$ will usually be several hundred we can obtain a good figure for $\sigma$ by calculating the standard deviation, $S$, of the $n$ observations. As noted before, for exponential calls this will be not far from the average holding time $\bar{t}$ which may be substituted for $\sigma$ if great accuracy is not required. In fact if the sampling is representatively made from a universe not strictly homogeneous, the better figure for $\sigma$ may be the average $\bar{t}$, instead of the standard deviation found in the sample since in so-called Poisson Sampling of stable but nonhomogeneous universes the standard error of the average may be somewhat reduced from $S/\sqrt{n}$.

We may now make the statement that for $n$ large the probability is $P$ that the true average holding time does not differ from that observed by more than $\pm z \dfrac{\bar{t}}{\sqrt{n}}$ seconds, where $P$ and $z$ are given in the table below.

TABLE II

| $P$ | $z$ | $P$ | $z$ |
|---|---|---|---|
| .50 | .6745 | .95 | 1.960 |
| .85 | 1.440 | .99 | 2.576 |
| .90 | 1.645 | .999 | 3.291 |

For example if we have measured the individual lengths of 900 calls which show an average of 150.3 seconds, we are then 99% sure that the true holding time average for the sampled universe lies closely within the range

$$150.3 \pm 2.576 \frac{150.3}{\sqrt{900}},$$

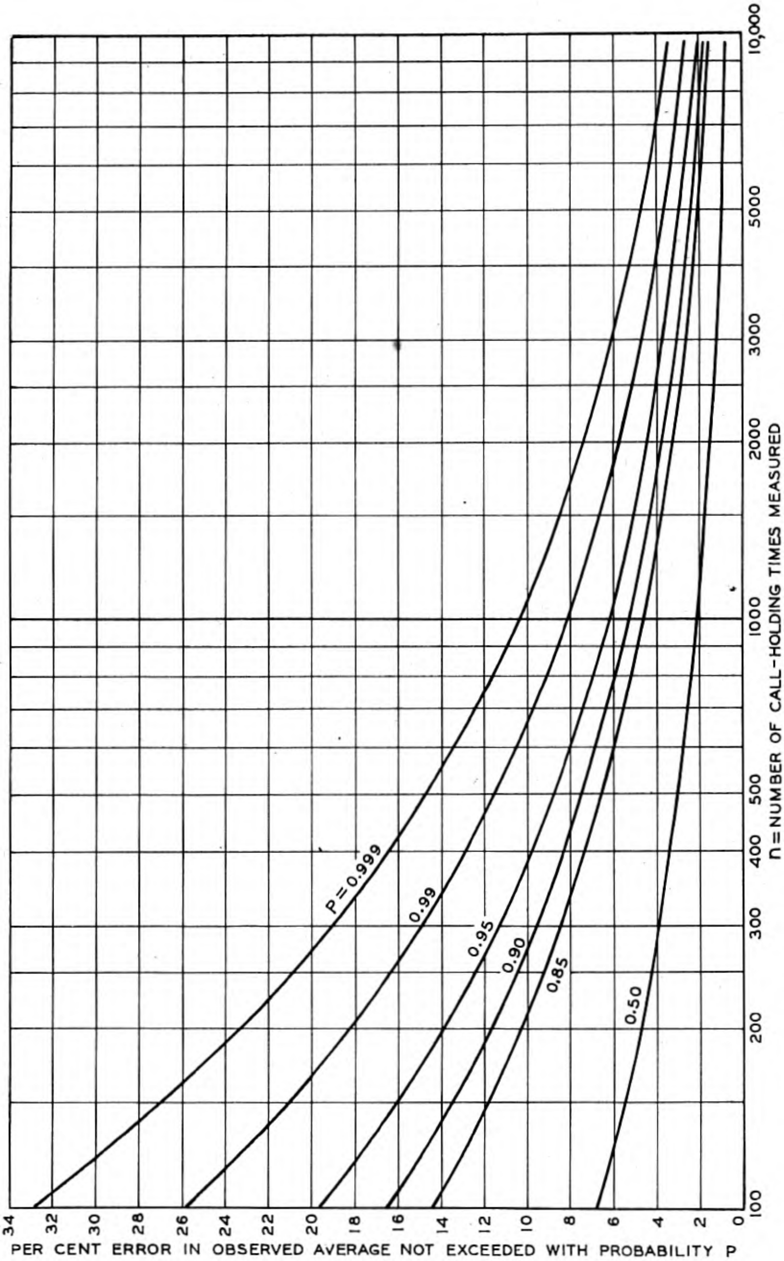



Fig. 5—Error in exponential-holding-time average determined by stop watch method

that is between 137.4 and 163.2 seconds. (The best single estimate, of course, is the observed average of 150.3 seconds.) Or conversely, if one should desire to determine the true average holding time within 5 seconds with a surety of $P = .90$, he may use an approximate value of the standard deviation (or the average holding time) based on past experience and substitute in

$$n = \left(\frac{1.645\bar{t}}{5}\right)^2,$$

to obtain the number of calls to be measured. If $\bar{t} = 150$ seconds here, $n = 2421$.

The same information is contained in Fig. 5 which gives the *per cent* error, $\pm 100 z_P/\sqrt{n}$, in the observed average not exceeded with probability $P$.

All this is based on the assumption that each of the $n$ call lengths is accurately enough measured so that no appreciable error is introduced from this source. Obviously there is no point in expending much effort in carefully "proportioning" a sample so as to be representative of the vagaries of the universe if each of the calls so chosen is not pretty accurately measured. This would be quite as futile as measuring very accurately the holding times of a number of calls chosen during some short time period which might turn out to be wholly untypical of certain of those important periods coming earlier or later. For these direct measurement cases it will probably be quite satisfactory if each call is measured with a maximum error of not over one-tenth of $\bar{t}/\sqrt{n}$. In our example of 900 calls this would be .501 seconds, that is measurement of each call to the nearest second.

## IV—Holding Times by Switch Count Methods

If each call's holding time is not measured with considerable accuracy it is immediately clear that additional calls must be observed in order to compensate therefor. This is the situation in the method of switch counts which is in effect a means for noting at regular intervals $i$ whether a particular call does or does not exist. Thus none of the calls are at all closely measured for their individual lengths. Other errors will also have to be considered since at the beginning of the period some switch counts are inevitably included on a number of calls from the preceding hour and at the end of the period some of the calls registered on the peg count meter will end beyond the period with the loss of part of their proper switch counts. As a result there are in this method three distinct sources of holding time error whose magnitudes we shall proceed to investigate in turn:

a. Errors at the start of the observation period;
b. Errors at the end of the observation period;
c. Errors at the beginning and end of each call.

The theoretical conclusions will be compared at various points with certain data available.

### *a. Errors at the Start of the Observation Period*

If the period of observation $T$ be divided into $r$ equal intervals of length $i$, and switch counts are made at the beginning (and end) of each interval, we shall have a total of $r + 1$ observations. The #1 count will give us immediately the number of calls extending into the period from the pre-

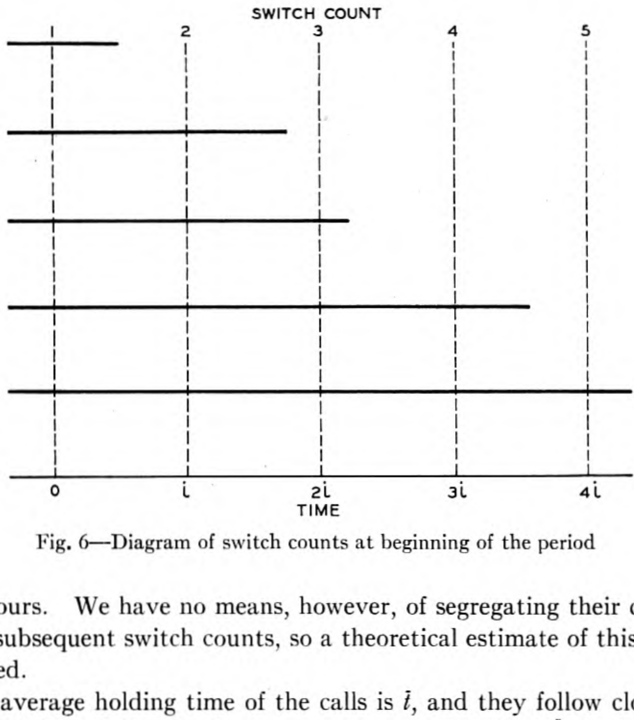



Fig. 6—Diagram of switch counts at beginning of the period

ceding hours. We have no means, however, of segregating their contributions to subsequent switch counts, so a theoretical estimate of this amount is required.

If the average holding time of the calls is $\bar{t}$, and they follow closely the exponential law of distribution, we may reason as follows.[3] Consider the case of a single call passing time 0 at the start of the observation period as in Fig. 6. Then the probability that it will be included only in switch count number 1, that is that it ends between time 0 and time $i$, is

$$p_1 = P(<i) = 1 - e^{-\frac{i}{\bar{t}}}.$$

[3] Of course we do not know $\bar{t}$ exactly since that is the ultimate object of our study; however, for the present purpose great accuracy will not be required, and $\bar{t}$ can usually be taken as the first estimate of holding time obtained by the switch count method without corrections.

Similarly the probability that exactly two switch counts will be contributed by such a call is

$$p_2 = P(>i) - P(>2i) = e^{-\frac{i}{\bar{i}}} - e^{-\frac{2i}{\bar{i}}}.$$

Likewise,

$$p_3 = P(>2i) - P(>3i) = e^{-\frac{2i}{\bar{i}}} - e^{-\frac{3i}{\bar{i}}},$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$$p_u = P[>(u-1)i] - P(>ui) = e^{-\frac{(u-1)i}{\bar{i}}} - e^{-\frac{ui}{\bar{i}}}.$$

If now there have been $m$ such calls observed on switch count number 1, we shall need to add $m$ variables of the type

$$f(u) = e^{-\frac{(u-1)i}{\bar{i}}} - e^{-\frac{ui}{\bar{i}}} = e^{-\frac{ui}{\bar{i}}}\left(e^{\frac{i}{\bar{i}}} - 1\right) = ce^{-\frac{ui}{\bar{i}}},$$

where $u$ may take all values from 1 to $r + 1$. The exact addition of these variables when $m$ is more than a small number, say 3 or 4, becomes quite complex. However, in such cases (which may be the rule) we revert to the method of combining their individual moments to obtain the moments (and parameters) of the resultant distribution. We find for a single variable,

$$\bar{u} = c\sum_{u=1}^{r+1} ue^{-u\frac{i}{\bar{i}}} = \frac{1}{1 - e^{-\frac{i}{\bar{i}}}}\left[1 - e^{-(r+1)\frac{i}{\bar{i}}}\right], \tag{3}$$

and

$$\sigma_u = \frac{e^{-\frac{i}{2\bar{i}}}}{1 - e^{-\frac{i}{\bar{i}}}}\left[1 - 2(r+1)\left(1 - e^{-\frac{i}{\bar{i}}}\right)e^{-r\frac{i}{\bar{i}}} - e^{-(2r+1)\frac{i}{\bar{i}}}\right]^{\frac{1}{2}}. \tag{4}$$

The factors shown in brackets in equations (3) and (4) will approach unity very closely in any practical applications of the present type; they will therefore be omitted in the subsequent analysis.

The mean and standard deviation of the sum of $m$ such variables are readily determined, of course, as[4]

[4] It is interesting to note that if the first switch count had been omitted so that only $\overline{m}$ could have been estimated from the average of the switch counts from #2 onward, we might have assumed a Poisson distribution for $m$, that is $p_m = \frac{\overline{m}^m e^{-\overline{m}}}{m!}$, and thereby have obtained an estimate of the switch counts contributed by calls from the preceding period as follows,

$$\bar{s}'_m \approx \frac{\overline{m}}{1 - e^{-\frac{i}{\bar{i}}}}, \tag{5'}$$

$$\sigma'_m \approx \frac{\sqrt{\overline{m}\left(e^{-\frac{i}{\bar{i}}} + 1\right)}}{1 - e^{-\frac{i}{\bar{i}}}}. \tag{6'}$$

$$\bar{s}_m = m\bar{u} = \frac{m}{1 - e}, \tag{5}$$

$$\sigma_m = \sqrt{m}\,\sigma_u = \sqrt{m}\,\frac{e^{-\frac{i}{2t}}}{1 - e^{-\frac{i}{t}}}. \tag{6}$$

In Fig. 7 is shown a comparison between this theory and the actual numbers of 1-minute switch counts contributed by these carry-over calls

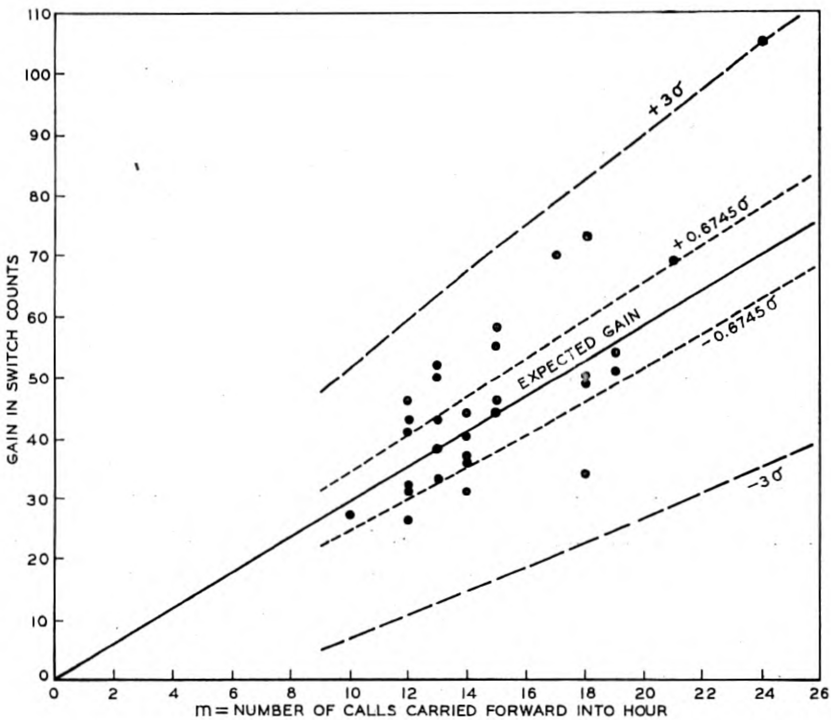



Fig. 7—Gains over true switch counts at the start of the observation period

in 30 hourly periods of observation on the Newark interoffice trunk group previously described. Since $\bar{m}$, the average number carried forward is about 15, it was thought that the sum-distribution of so many variables would rather closely approach normality. On this assumption the ±50% and ±99.73% "control" lines have been drawn on Fig. 7. One point falls on the latter limit lines, while 16 of the 30 fall within the 50% lines, thus providing a gratifying corroboration of the theory.

### *b. Errors at the End of the Observation Period*

If switch counts have been made at regular intervals $i$ so that the $r + 1$st count occurs at the exact end of the period for which the number of originating calls has been registered, then the situation closely resembles that at the start of the period. The $r + 1$st observation tells us immediately how many calls are continuing into the next hour. A particular one of these calls may extend to the areas 0 to $i$, $i$ to $2i$, $2i$ to $3i$, etc., measured beyond the end of the period as in Fig. 8. A call ending in the interval $2i$ to $3i$,

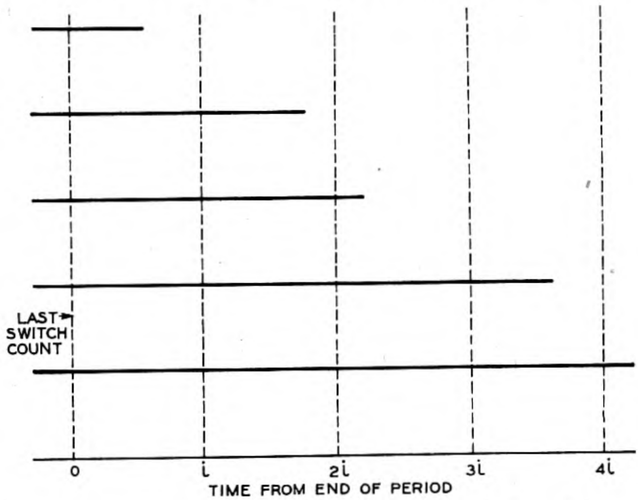



Fig. 8—Diagram of switch counts at the end of the period

for instance, would fail to have two switch counts marked up if the counting stopped at 0. Then the probability of losing exactly zero switch counts is

$$p_0 = P(<i) = P(>0) - P(>1i) = 1 - e^{-\frac{i}{\bar{i}}},$$

and in general

$$p_v = P(>vi) - P[>(v + 1)i] = e^{-\frac{vi}{\bar{i}}} - e^{-\frac{(v+1)i}{\bar{i}}} = e^{-\frac{vi}{\bar{i}}}\left(1 - e^{-\frac{i}{\bar{i}}}\right)$$

$$= c'e^{-v\frac{i}{\bar{i}}} \tag{7}$$

where $v$ varies from $o$ to $\infty$. The average and standard deviation of a single variable will be[5]

$$\bar{v} = \sum_{v=0}^{\infty} vp_v = \frac{e^{-\frac{i}{\bar{i}}}}{1 - e^{-\frac{i}{\bar{i}}}}, \tag{8}$$

[5] The value $\bar{v}$ is one less than $\bar{u}$ shown in equation (3) after neglecting the minute correction factor, since each call there received a switch count at 0 time which is omitted here. The standard deviation is identical with equation (4) without the correction factor, since a constant deduction has simply been made on each call.

and

$$\sigma_v = \frac{e^{-\frac{i}{2\bar{i}}}}{1 - e^{-\frac{i}{\bar{i}}}}. \tag{9}$$

The frequency distribution of the sum of $w$ such discrete variables $v$ is readily found to be

$$f(v_w) = \left(1 - e^{-\frac{i}{\bar{i}}}\right)^w \frac{(w + v_w - 1)!}{(w - 1)!\, v_w!} e^{-v_w \frac{i}{\bar{i}}}.$$

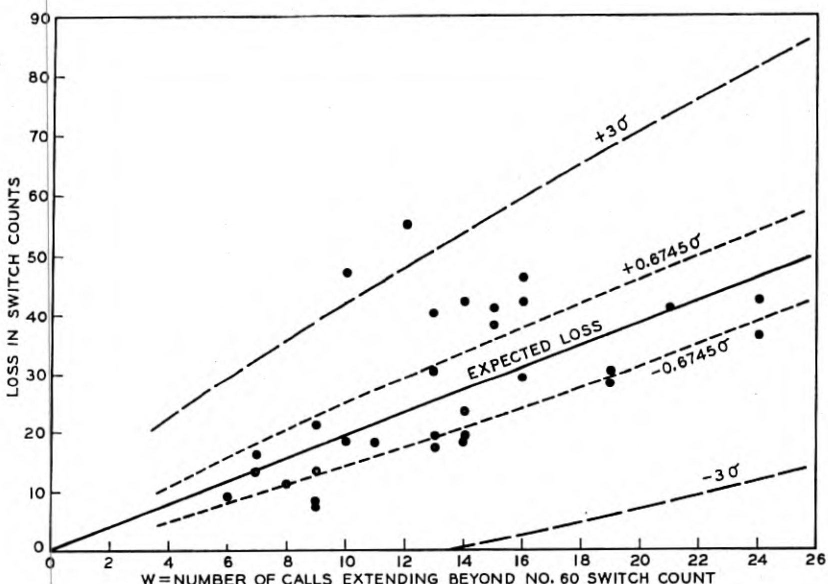



Fig. 9—Loss of true switch counts at the end of the observation period

The parameters of this sum distribution are

$$\bar{s}_w = w\bar{v} = \frac{we^{-\frac{i}{\bar{i}}}}{1 - e^{-\frac{i}{\bar{i}}}}, \tag{10}$$

$$\sigma_w = \sqrt{w}\,\sigma_v = \sqrt{w}\,\frac{e^{-\frac{i}{2\bar{i}}}}{1 - e^{-\frac{i}{\bar{i}}}}. \tag{11}$$

A check on these last formulas is shown in Fig. 9 where the numbers of switch counts lost on $w$ calls carried beyond the last (or $r$ + 1st) switch count are recorded for 30 busy hour periods. Two of the hours show results outside the theoretical $\pm 3\ \sigma$ normal curve limits but the falling of almost exactly half of the points within the ±50% lines reassures us that the estimated parameters are probably quite good. (If instead we had calculated, say, ±99% limit lines based on the skew distribution $f(v_w)$ just derived, it seems likely that even the two "unusual" points of Fig. 9 might have fallen in the "reasonable" range.)

At this point it will be well to point out that in making switch counts considerable care needs to be exercised in two directions. First, switch counts #1 and #$r + 1$ should coincide very closely with the beginning and end, respectively, of the observation period, the intermediate counts of course being uniformly spaced. Second, each count should be taken as quickly as possible so that a substantially instantaneous reading of calls in progress is obtained. These two desiderata can usually be attained readily in schemes using mechanical, electrical or photographic means for recording the switch counts. Counts made by observers, however, may be subject to highly variable errors since in some cases a substantial portion of the interval $i$ may be required to complete a count. Such uncertainties naturally increase the end-effect errors, and, consequently, the overall error in an average-holding-time determination.

To estimate the magnitude of the increased errors resulting from failure to meet the above switch count specifications would require a special study for each kind and type of failure; these would probably differ in every application. An idea of the sensitiveness of switch counting to such irregularities and the likely order of magnitude of the increased errors may be gained by examining certain of the Newark data. Here the last switch count in many of the hours, although taken instantly, failed to coincide well with the end of the observation period $T$. The last count fell at points varying from a little after the period closed to nearly an interval $i$ ahead of this instant as shown in Fig. 10. In most hours this permitted a small number of calls to mark up on the peg count register after the last switch count was taken.

As a result, if corrections were not made, the estimate of switch counts lost on calls extending beyond the end of the hour would have omitted about a third of those properly included, with a consequent lowering of the average holding time estimate by approximately one per cent.

If the time $j$ which has elapsed between the last switch count and the end of the period $T$ is known, certain corrections for this particular irregularity can be attempted. We shall indicate the formulas required since they will be useful in an analysis of the Newark data. If an average of $\alpha = \frac{j}{i} a$ calls are assumed to originate in the interval $j$, and they follow a Poisson distribution $\frac{\alpha^x e^{-\alpha}}{\underline{/x}}$, then the expected number of switch counts lost is found to be

$$E = \alpha \frac{\frac{\bar{t}}{j}\left(e^{\frac{i}{\bar{i}}} - 1\right)\left(1 - e^{-\frac{j}{\bar{i}}}\right)e^{\frac{j-2i}{\bar{i}}}}{\left(1 - e^{-\frac{i}{\bar{i}}}\right)^2}, \tag{12}$$

and the standard deviation is

$$\sigma' = \sqrt{\alpha}\left[\frac{\bar{t}}{j}\left(e^{\frac{i}{\bar{i}}} - 1\right)\left(1 - e^{-\frac{j}{\bar{i}}}\right)e^{\frac{j-2i}{\bar{i}}}\ \frac{1 + e^{-\frac{i}{\bar{i}}}}{\left(1 - e^{-\frac{i}{\bar{i}}}\right)^3}\right]^{\frac{1}{2}}. \tag{13}$$

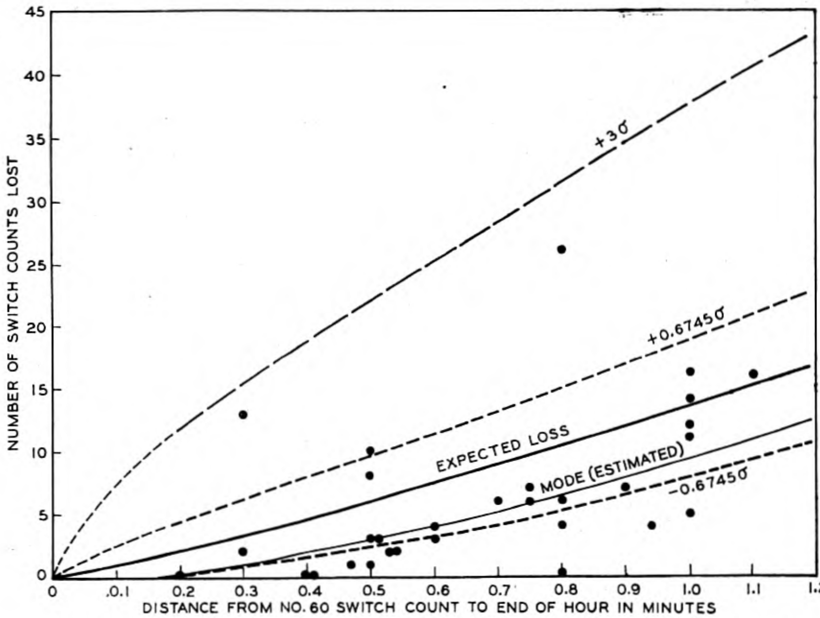



Fig. 10—Switch counts lost on calls which originated after the last switch count

If $j$ is small, $\alpha$ will be correspondingly small making the average switch counts lost, $E$, small. The distribution of lost switch counts will then be very skew since the case of zero lost will be prominent. For larger $j$'s the distribution should assume a unimodal form, gradually becoming less skew. A number of Newark busy hours were studied in this fashion, and the results are compared with theory on Fig. 10. The agreement is seen to be fair and a modal line estimated by eye falls substantially below the expected mean corroborating the decided skewness just predicted. The wide fluctuations in lost switch counts, each one of which if incorrectly estimated results in a considerable error (in the present example about .2 second) in the estimate of the average holding time, will serve to indicate the desirability where possible of eliminating altogether this and other supplementary errors by seeing that individual counts are taken very quickly, and that the first and last switch counts coincide closely with the ends of the observation period.

### *c. Errors in Measuring Each Call*

Due to the method of counting the switches at finite intervals, an exact measurement of the length $t$ of any one call will seldom if ever be made. We shall attempt in what follows to determine the magnitude and char-

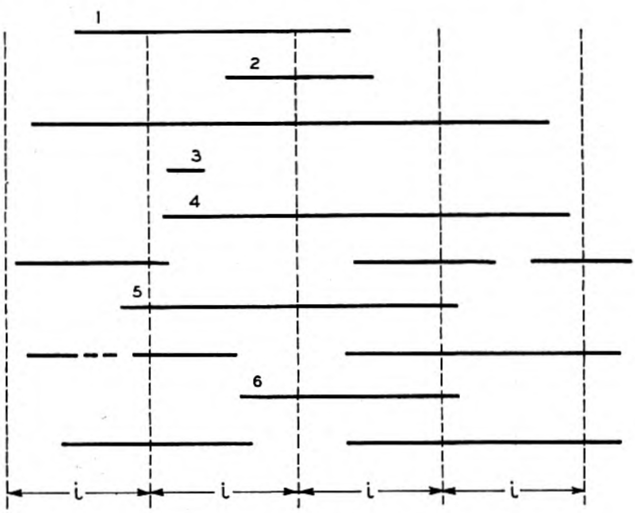



Fig. 11—Typical field of switch counts

acteristics of these errors in measuring individual calls. In any field of switch counts, such as in Fig. 11, there will be calls of types #1 and #2 which receive about one switch count for each $i$ call seconds of length. There will likewise be many others such as #3 and #4 which will be substantially undercounted, and about as many others, #5 and #6, which will be overcounted. We shall proceed with certain special cases, and then obtain the general result desired.

*Case 1. $t$ lies between 0 and $i$*

If the holding times $t$ follow some law of fluctuation $f(t)$, a certain proportion of them will have lengths lying in the range $t = 0$ to $t = i$. Such a call will either cross one of the switch count points, or it will not. Upon the assumption of a random instant of origination the probability of its crossing will be simply $\frac{t}{i}$. That is, if the origination point falls within $t$ seconds to the left of a switch count, the call will be marked up; if not it will not be counted. If it is marked up there will occur for that call a plus estimation error of $x = i - t$ seconds since we shall eventually assume that every switch count infers $i$ call seconds of use on the trunk group. Likewise there is a probability of $\frac{i - t}{i}$ that the call will not be counted with the resultant negative error of $x = -t$ seconds. In summary the total probability of a positive error of $x = i - t$ is

$$f(t)\, dt\, \frac{t}{i};$$

and for a negative error of $x = -t$,

$$f(t)\, dt\, \frac{i - t}{i}.$$

*Case 2. $t$ lies between $i$ and $2i$*

Such a call may be included either once or twice in the switch counts. The probability that it is counted twice is $\frac{t - i}{i}$ with a resultant plus error of $x = 2i - t$. The probability that it is counted but once is $1 - \frac{t - i}{i} = \frac{2i - t}{i}$, with the corresponding negative error of $x = i - t$. The overall probabilities of course will be formed by weighting these as in the first case with the probability, $f(t)dt$, that the holding time of length $t$ to $t + dt$ actually occurs.

*General Case. $t$ lies between $qi$ and $(q + 1)i$*

It will readily be seen that by extending the reasoning of the two cases above to the case of $t$ lying between $qi$ and $(q + 1)i$ we shall have a plus error (due to the call being marked up $q + 1$ times) of $x = (q + 1)i - t$, with a probability of occurrence of $\frac{t - qi}{i}$, and a negative error (the call marked up $q$ times) of $x = qi - t$ with a probability of $\frac{(q + 1)i - t}{i}$.

Summarizing the above cases, a *negative* error of size $x$ can occur in a great number of ways, due to $t$ taking the values $-x$, $i - x$, $2i - x$, $\cdots$ $qi - x$, $\cdots$ with the corresponding probabilities of occurrence of the call lengths, $f(-x)$, $f(i - x)$, $f(2i - x)$, $\cdots$ $f(qi - x)$ $\cdots$, respectively. In addition each time such a call length does occur we must introduce the contingent probability $\frac{i+x}{i}$ that a negative and not a positive error will occur. The total probability of making an error of $x$, where $x \lesseqgtr 0$, on any call is then,

$$p_{x \lesseqgtr 0}(x)\,dx = \frac{i+x}{i}\,[f(-x) + f(i-x) + f(2i-x) + \cdots]\,dx. \quad (14)$$

Similarly we find the total probability of making a positive error of magnitude $x$, on any call, as

$$p_{x \gtreqless 0}(x)\,dx = \frac{i-x}{i}\,[f(i-x) + f(2i-x) + f(3i-x) + \cdots]\,dx. \quad (15)$$

It will now be of interest to apply equations (14) and (15) to some particular types of holding time distributions.

*(a). Constant Holding Times, $t = h$*[6]

If $t$ is constant and equal to $h$, it will necessarily fall within some one of the special cases enumerated above. Suppose $h$ lies between $qi$ and $(q+1)i$. There will be but one value of the error $x_1$ possible in the negative range and it will equal $qi - h$, with a corresponding single value $x_2$ in the positive range equal to $(q+1)i - h$. It will be seen that equations (14) and (15) reduce simply to

$$p_{x_1 \leq 0}(x_1) = p(qi - h) = \frac{i+x_1}{i} f(qi - x) = \frac{i+x_1}{i} f(h) = \frac{i+x_1}{i}, \quad (16)$$

and

$$\begin{aligned} p_{x_2 \geq 0}(x_2) &= p[(q+1)i - h] \\ &= \frac{i-x_2}{i} f[(q+1)i - x_2] = \frac{i-x_2}{i} f(h) = \frac{i-x_2}{i}. \end{aligned} \quad (17)$$

The mean and standard deviation of this two-valued variable are found to be

$$\bar{x} = 0, \quad (18)$$

$$\sigma_x = \sqrt{-x_1(i+x_1)} = \sqrt{(i-x_2)x_2} = \sqrt{-x_1x_2}. \quad (19)$$

It may be noted that $\sigma_x$ attains a maximum of $i/2$ when $x_2 = i/2$, and approaches 0 for $x = 0$. This is of importance when one has to choose an observation interval $i$ for switch counting constant or relatively constant holding times.

[6] The particular error distributions for cases *a* and *c* were obtained by G. W. Kenrick in 1923.

*Example*: An hour with 372 calls having a constant holding time per call of $h = 131.8$ seconds was subjected to a 60 second switch count study, records being kept of the errors in measurement on individual calls. As shown in Fig. 12, 284 or 76.3% gave counts of "2" with an error of $120 - 131.8 = -11.8$ seconds. The remaining 88 calls, or 23.7% received counts

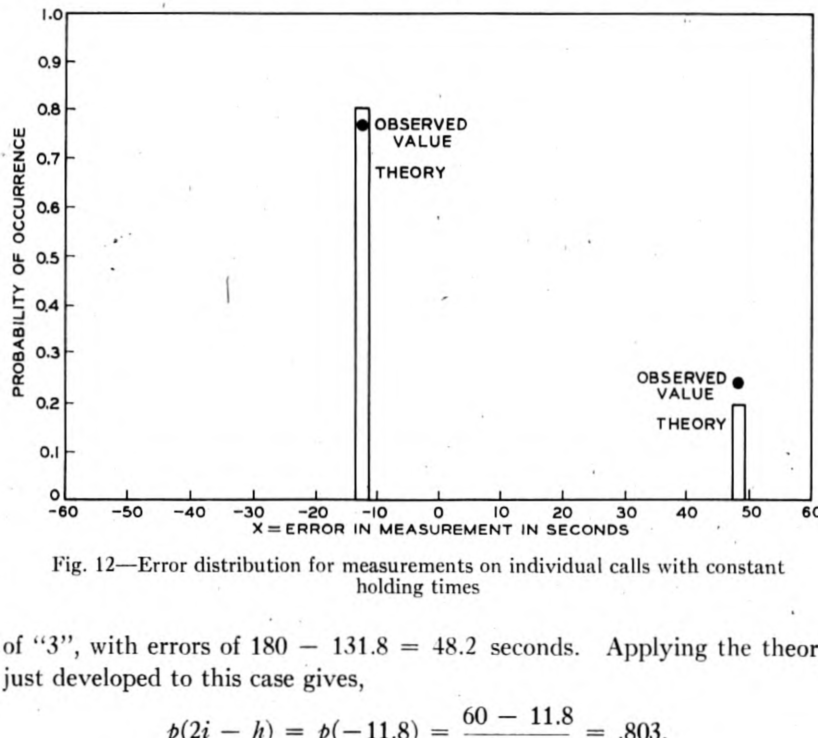



Fig. 12—Error distribution for measurements on individual calls with constant holding times

of "3", with errors of $180 - 131.8 = 48.2$ seconds. Applying the theory just developed to this case gives,

$$p(2i - h) = p(-11.8) = \frac{60 - 11.8}{60} = .803,$$

$$p(3i - h) = p(48.2) = \frac{60 - 48.2}{60} = .197.$$

As indicated on Fig. 12, these theoretical values check very satisfactorily with the observations. The observed average holding time = 134.2 seconds as against the true value of 131.8 seconds; the error of 2.4 seconds is quite compatible with $\sigma_x = \sqrt{11.8(48.2)} = 23.85$ seconds and the $n = 372$ calls observed.

(*b*). *Equally Likely Distribution of Holding Times between Adjacent Multiples of i*

Imagine a holding time distribution of any general form but with a constant probability of occurrence between adjacent pairs of multiples of $i$, such as in Fig. 13. Such a distribution would probably occur but rarely, if ever, in practice. However, if the intervals $i$ are short compared to the average holding time $\bar{l}$, such an assumption may not introduce any serious discrepancy in whatever form is simulated.[7]

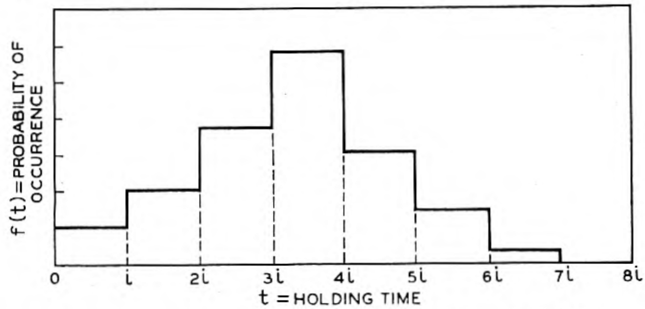



Fig. 13—A varying holding time with a number of equally likely ranges

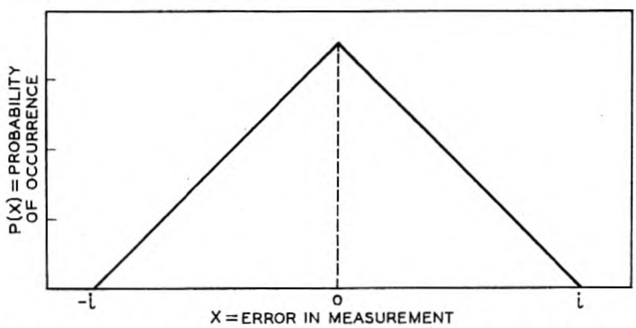



Fig. 14—Distribution of call measurement errors for "equally likely" holding time distributions

In this case it is obvious upon inspection that the sum of the terms in the brackets in equation (14) is a constant for all values of $x$, and likewise in equation (15), and that they equal each other. Hence

$$p_{x \leqq 0}(x) = K \frac{i + x}{i}, \tag{20}$$

and

$$p_{x \geqq 0}(x) = K \frac{i - x}{i}, \tag{21}$$

[7] The analytics of the allied case in which the errors at the ends of a call were assumed to fall equally likely between $\pm i/2$, were discussed by E. C. Molina in an unpublished memorandum dated September 7, 1920.

give the isosceles triangular distribution of errors on individual calls shown in Fig. 14. In this the average error is

$$\bar{x} = 0, \tag{22}$$

and the standard deviation is

$$\sigma_x = .408i. \tag{23}$$

*(c). Holding Times Exponentially Distributed,* $f(t) = ke^{-kt}$, *where* $k = \frac{1}{\bar{t}}$

With holding times of the exponential type the sum of the terms in the brackets of equations (14) and (15) will depend on the particular magnitudes of the errors $x$ assumed. If in equation (14), we substitute exponential expressions for the $f$-functions, we have

$$\begin{aligned} p_{x\lessgtr 0}(x)\,dx &= \frac{i+x}{i}\left(ke^{-k(-x)} + ke^{-k(i-x)} + ke^{-k(2i-x)} + \cdots\right)dx \\ &= \frac{i+x}{i}\,ke^{kx}\left(1 + e^{-ki} + e^{-2ki} + \cdots\right)dx \\ &= \frac{i+x}{i}\,ke^{kx}\,\frac{1}{1-e^{-ki}}\,dx \\ &= k'\,\frac{i+x}{i}\,e^{\frac{x}{\bar{t}}}\,dx, \end{aligned} \tag{24}$$

where

$$k' = \frac{1}{\bar{t}\left(1 - e^{-\frac{i}{\bar{t}}}\right)}.$$

Similarly we find

$$p_{x\gtrless 0}(x)\,dx = k'\,\frac{i-x}{i}\,e^{-\frac{i-x}{\bar{t}}}\,dx. \tag{25}$$

The mean and standard deviation of this unusual-shaped distribution of $x$ are found to be

$$\bar{x} = 0, \tag{26}$$

$$\sigma_x = \sqrt{\bar{t}}\,\sqrt{\frac{(2\bar{t}+i)e^{-\frac{i}{\bar{t}}} - 2\bar{t} + i}{1 - e^{-\frac{i}{\bar{t}}}}}. \tag{27}$$

In Fig. 15 is shown the distribution of the individual errors found by 60-second switch counts on 746 varying holding time calls (2 hours on the

Newark group). Their true average holding time was 131.45 seconds. The mean error was found to be +1.84 seconds and the standard deviation 25.55 seconds. The corresponding theoretical distribution is found to have a standard deviation of 24.56 seconds with a mean, of course, of zero. The theoretical distribution is superposed on the data of Fig. 15 and is seen to give quite a good fit.

It is interesting that the theoretical average error for each of these three widely dissimilar holding time distributions should be zero, while their standard deviations and analytical forms assume quite different characteris-

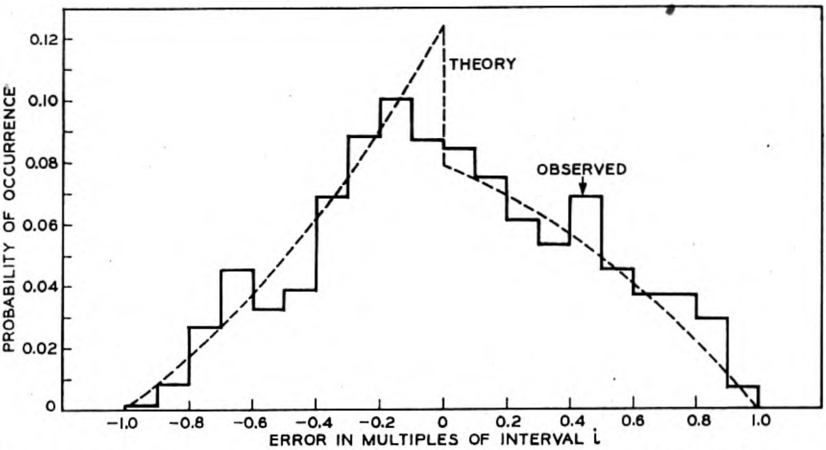



Fig. 15—Distribution of call measurement errors for exponential holding times

tics.[8] A comparison of the $\sigma$'s obtained from equations (19), (23) and (27) for a typical choice of values, $\bar{t}$ = 145 seconds, $i$ = 60 seconds, gives

$$\sigma_x \text{ constant h.t.} = 29.58 \text{ seconds},$$

$$\sigma_x \text{ ``Equally likely'' h.t.} = 24.48 \text{ seconds},$$

$$\sigma_x \text{ Exponential h.t.} = 24.03 \text{ seconds}.$$

The $\sigma_x$ constant h.t. is largely a function of whether $\bar{t}$ is closely a whole multiple of $i$; comparing the values for the other two $\sigma_x$'s indicates it is slightly advantageous that most of the variable holding time calls to be switch counted in practice are of a roughly exponential form.

*The Total Error on n Calls*

We shall now attempt to combine the errors from the three sources just discussed and formulate some general conclusions for making the most

[8] It may readily be shown that the average error will be zero for any assumed holding time distribution, by noting that each length of call therein may momentarily be segregated and considered under paragraph "a" as a constant holding time.

of the switch count method of estimating average holding times. Suppose we have completed a succession of $r + 1$ switch counts spaced uniformly by the interval $i$. The period covered will then be $ri$ units long, and we shall assume that the call count has covered exactly the same total interval so as to eliminate certain of the correction difficulties described under IV-b above. Suppose the first switch count (at the beginning of the period) showed $m$ calls up, the last switch count (at the exact end of the period) showed $w$ calls up, and the sum of all of the $r + 1$ counts, *including* the first, totalled the number $s$.

As we found in equations (5) and (6) the average correction in the number $s$ to be made on account of calls from the previous period being switch counted is

$$\bar{s}_m = \frac{m}{1 - e^{-\frac{i}{\bar{t}}}},$$

and the standard deviation in the correction here is

$$\sigma_m = \sqrt{m}\,\frac{e^{-\frac{i}{2\bar{t}}}}{1 - e^{-\frac{i}{\bar{t}}}}.$$

At the end of the hour the corresponding correction was found in equations (10) and (11) to have an average of

$$\bar{s}_w = \frac{we^{-\frac{i}{\bar{t}}}}{1 - e^{-\frac{i}{\bar{t}}}},$$

and a standard deviation of

$$\sigma_w = \sqrt{w}\,\frac{e^{-\frac{i}{2\bar{t}}}}{1 - e^{-\frac{i}{\bar{t}}}}.$$

These are quite independent corrections if the observation period is several times the average holding time, the usual case. Then our best estimate of the number of switch counts we would have obtained if all (and only) those associated with the $r$ calls originated in the period had been counted is,

$$s' = s - \bar{s}_m + \bar{s}_w, \tag{28}$$

and the standard deviation of $s'$ is

$$\sigma_{s'} = \sqrt{\sigma_m^2 + \sigma_w^2}. \tag{29}$$

We now obtain immediately a preliminary figure for the average holding time of the $n$ calls as

$$\overline{t_1'} = \frac{s' i}{n}, \tag{30}$$

and a standard deviation of this average by

$$\sigma_{\bar{t}_1'} = \frac{\sigma_{s'} i}{n}. \tag{31}$$

This last standard deviation comprehends the uncertainty in the holding time average caused by our inability to measure exactly the number of switch counts which properly should be associated with the $n$ calls. We must now modify this measure of dispersion by the added fluctuation inherent in the method of switch counting at finite intervals. These variations were found to cause no change in the "expected" or most likely value of the average holding time (as shown by $\bar{x} = 0$ in equations (18), (22) and (26)), but the $\sigma_x$'s of equations (19), (23) and (27) showed sizeable uncertainties which must be included. Since $\sigma_x$ is for errors in the measurement of individual calls we obtain the standard error of the average as

$$\sigma_{\text{avg}} = \frac{\sigma_x}{\sqrt{n}}. \tag{32}$$

This error is built up on each call throughout the period and hence is practically independent of those errors arising at the ends of the hour. Their joint effects are additive so we obtain the estimates of the final parameters of the average holding time as

$$\overline{t'} = \overline{t_1'} + 0 = \frac{(s - \bar{s}_m + \bar{s}_w)i}{n}, \tag{33}$$

$$\sigma_{\bar{t}'} = \sqrt{\sigma_{\bar{t}_1'}^2 + \sigma_{\text{avg}}^2} = \frac{1}{n}\sqrt{(\sigma_m^2 + \sigma_w^2)i^2 + n\sigma_x^2}. \tag{34}$$

If the holding times are exponential these equations may be written as

$$\overline{t'} = \frac{i}{n}\left[s + \frac{we^{-\frac{i}{\bar{t}}} - m}{1 - e^{-\frac{i}{\bar{t}}}}\right], \tag{35}$$

$$\sigma_{\bar{t}'} = \frac{1}{n\left(1 - e^{-\frac{i}{\bar{t}}}\right)} \times \sqrt{\left((m + w)i^2 + n\bar{t}(2\bar{t} + i)\left(1 - e^{-\frac{i}{\bar{t}}}\right)\right)e^{-\frac{i}{\bar{t}}} - n\bar{t}(2\bar{t} - i)\left(1 - e^{-\frac{i}{\bar{t}}}\right)}. \tag{36}$$

It may be noted that the unknown average holding time $\bar{t}$ enters into both equations (35) and (36). They are not very sensitive to this value, however, and a first approximation obtained from $\bar{t} = \frac{(s - m)i}{n}$ will usually suffice. Further refinement may be obtained by recalculating using the new value $\bar{t}'$ found from equation (35). The form of the distribution represented by the parameters of equations (35) and (36) is not known of

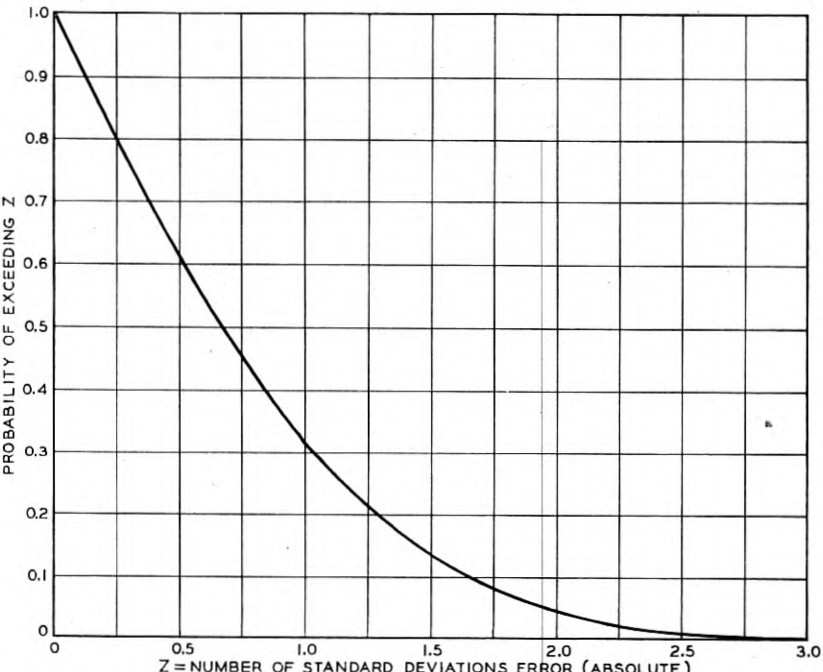



Fig. 16—Cumulative distribution of overall errors in average holding time, regardless of sign

course, but since the errors are essentially the sum of three primary error distributions which are inclined to be unimodal themselves (except for very small trunk groups), we shall probably be not far wrong to assume the normal form for $\bar{t}'$. If the magnitude only of the errors in the estimate $\bar{t}'$ of the unknown true holding time $\bar{t}$ of the $n$ calls under observation is desired, we can readily construct a distribution of these discrepancies Figure 16 is the theoretical cumulative half-normal frequency curve, and when $\sigma$ is equated to $\sigma_{\bar{t}'}$ found from equation (36), the probability of exceeding any given error in the holding time estimate may be read off directly.

*Example:* Suppose we have made 60-second switch counts on a group of

TA

SWITCH COUNT DAT

| No. | At Beginning of the Hour | | | At End of the Hour | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | Due to calls which Pass #60 Sw. Count | | Due to Calls Originating betwe #60 S.C. and End of the Hou | | |
| | No. Calls Carried Forward $m$ | Expected Switch Count Gain** $m \frac{1}{1-e^{-c}}$ | St. Dev. of Total Gain** $\sqrt{m}\frac{e^{-\frac{c}{2}}}{1-e^{-c}}$ | No. Calls Beyond #60 Count $w$ | Distance between #60 and end of Hour (min.) $j$ | Expected Loss** $w\frac{e^{-c}}{1-e^{-c}}$ | St. Dev. of Total Loss** $\sqrt{w}\frac{e^{-\frac{c}{2}}}{1-e^{-c}}$ | Avg. No. Expected $\alpha = \frac{j}{t}a$; $(a = 13.44)$ | Expected S.C. Loss on these calls (Equation 12) | St. De of Tot S.C. L (Equat 13) |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| * | 12 | 35.13 | 8.23 | 9 | .4 | 17.34 | 7.13 | 2.24 | 4.71 | 4.78 |
| 2 | 19 | 55.61 | 10.35 | 24 | .5 | 46.25 | 11.64 | 2.80 | 5.99 | 5.39 |
| 3 | 12 | 35.13 | 8.23 | 8 | 1.1 | 15.42 | 6.72 | 6.16 | 15.05 | 8.55 |
| 4 | 15 | 43.91 | 9.20 | 13 | .9 | 25.05 | 8.56 | 5.04 | 11.78 | 7.56 |
| 5 | 10 | 29.27 | 7.51 | 13 | 1.0 | 25.05 | 8.56 | 5.60 | 13.40 | 8.07 |
| 6 | 18 | 52.69 | 10.08 | 9 | 1.0 | 17.34 | 7.13 | 5.60 | 13.40 | 8.07 |
| 7 | 14 | 40.98 | 8.89 | 10 | 1.0 | 19.27 | 7.51 | 5.60 | 13.40 | 8.07 |
| 8 | 13 | 38.05 | 8.56 | 7 | .8 | 13.49 | 6.28 | 4.48 | 10.23 | 7.05 |
| 9 | 18 | 52.69 | 10.08 | 15 | .7 | 28.91 | 9.20 | 3.92 | 8.78 | 6.53 |
| 0 | 15 | 43.91 | 9.20 | 7 | .6 | 13.49 | 6.28 | 3.36 | 7.36 | 5.98 |
| 1†* | 19 | 55.61 | 10.35 | 24 | .2 | 46.25 | 11.64 | 1.12 | 2.24 | 3.30 |
| 2 | 24 | 70.26 | 11.64 | 10 | .5 | 19.27 | 7.51 | 2.80 | 5.99 | 5.39 |
| 3 | 13 | 38.05 | 8.56 | 19 | 1.0 | 36.61 | 10.35 | 5.60 | 13.40 | 8.07 |
| 4‡ | 19 | 55.61 | 10.35 | 10 | .3 | 19.27 | 7.51 | 1.68 | 3.45 | 4.09 |
| 5 | 13 | 38.05 | 8.56 | 14 | .5 | 26.98 | 8.89 | 2.80 | 5.99 | 5.39 |
| 6 | 15 | 43.91 | 9.20 | 12 | 1.1 | 23.13 | 8.23 | 6.16 | 15.05 | 8.55 |
| 7 | 21 | 61.47 | 10.88 | 11 | .8 | 21.20 | 7.88 | 4.48 | 10.23 | 7.05 |
| 8 | 12 | 35.13 | 8.23 | 14 | .75 | 26.98 | 8.89 | 4.20 | 9.50 | 6.79 |
| 9 | 13 | 38.05 | 8.56 | 8 | .6 | 15.42 | 6.72 | 3.36 | 7.36 | 5.98 |
| 0 | 12 | 35.13 | 8.23 | 21 | .5 | 40.47 | 10.88 | 2.80 | 5.99 | 5.39 |
| 1 | 14 | 40.98 | 8.89 | 16 | .5 | 30.83 | 9.50 | 2.80 | 5.99 | 5.39 |
| 2 | 17 | 49.76 | 9.79 | 14 | .94 | 26.98 | 8.89 | 5.26 | 12.43 | 7.77 |
| 3 | 12 | 35.13 | 8.23 | 13 | .53 | 25.05 | 8.56 | 2.97 | 6.41 | 5.58 |
| 4‡ | 14 | 40.98 | 8.89 | 14 | .8 | 26.98 | 8.89 | 4.48 | 10.23 | 7.05 |
| 5‡ | 18 | 52.69 | 10.08 | 9 | .47 | 17.34 | 7.13 | 2.63 | 5.61 | 5.22 |
| 6* | 18 | 52.69 | 10.08 | 6 | .8 | 11.56 | 5.82 | 4.48 | 10.23 | 7.05 |
| 7 | 13 | 38.05 | 8.56 | 13 | .53 | 25.05 | 8.56 | 2.97 | 6.41 | 5.58 |
| 8 | 14 | 40.98 | 8.89 | 16 | .3 | 30.83 | 9.50 | 1.68 | 3.45 | 4.09 |
| 9 | 14 | 40.98 | 8.89 | 19 | 1.0 | 36.61 | 10.35 | 5.60 | 13.40 | 8.07 |
| 0 | 12 | 35.13 | 8.23 | 15 | .4 | 28.91 | 9.20 | 2.24 | 4.71 | 4.78 |
| 1 | 15 | 43.91 | 9.20 | 16 | .75 | 30.83 | 9.50 | 4.20 | 9.50 | 6.79 |

es: * In these hours additional errors are present because #1 switch count did not coincide well w time 0.

† Hour No. 11 was only 59 minutes long.

‡ In each of these hours 1 pen registered call lasting for over 50 minutes was excluded as a trou condition.

** $c$ in the formulas at column headings stands for $\frac{i}{t}$, and was estimated as .418055 from $i = 60.2$ a $t = 144$.

III
ɴ 31 Hours, Newark

| Switch Counts in the Hour | | | Length of S.C. Interval $i = 60j \frac{60-j}{59}$ (seconds) | No. Calls Originated in the Hour $n$ | Estimated Average Holding Time $= \frac{s' \cdot i}{n}$ (seconds) | Std. Dev. of Avg. Holding Time due to Variation in No. Sw. Counts $\frac{(14) \cdot i}{n}$ | Std. Dev. of Avg. Holding Time due to Errors in Measuring Each Call $= \frac{\sigma_x}{\sqrt{n}}$ (Equation 27) | Total Std. Dev. of Avg. Hold. Time $= [(18)^2 + (19)^2]^{1/2}$ | Actual Measured Average Holding Time in the Hour (seconds) | Error in Estimated Holding Times in Multiples of the Theoretical Std. Dev. $\frac{(21)-(17)}{(20)}$ | Perc Err (21) − ( (21 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| No. witch ounts bs'd. $s$ | Corr. No. $s' = s -$ (3) + (7) + (10) | St. Dev. of Corr. Total S.C. $[(4)^2 + (8)^2 + (11)^2]^{1/2}$ | | | | | | | | | |
| 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 869 | 856 | 11.89 | 60.61 | 355 | 146.15 | 2.03 | 1.31 | 2.40 | 142.0 | 1.73 | 2.9 |
| 837 | 834 | 16.48 | 60.51 | 359 | 140.57 | 2.78 | 1.30 | 3.05 | 139.9 | .22 | .4 |
| 850 | 845 | 13.63 | 59.90 | 371 | 136.80 | 2.21 | 1.28 | 2.56 | 133.0 | 1.49 | 2.8 |
| 848 | 841 | 14.67 | 60.10 | 328 | 154.10 | 2.69 | 1.36 | 3.02 | 147.2 | 2.28 | 4.6 |
| 837 | 846 | 13.96 | 60.00 | 311 | 163.22 | 2.69 | 1.40 | 3.04 | 159.9 | 1.09 | 2.0 |
| 868 | 846 | 14.74 | 60.00 | 372 | 136.45 | 2.38 | 1.28 | 2.71 | 131.8 | 1.72 | 3.5 |
| 774 | 766 | 14.16 | 60.00 | 315 | 145.90 | 2.70 | 1.39 | 3.04 | 149.0 | −1.02 | 2.0 |
| 842 | 828 | 12.75 | 60.20 | 332 | 150.14 | 2.31 | 1.35 | 2.68 | 143.7 | 2.40 | 4.4 |
| 793 | 778 | 15.13 | 60.30 | 350 | 134.04 | 2.60 | 1.32 | 2.92 | 138.0 | −1.36 | 2.8 |
| 841 | 818 | 12.64 | 60.41 | 308 | 160.44 | 2.48 | 1.40 | 2.84 | 157.8 | .93 | 1.6 |
| 865 | 858 | 15.92 | 60.81 | 343 | 152.11 | 2.82 | 1.33 | 3.09 | 149.0 | 1.01 | 2.0 |
| 901 | 856 | 14.86 | 60.51 | 374 | 138.49 | 2.40 | 1.28 | 2.73 | 131.1 | 2.71 | 5.6 |
| 811 | 823 | 15.67 | 60.00 | 341 | 144.81 | 2.76 | 1.34 | 3.07 | 144.0 | .26 | .5 |
| 801 | 766 | 13.43 | 60.71 | 318 | 146.24 | 2.56 | 1.38 | 2.88 | 144.5 | .60 | 1.2 |
| 809 | 804 | 13.47 | 60.51 | 351 | 138.60 | 2.32 | 1.32 | 2.66 | 141.7 | −1.16 | 2.1 |
| 802 | 796 | 15.01 | 59.90 | 305 | 156.33 | 2.95 | 1.41 | 3.28 | 158.7 | − .72 | 1.4 |
| 839 | 809 | 15.17 | 60.20 | 337 | 144.52 | 2.71 | 1.34 | 3.02 | 141.8 | .90 | 1.9 |
| 792 | 793 | 13.88 | 60.25 | 332 | 143.91 | 2.52 | 1.35 | 2.86 | 141.7 | .77 | 1.5 |
| 319 | 804 | 12.42 | 60.41 | 313 | 155.17 | 2.40 | 1.39 | 2.76 | 151.8 | 1.22 | 2.2 |
| 780 | 791 | 14.67 | 60.51 | 311 | 153.90 | 2.85 | 1.40 | 3.16 | 157.5 | −1.14 | 2.2 |
| 305 | 801 | 14.08 | 60.51 | 380 | 127.55 | 2.24 | 1.26 | 2.56 | 128.9 | − .53 | 1.0 |
| 363 | 853 | 15.34 | 60.06 | 303 | 169.08 | 3.04 | 1.42 | 3.36 | 160.5 | 2.55 | 5.3 |
| 734 | 730 | 13.12 | 60.48 | 290 | 152.24 | 2.74 | 1.45 | 3.08 | 158.0 | −1.87 | 3.6 |
| 571 | 667 | 14.41 | 60.20 | 287 | 139.91 | 3.02 | 1.46 | 3.35 | 136.5 | 1.02 | 2.5 |
| 345 | 815 | 13.40 | 60.54 | 321 | 153.71 | 2.53 | 1.38 | 2.87 | 153.5 | − .07 | .1 |
| 348 | 817 | 13.60 | 60.20 | 338 | 145.51 | 2.42 | 1.34 | 2.77 | 145.8 | − .10 | .2 |
| 305 | 798 | 13.33 | 60.48 | 324 | 148.96 | 2.49 | 1.37 | 2.83 | 150.5 | − .55 | 1.0 |
| 766 | 759 | 13.64 | 60.71 | 316 | 145.82 | 2.62 | 1.39 | 2.94 | 150.0 | −1.42 | 2.7 |
| 755 | 764 | 15.85 | 60.00 | 340 | 134.82 | 2.80 | 1.34 | 3.11 | 137.6 | − .90 | 2.0 |
| 783 | 781 | 13.23 | 60.61 | 337 | 140.46 | 2.38 | 1.34 | 2.72 | 1[illegible]5.2 | −1.74 | 3.2 |
| 775 | 771 | 14.86 | 60.25 | 327 | 142.06 | 2.74 | 1.36 | 3.06 | 145.5 | −1.13 | 2.3 |

20 trunks for a period of one hour, finding $m = 7$ and $w = 13$ as the number of calls up at the beginning and the end of the hour, respectively. Suppose also we have a total of $s = 680$ switch counts, which includes the first count of 7 at time zero. If the register recording number of calls originated in the hour reads 282, what is the best estimate of the average holding time of the $n$ calls, and what is the probability that the true holding time is within 3 seconds of this estimate?

We find our initial estimate for $\bar{t}$ from

$$\frac{(s - m)i}{n} = \frac{(673)60}{282} = 143.19 \text{ seconds.}$$

Substituting in (35) and (36) we find

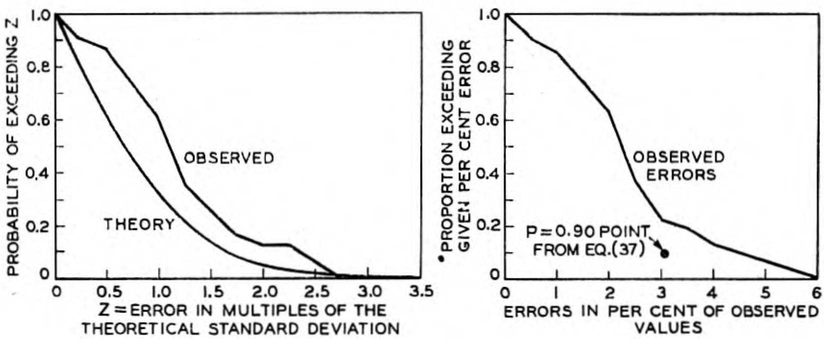
$$\bar{t}' = 145.65 \text{ seconds,}$$

$$\sigma_{\bar{t}'} = 2.685 \text{ seconds.}$$

Then from Fig. 16 we read that the probability that this estimate of $\bar{t}$ is more than 3 seconds, that is $\frac{3}{\sigma_{\bar{t}'}} = 1.12$ standard deviations, in error is .263. Likewise we may read that the probability is .94 that the error in the average is *not* over 5 seconds.

As something of a final and overall comparison of theory and observation, the actual errors in holding times when estimated by the switch count method for the 31 busy hours in Newark have been tabulated in Table III. The analysis of these pen register records was complicated by the fact that the intervals $i$ varied somewhat from switch count to switch count, and even more from hour to hour, so that the last switch count often came near the midpoint of the 60th minute. To some extent these irregularities of counting correspond more closely to the timing variations in manual switch counts than if they had been taken by machine at perfectly uniform intervals. Such corrections as could be managed by the application of equations (12) and (13) were made to the individual hours. In spite of these precautions the actual errors were somewhat larger than those which could be explained by theory although all large discrepancies were run down and accounted for. The absolute errors are shown plotted in Fig. 17a in terms of the theoretical standard deviations for each hour and in 17b in per cent of the observed holding times. This case will again serve to show that the switch count method is quite sensitive to variations from a perfect application of the rules, and that very considerable care is required to remain within the error limits indicated by the theory.

It is interesting to see what portion of the error is contributed by the two end effects and what by the errors made through "measuring" the calls by switch counts at finite intervals. Hence on Fig. 18 is shown the overall error distribution for the illustrative example given above with the distributions of its two elements as well. The method of combining standard deviations (equation (34)) explains why the two error distributions of Fig.




(a) (b)

Fig. 17—Comparison of observed and theoretical overall errors

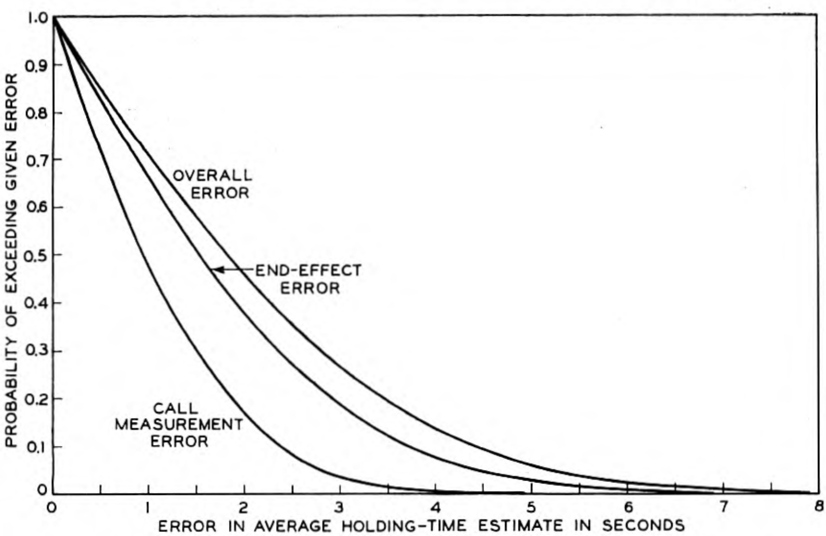



Fig. 18—Typical overall distribution of errors in estimating average holding times

18 do not add directly to give the overall error curve. We may immediately draw the important conclusion from Fig. 18 that the errors due to the uncertainties at the ends of the observation period may in certain cases be considerably greater than those due to the lack of precise measurement of the length of each individual call.

The relative prominence of these two errors will not change very rapidly with different sizes of observation periods (that is, the number of calls, $n$), since the end effects' standard deviation (equation 31) will vary inversely as $n$, while the standard deviation due to the error in measuring individual calls (equation 32) will vary inversely as the $\sqrt{n}$. Doubling the length of the observation period would then decrease the first $\sigma$ to one-half, and the second $\sigma$ to .707 of its former value.

Equation (36) shows that

$$\sigma_{\bar{t}'} = f(m, w, n, \bar{t}, i).$$

We will not know $m$ and $w$ before making the switch counts but we can probably substitute the average value of their sum which is $2n\bar{t}/T$, without seriously disturbing the average value of $\sigma_{\bar{t}'}$. This gives

$$\sigma_{\bar{t}'} \approx \frac{\sqrt{\frac{\bar{t}}{n}}}{1 - e^{-\frac{i}{\bar{t}}}} \sqrt{\frac{2i^2 e^{-\frac{i}{\bar{t}}}}{T} + \left[(2\bar{t} + i)e^{-\frac{i}{\bar{t}}} - 2\bar{t} + i\right]\left(1 - e^{-\frac{i}{\bar{t}}}\right)}, \quad (37)$$

where now $\sigma_{\bar{t}'}$ is a function of only four variables, $n$, $\bar{t}$, $i$ and $T$.

It is of interest to compare the errors predicted by equation (37) with those found by the theory formulated for the particular $m$ and $w$ observations found in each hour's observations at Newark. An average of $n$ = 332 calls per hour was observed with an average holding time in the order of $\bar{t}$ = 145 seconds. The switch count interval, $i$, was approximately 60 seconds, and the observation period was $T$ = 3600 seconds. If we take $P = .90$, the per cent error corresponding will be $\pm 100\ (1.645)\ \sigma_{\bar{t}'}/\bar{t}$. Using the above estimate of $\sigma_{\bar{t}}$, we find an error of about 3.05 per cent, or $\pm .0305$ (145) = $\pm 4.42$ seconds. This point has been plotted on Fig. 17b and bears about the same relationship to the observed errors as does the theoretical curve in Fig. 17a in which the comparison takes into account the actual calls carried beyond the start and end of each hour. The discrepancy in Fig. 17b is largely accounted for by the same discussion given heretofore for Fig. 17a.

*The Overall Error in Estimating the Average Holding Time*

The engineer who has the problem of devising a switch count schedule will want to be able to estimate at least roughly the order of accuracy he will actually obtain in the average holding time found from the data in a number of observation periods. Up to this point in this section we have concerned ourselves with discovering only the errors inherent in measuring the average length of a *particular n calls* of the exponential type in an observation period of length $T$. As we saw in section III, even when such an average length $\bar{t}$ is accurately known for the $n$ calls, it may not exactly or even closely coincide with the true average of the universe of calls of which the $n$ are presumed to form a random sample. The errors we have just studied and those described in section III must now be combined to give us a measure of the overall accuracy of the switch count method.

Equation (2), when applied to the exponential holding times we are here concerned with, gives us for the standard error of the average in a sample

$$\sigma_{\text{sampling}} = \frac{\sigma}{\sqrt{n}} \approx \frac{\bar{t}}{\sqrt{n}}. \tag{38}$$

This error is independent of that represented by $\sigma_{\bar{t}'}$, so we may determine the overall ($oa$) standard error by

$$\sigma_{oa} = \sqrt{\sigma_{\text{sampling}}^2 + \sigma_{\bar{t}'}^2}. \tag{39}$$

We shall be particularly interested in knowing how much the value of $\sigma_{\bar{t}'}$ contributes to the overall standard deviation, $\sigma_{oa}$. This may be conveniently expressed by writing the ratio

$$q = \frac{\sigma_{oa}}{\sigma_{\text{sampling}}} = \sqrt{1 + \frac{\sigma_{\bar{t}'}^2}{\sigma_{\text{sampling}}^2}}$$

$$= \sqrt{1 + \frac{\frac{2i}{\bar{t}}\frac{i}{T}e^{-\frac{i}{\bar{t}}} + \left[\left(2 + \frac{i}{\bar{t}}\right)e^{-\frac{i}{\bar{t}}} - 2 + \frac{i}{\bar{t}}\right]\left(1 - e^{-\frac{i}{\bar{t}}}\right)}{\left(1 - e^{-\frac{i}{\bar{t}}}\right)^2}}. \tag{40}$$

Now it is readily seen from (40) that $q$ depends on $\bar{t}$, $i$ and $T$. Hence if $T$ is held constant we may plot curves between $q$, $i$ and $\bar{t}$ as shown on Fig. 19. What is more, if $T$ is varied, say increased by a factor $k$, equation (40) shows that if $i$ and $\bar{t}$ are also increased by the same factor the values of $q$ resulting may still be read directly from Fig. 19.

For example: If approximately 100-second exponential calls are to be switch counted for an hour with observation intervals of 120 seconds, we read on Fig. 19 that the *overall* standard error (or $P = .50, .90, .99$, etc. error) in estimating the true average holding time is $q = 1.134$ times the basic standard error resulting from taking a random sample of $n$ calls from a very large universe of calls. That is to say, the residual sampling error present in a stop watch measurement of the $n$ calls is increased by 13.4 per cent due to our resort to switch count methods.

If a continuous period of $T = 2$ hours (i.e. $k = 2$) is switch counted in just the same manner and under the same conditions, we should now read on the $\bar{t} = 50$ seconds curve opposite $i = 60$ seconds, giving us an increase over the basic sampling error of 12.4 per cent. This meets one's common

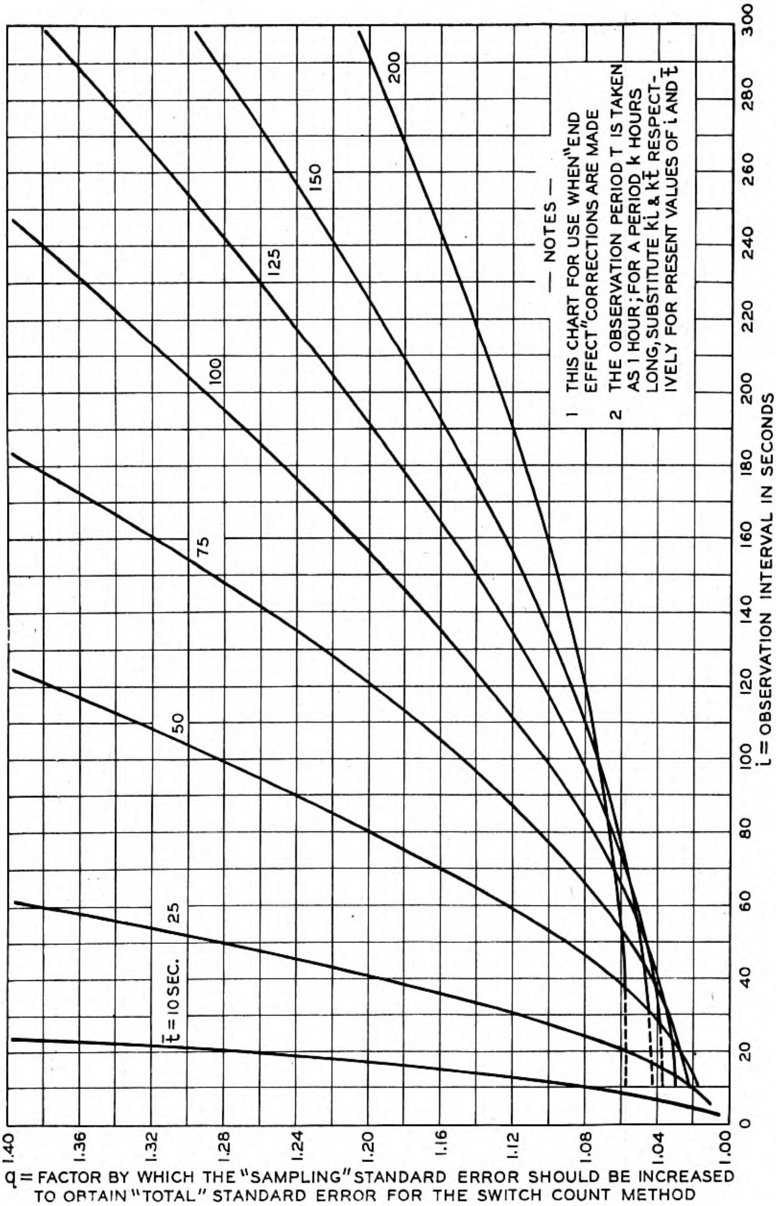



Fig. 19—Increase in "overall error" over "sampling error" in the estimate of holding time averages by switch counts (with "end effect" corrections)

sense conclusions since, as previously suggested, when the observation period is increased by a factor $k$,

a. The "sampling" error decreases as $\frac{1}{\sqrt{k}}$, since we are looking at $k$ times as many calls,

b. The error of measurement on the individual calls decreases also as $1/\sqrt{k}$, and for the same reason, and

c. The "end effect" errors are unchanged in actual magnitude but are now prorated over $k$ times as many calls; hence, the effect of this element decreases as $1/k$.

The overall effect then as $k$ increases is for $\sigma_{oa}$ to decrease faster than $\sigma_{\text{sampling}}$ as we have just seen in the example. Thus not only is the sampling error decreased by lengthening the observation period, but the overall error decreases even faster. It is clear then that an observation period as long as is consistent with a "controlled" universe of holding times is to be preferred.

Further important deductions may be drawn from a study of Fig. 19. If we wish to minimize the effect of errors introduced by a use of the switch count method we shall need to select our observation interval $i$ so that it will be relatively small compared with the average holding time. Apparently we should do well to keep $\frac{\bar{l}}{i} \geqq 1.5$; the higher this ratio the better, although the improvement beyond, say, 2.0, is slow. Roughly, for commonly observed local subscriber holding times if the holding time is at least twice the observation interval the increase in error occasioned by the switch count method over the stop watch method need not exceed 7 per cent.

Fig. 19 has been constructed for use when $\bar{l}$ is estimated as $\bar{l}'$ from equation (35),

$$\bar{l}' = \frac{(s - \bar{s}_m + \bar{s}_w)i}{n} = \frac{i}{n}\left[s + \frac{w\,e^{-\frac{i}{\bar{l}}} - m}{1 - e^{-\frac{i}{\bar{l}}}}\right], \tag{35}$$

in which $s$ is the sum of $r + 1$ switch counts (which includes counts at both the extreme ends of the period), $m$ is the count at the beginning, and $w$ the count at the end of the period.

It will be of interest to estimate something of the enlarged error when $\bar{l}'$ is found merely by taking

$$\bar{l}' = \frac{(s - m)i}{n} \tag{41}$$

which entirely neglects the special information contained in the first and last switch counts. $\left(\text{If only } r\left(= \frac{T}{i}\right)\right.$ counts are made they should be at the *end* of each $i$-interval, the last one coming at the exact end of the whole period.) This is the common case in which we merely sum all the switch counts, multiply by the counting interval and divide by the number of calls shown on the peg count meter as originating in the period $T$. The standard error for each end effect will then be approximately that given by $\sigma'_m$ in equation (6') where no attention is paid to the end switch count values of $m$ and $w$. Substituting $\sigma'_m$ for $\sigma_m$ and $\sigma_w$ in (29) and following the same analysis as before gives for $q'$ (instead of $q$),

$$q' = \sqrt{1 + \frac{\frac{2i}{\bar{l}}\frac{i}{T}\left(e^{-\frac{i}{\bar{l}}} + 1\right) + \left[\left(2 + \frac{i}{\bar{l}}\right)e^{-\frac{i}{\bar{l}}} - 2 + \frac{i}{\bar{l}}\right]\left(1 - e^{-\frac{i}{\bar{l}}}\right)}{\left(1 - e^{-\frac{i}{\bar{l}}}\right)^2}}. \quad (42)$$

A plot of this last expression is given in Fig. 20. By comparing points on Fig. 20 with corresponding ones on Fig. 19, one obtains an idea of the increase of error due to failure to correct the switch counts for the end effects as indicated in equation (35). For example, with 100-second calls switch counted at 120-second intervals we find $q = 1.134$ while $q' = 1.203$, indicating quite a marked increase in the overall error. The particular errors resulting in any given circumstance coupled with the cost of making the end effect corrections will determine the practical desirability of which method to adopt, that is whether the factor for increasing the basic sampling standard error shall be read from Fig. 19 or from Fig. 20.

Finally, a chart has been drawn up as Fig. 21, by which a measure of the overall error in estimating the unknown true holding time may readily be determined. The right hand section of the chart is a redrawing of Fig. 5 given in section III for the sampling errors of individually measured calls. Scale $A$ is carried across and reproduced at $C$ permitting the small nomograph $B$ $C$ $D$ to give easily the product of the sampling error and the $q$ (or $q'$) factor at $D$. The left hand chart is based simply on the fact that the overall error decreases inversely as the square root of the number of periods switch counted. From it the number of periods required to obtain any desired accuracy can be read.

The estimate of the average holding time will be found from a simple average of the estimates made for individual observation periods,

$$\bar{l}' = \frac{\bar{l}'_1 + \bar{l}'_2 + \cdots + \bar{l}'_g}{g}. \quad (43)$$

If a certain per cent error in the estimated average holding time is obtained for a single period the improvement for the combination of $g$ periods

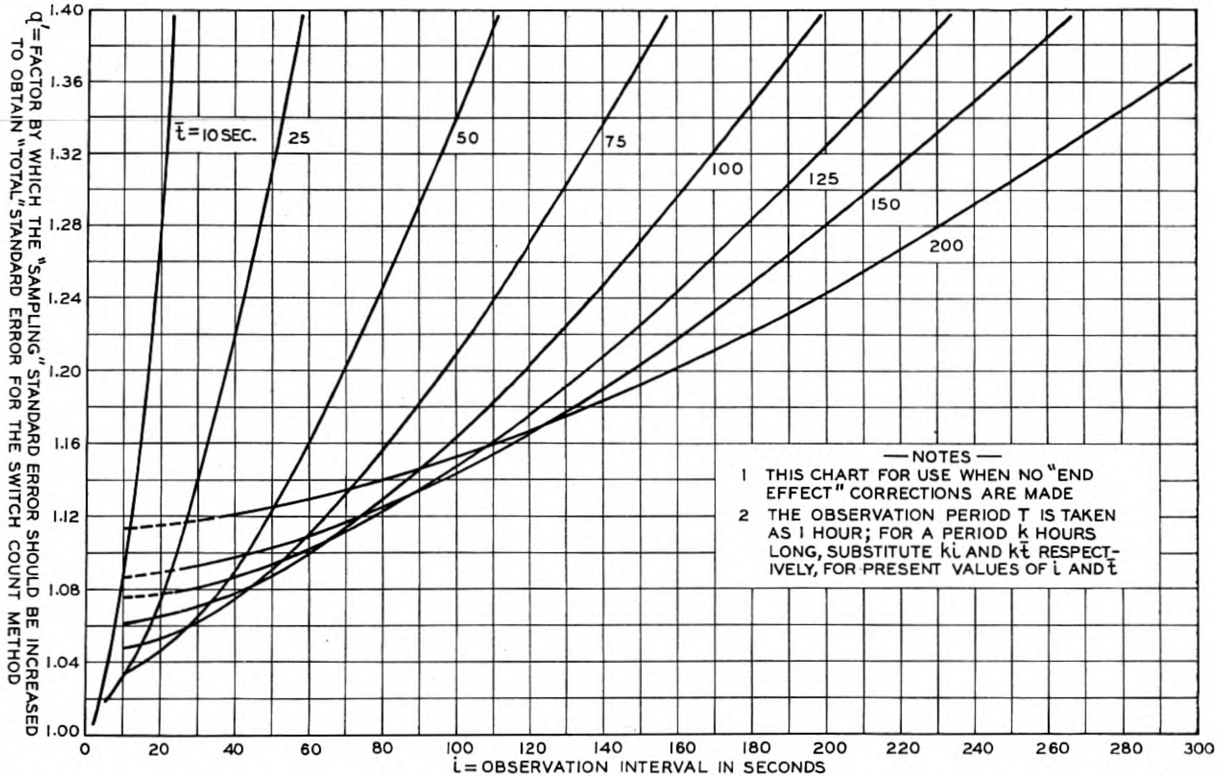



Fig. 20—Increase in "overall error" over "sampling error" in the estimate of holding time averages by switch counts (no "end effect" corrections)

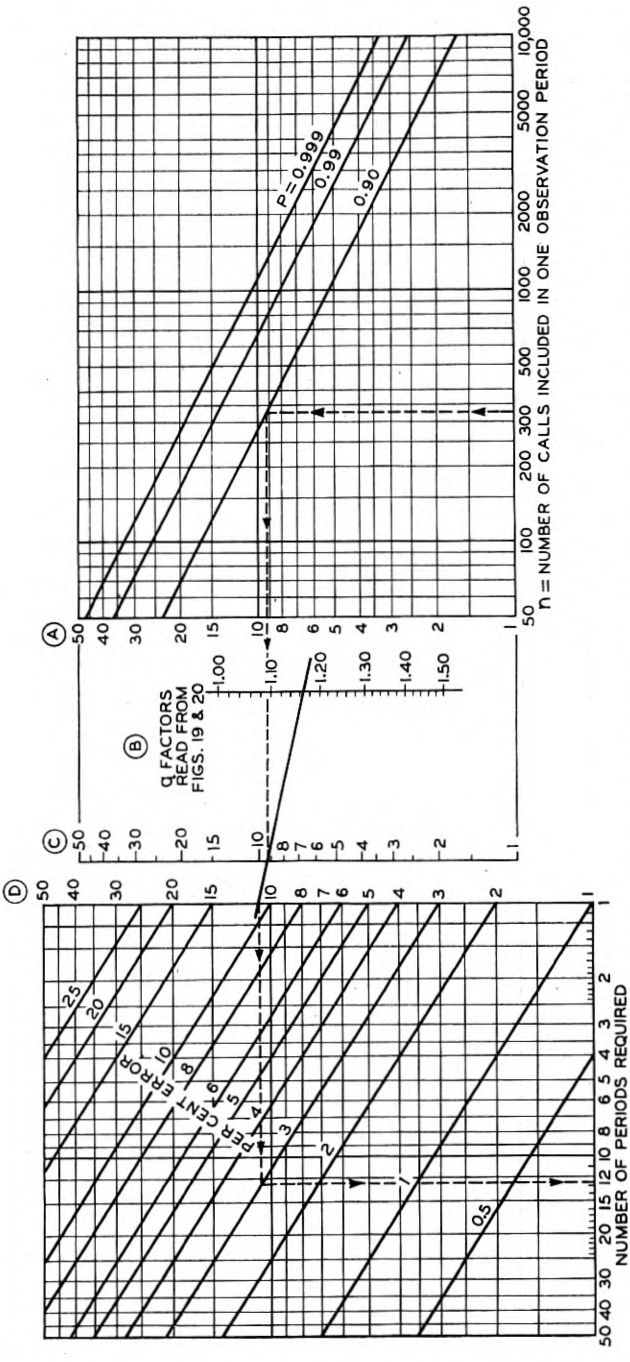



Fig. 21—Determination of number of observation periods required to produce any desired accuracy in estimating holding times by switch counts

is then,

$$\frac{1}{\sqrt{g}} \text{ (single period error).}$$

*Illustrative Example.* If calls of approximately 135 seconds holding time are to be switch counted at 3 minute intervals by 1 hour periods, how many such periods will be required to give us an assurance of $P = .90$ that the resulting holding time estimate does not differ from the true by more than 3 per cent? Assume an average of 325 calls per hour over the group, and that end effect corrections (a) will be made, and (b) will not be made.

*Solution* (a) We first read on Fig. 19 that $q \approx 1.165$. Turning to Fig. 21, we find that opposite $n = 325$ and $P = .90$ we have an error of 9.2 per cent, which we carry over to the $C$ scale of the nomograph. Laying a straight-edge across this point and $q = 1.165$ determines a point on $D$ somewhat above the 10 per cent line. Projecting this point across to the desired 3 per cent error line and dropping to the lower edge of the chart we find that 12.7, say 13, such one-hour observation periods will be necessary to meet the accuracy specifications of the problem. This is the effort required when the end effect corrections are made.

(b) If the end effect corrections are to be ignored, we determine our factor from Fig. 20 to be $q' \approx 1.268$. Proceeding on Fig. 21 exactly as before we find that the number of one-hour observation periods required is increased to 15.1, say 15. Thus a failure to make the end effect corrections causes us to increase our switch counts by about 20 per cent. It is by such comparisons as these that one decides the practical desirability of making the end effect corrections.

### *Summary of Switch Count Procedures for Exponential Holding Times*

The choice of the size of unit observation period should rest primarily on the considerations discussed in Section II, that is the homogeneity of the holding time data from hour to hour, day to day, etc. Other things being equal, we should select the longest period consistent with the view that enough periods must be included so that representative sampling of all known or suspected major variations in holding time character is accomplished. The length of the switch count interval will likely already have been decided by the equipment at hand or by other considerations. If a choice is available, however, a short interval will produce more reliable results than a long one, by the amounts indicated on Figs. 19 and 20. With these matters decided, Fig. 21 is consulted to see how many periods must be switch counted in order to obtain the desired accuracy.

Having actually made the switch counts, exercising the cautions we have mentioned, the average holding time $\bar{t}'$ for each period is obtained from equation (35) if the correction of the counts for the end effects in each period is made. If no such corrections are to be made, equation (41) is used instead of (35). The arithmetic mean, equation (43), of the values obtained in the various periods will then be the best available estimate of the unknown true average holding time. The reliability of this figure should be substantially that which the schedule was designed to produce.

*Switch Count Errors for Non-Exponential Holding Times*

If switch counts are made on calls with other than exponential holding times the resultant errors may be greater or less than those shown by Figs. 19, 20 and 21. The comparison of typical $\sigma_x$'s calculated from equations (23) and (27) would suggest that for varying holding times the error in the measurement of individual calls is not greatly dependent on the form of the holding time distribution *as long as the average call length covers several intervals* $i$. In such a case the charts developed for exponential holding times can probably be used with little allowance for the discrepancy present.

On the other hand for calls with an unusual or extreme fluctuation about an average $\bar{t}$ less than $i$, the errors due to assuming the situation to be equivalent to the exponential case may be no longer negligible. The only procedure then would appear to be either to work out the errors actually present, reverting to the basic error equations (14) and (15), and approximating the new end effect corrections, or to revise the switch counting program to materially shorten the interval $i$.

For relatively constant holding times the value of $\sigma_x$ can be reduced to a small figure by choosing the switch count interval $i$ so that it is contained in the average holding time $\bar{t}$ closely a whole number of times. Then by equation (19) the error in individual call measurements must, of necessity, be small in nearly every instance. It will be noted that the above specification permits choosing $i = \bar{t}$; moreover, it may readily be seen that in this case the end effect corrections will tend to disappear, giving a highly accurate measurement with relatively few observations. Just how many, of course, will depend on how constant is the quantity measured, and how closely the switch count interval $i$ approaches the true average $\bar{t}$.

## V—General Summary

The general problem of determining the average holding times of subscribers' or other calls by sampling methods has been discussed. The need for a proper apportionment of the sample is emphasized and examples are given from telephone experience to illustrate typical analysis procedures. Methods for estimating the reliability of these sampling results for both directly measured holding times and for switch count studies are given along with various curves and charts calculated to assist the traffic engineer in devising a working schedule for the sampling of holding times, particularly those of an exponential character.

# Electrical and Mechanical Analogies

By W. P. MASON

## Introduction

During the past few years, apparatus which transfers electrical into acoustical or mechanical energy has received wide application. This came about through the popular use of radios, phonographs, public address systems, and sound motion pictures. While the fundamental principles of such electro-mechanical or electro-acoustic transducers have been known for decades, it is safe to say that the rapid progress and excellent design obtained have been due in a large part to the knowledge derived from the related subject of electrical network theory.

Two examples may be cited to show the nature and extent of the improvement. Barton in his "Theory of Sound" (1914) cites measurements on the efficiency of acoustic foghorns operated from an electrical source of power and finds that the efficiency of conversion from electrical into acoustic energy is less than one per cent. Today large loud speakers have been developed which can be used for similar purposes and these have efficiencies of conversion greater than 50 per cent. Another and more striking example is the mechanical phonograph. From the days of its invention by Edison, mechanical phonograph reproducers had been constructed from such mechanical units as needles, diaphragms, horns, and their connecting mechanical elements. As late as 1925 the best of such units was capable of reproducing only three octaves. About this time, another mechanical phonograph[1] was constructed from the same sort of elements, but with their dimensions and relationships designed according to relations developed in electrical network theory, and the resulting structure was able to reproduce a frequency band corresponding to five octaves with greater uniformity and an increase in the efficiency of conversion.

The type of electrical network which has received the greatest application in the design of mechanical and acoustical systems is the electrical wave filter invented by Dr. G. A. Campbell. This may seen surprising at first sight, since the filter is usually regarded as a device for attenuating unwanted frequency bands while passing other frequency bands which it is desired to receive. The filter has two properties which make it of interest in electro-acoustic transducer systems. These are: first, the filter is able to coordinate

[1] Maxfield and Harrison, *Bell Sys. Tech. Jour.*, Vol. 5, No. 3, p. 493, 1926.

the action of several resonant elements to produce a device with a uniform transmission over a wide frequency range; and second, the dissipationless filter, with matched impedance terminations, is a device which delivers to its output all of the energy impressed upon it over the widest possible frequency range consistent with the elements composing it. These properties of the filter have been made use of in purely electrical networks to determine the largest band width a vacuum tube with known characteristics can have and still deliver a specified gain at a specified impedance level. Applied to electro-mechanical transducer systems, the filter theory shows how to combine resonant mechanical or electro-mechanical elements to produce a uniform conversion of electrical to mechanical energy, or vice versa, over a wide frequency range. Also, it is able to determine the greatest band width that can be obtained without loss of efficiency for any type of conversion element.

This transfer of knowledge from one branch of science, electrical network theory, to another branch of science dealing with mechanical and electro-mechanical structures is one example of a long line of such interchanges that have been going on for over a hundred years. These interchanges are made possible by the fundamental analogies which exist between electrical and mechanical systems and which rest finally on the fact that electrical motions and mechanical motions satisfy the same type of differential equations. Since such analogies have been very productive in the past and are likely to continue to be so in the future, it seems worthwhile to examine their foundation and development.

## Early Borrowings of Electrical from Mechanical Theory

The equations of motion of mechanical bodies and mechanical media were developed and studied long before the equations for electrical wave propagation were derived. Under these circumstances it is natural that attempts should have been made to explain electrical wave propagation as a mechanical phenomenon. The view that electrical actions are ultimately dynamic was one whose development in the hands of Maxwell led to notable advances in the science, and it was the view toward which most of the early authorities leaned. In support of this point of view Maxwell showed that the forces on any system of charged bodies could be attributed to a system of stresses in the medium in which they are embedded. Since magnetic energy is associated with the presence of charge in motion while electro-static energy is present for charges at rest, an identification was made between kinetic and magnetic energy and between electro-static and potential energy. Applied to a concentrated system, this point of view indicates that an inductance is the analogue of a mass, while a capacitance is the analogue of a spring.

A case for which this point of view bore useful results was the case of anomalous dispersion in optics. It was found experimentally that when light was sent through certain substances the velocity of propagation depended markedly on the frequency in the neighborhood of a certain critical frequency. Below this critical frequency the velocity decreased as the frequency approached it, going rapidly toward zero as the critical frequency was approached. Above this critical frequency the phase velocity was greater than the velocity of light in the material and gradually approached that value for high frequencies. At the critical frequency a large absorption of light occurred. This was first explained by Sellmeier as being due to some element in the medium having a resonant frequency at the critical frequency In obtaining his equations he used a mechanical model in which the resonant elements were spaced at equal intervals and excited by waves propagated by virtue of the mass and elasticity of the substance.

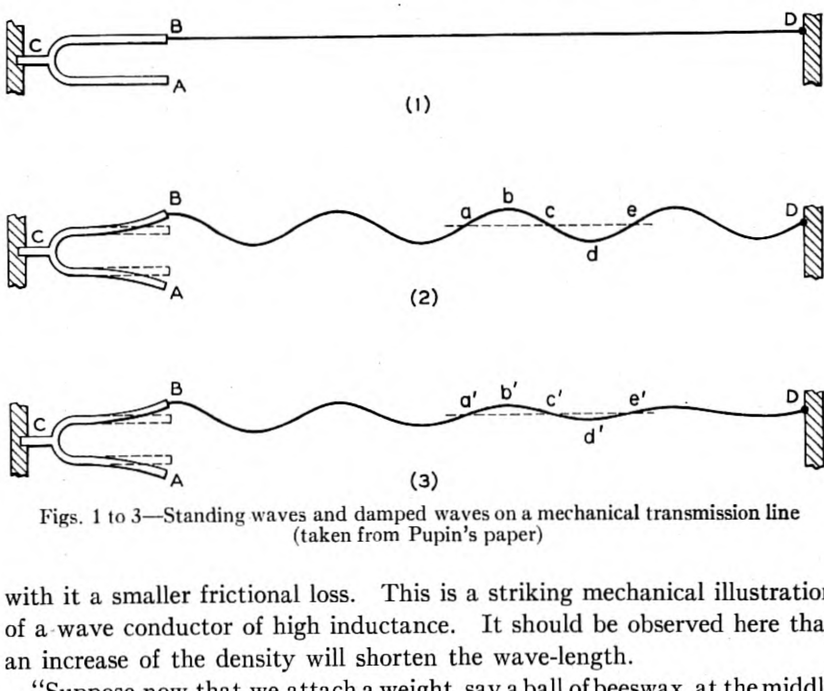
The case of greatest interest from the communication viewpoint is the influence of mechanical theory on the theory of the loaded transmission line. Wave propagation in a mechanical bar or stretched string has similar characteristics to that of a dissipationless electrical line, but when the effect of series resistances and shunt leakances were taken account of, effects appeared for the electric line which had not previously been studied in mechanical systems. These were high attenuation, which cut down the amount of power delivered to the output, and distortion, which caused the shape of the signal received at the end of the line to be different from that sent into the line. Heaviside showed that the distortion could be removed by having a certain relationship between the inductance, capacitance, resistance and leakance, and moreover that a smaller attenuation and a lower distortion would result, if an inductance were uniformly distributed along the line.

It was not a practical matter, however, to put in extra inductance at every point of the line so Heaviside suggested and tested out the effect of placing inductances at discrete points along the line, and found no beneficial results. It was not until Campbell and Pupin independently showed that the inductances had to have discrete values and be placed at definite separations that any progress was made in approaching the desired conditions.

Pupin's method of arriving at the solution is well illustrated by the following extract from his paper.[2] "The main features of the theory are extremely simple and can be explained by a simple mechanical illustration. Consider the arrangement of Fig. 1. A tuning fork has its handle C rigidly fixed. To one of its prongs is attached a flexible inextensible cord BD. One terminal of the cord is fixed at D. Let the fork vibrate steadily, the vibration being maintained electromagnetically or otherwise. The motion of the cord will be a wave motion If the frictional resistances opposing the motion

[2] "Wave Transmission Over Non-uniform Cables and Long Distance Air Lines," M. I. Pupin, *Trans. A. I. E. E.*, Vol. XVII, May 19, 1900.

of the cord are negligibly small the wave motion will be approximately that of stationary waves as in Fig. 2. The direct waves coming from the tuning fork and the reflected waves coming from the fixed point D will have nearly equal amplitudes and by their interference form approximately stationary waves. If, however, the frictional resistances are not negligibly small, then there will be dissipation of the propagated wave energy. Hence, the direct and the reflected waves will not result in stationary waves. The attenuation of the wave is represented graphically in Fig. 3. Experiment will show that, other things being equal, increased density of the string will diminish attenuation, because a larger wave requires a smaller velocity in order to store up a given quantity of kinetic energy and a smaller velocity brings




Figs. 1 to 3—Standing waves and damped waves on a mechanical transmission line (taken from Pupin's paper)

with it a smaller frictional loss. This is a striking mechanical illustration of a wave conductor of high inductance. It should be observed here that an increase of the density will shorten the wave-length.

"Suppose now that we attach a weight, say a ball of beeswax, at the middle point of the string, in order to increase the vibrating mass. This weight will become a source of reflection and less wave energy will reach the point D than before. The efficiency of transmission will be smaller now than before the weight was attached. Subdivide now the beeswax into three equal parts and place them at three equidistant points along the cord. The efficiency of wave transmission will be better now than it was when all the wax was concentrated at a single point. By subdividing still further the efficiency will be still more improved; but a point is soon reached when further subdivision produces an inappreciable improvement only. This point is reached when the cord thus loaded vibrates very nearly like a uniform cord of the same mass, tension, and frictional resistance."

Campbell, who first arrived at the design formulae for the coil loaded line, was guided by the solution, given by Lagrange over 100 years earlier, of the propagation of a wave along a string loaded with masses at discrete intervals, and a generalization of it made by Charles Godfrey,[3] for he states[4] "For the method of treatment which I first employed I am indebted to an interesting article by Mr. C. Godfrey on the 'Discontinuities of Wave Motion Along a Periodically Loaded String.'" The spacing of the coils arrived at is the same as the spacing of the massive beads along a string, namely, that $\pi$ coils or beads occur per actual wave-length of the highest frequency to be transmitted. The added result not given in the mechanical case was that the addition of such coils reduced the attenuation and decreased the distortion.

The different points of view of the two inventors are well illustrated by the quotations. Pupin was attempting to obtain a system which approached a uniform line while Campbell was investigating the propagation characteristics of the structure without particular regard as to whether the transmission was the same as that which would be provided by an equal amount of inductance uniformly distributed along the line. It was the broader point of view of Campbell which proved of wide significance and which in particular led to the invention of the electrical wave filter.

## Development of Electrical Network Theory

The first structures capable of transmitting bands of frequencies and attenuating all other frequencies were mechanical structures, although this was not generally realized at the time or made use of. The first structure of this sort was the string loaded with massive beads, which was first studied by Lagrange. Introducing the approximation that the mass of the string could be neglected, Lagrange showed that all of the natural frequencies of the device came below a certain critical frequency $f_c$. Routh[5] after discussing Lagrange's solution, points out that there may be a period of excitation of the string which is "so short that no motion of the nature of a wave is transmitted along the string."[6] An acoustic forerunner of the wave filter was the combination of two tubes of different lengths first proposed by

[3] *Phil. Mag.* 45, 456 (1898).
[4] See "Collected Papers of George A. Campbell," page 16.
[5] "Advanced Rigid Dynamics," p. 260, Paragraph 411 (1892).
[6] It should be noted that all mechanical filter theory of this time had to do with the natural resonances of unterminated filters or the transmission through misterminated sections. The idea of matched impedance terminations to introduce power into and absorb power from a filter is a development of electrical network theory.

Herschel. This structure passes low frequencies but attenuates strongly frequencies for which the difference in path length of the two tubes is an odd number of half wave-lengths.

It is interesting to note that all of the fore-runners of the filter were of the continuously distributed type which had their elements distributed uniformly along the length of the device. All such devices have an infinite number of pass bands, usually harmonically related. This is true also of the dissipationless loaded line considered as a filter structure. It was only by such abstractions as neglecting the weight of the loaded string that single pass bands were obtainable.

The low-pass electrical filter grew out of a network to simulate the operation of a long cable. By using series coils to simulate the loading and the distributed inductance of the line, condensers to simulate the distributed capacitance of the line, series resistances to simulate the ohmic resistance of the line and the resistance of the coils, and shunt resistances to simulate the leakance of the line, Campbell was able to obtain in a small space, a device which had the same propagation characteristics up to the cut-off frequency as a long section of loaded cable. Furthermore, by making the resistance small, he was able to obtain a frequency range from zero frequency up to a cut-off frequency $f_c$ with small attenuation, and a high attenuation at higher frequencies, and thus obtained the first true low-pass filter. He also put his filter to practical use for he says[7] "I have made use of these results by employing artificial loaded lines for cutting out harmonics in generator circuits. The harmonics may all be cut down as far as desired by the use of a sufficient number of sections, while the attenuation of the fundamental can be reduced at pleasure by decreasing the resistance."

Having developed the fundamental idea of a filter as a device for transmitting without loss one frequency range and attenuating all other frequencies, he went on to extend the idea to other types of filters in which different frequency ranges were passed. The band-pass filter was already realized in 1903 for Campbell says[7] "Combining condensers and inductances, we may make a system which will not only cut out higher frequencies, but also all frequencies below a certain limit." The high-pass and band elimination filters followed shortly after. With the invention of the electrical wave filter, electrical network theory can be considered as well started.

The science of electrical networks did not progress much farther for a number of years. This was due primarily to the lack of application for any of the structures developed. However, with the development of carrier current transmission on telephone lines, the necessary stimulus was received. Carrier current systems were an adaptation of radio communication systems

[7] "On Loaded Lines in Telephonic Transmission," *Philosophical Magazine*, Ser. 6, Vol. 5, pp. 313–330, March 1903.

to wire lines, with the line taking the place of the ether as the wave transmitting medium. Previously, radio systems had been developed which would transmit messages in definite frequency ranges. These transmitting ranges were selected from other frequency ranges by means of electrical tuned circuits, which were themselves a borrowing from the acoustic resonators of Helmholtz devised many years earlier. Tuned circuits are not advantageous for selecting out channels in a carrier system, because with them it is not possible to regulate the band width received or to get the necessary discrimination between the pass band and the attenuated region. It was found, however, that filters could meet these requirements and consequently they were applied in separating the channels of the first carrier systems.

This use stimulated the further development of electrical network theory. Filters with sharper discriminating properties, composite filters containing sections of like image impedances but different attenuation characteristics, transforming filters, impedance corrected filters for reducing reflections, filters using mutual inductances, attenuation and phase correcting networks are among the later developments. These investigations were carried out by a large number of individuals among whom may be mentioned Bartlett, Bode, Carson, Cauer, Foster, Fry, Guillemin, Johnson, Norton, Shea, Wagner, and Zobel. Electrical network theory has progressed to such an extent that it is now possible to select substantially any desired frequency range, with very little of the frequency range wasted in obtaining the desired selectivity, and to control the amplitude and phase of the currents received over long distance lines so that a high degree of fidelity of the received signal can be maintained.

## Borrowings of Mechanical Theory from Electrical Network Theory

While this development of electrical theory was progressing, very little development of a parallel nature was being carried out for mechanical theory, due probably to the lack of a corresponding stimulus. With the advent of the vacuum tube, public address system, and radio broadcasting, however, a demand developed for loud speakers and related equipment. It was shortly realized that the parallel developments of electrical network theory provided a base for the design of such equipment. One of the first to recognize this possibility was Professor A. G. Webster,[8] who pointed out the usefulness of the concept of impedance in mechanical systems. He applied the concept to the phonograph and developed the first theory of the action of acoustic horns. After this occurred the widespread application of the

[8] A. G. Webster, "Acoustic Impedance, and The Theory of Horns and of the Phonograph," *Nat'l. Acad. of Science*, Vol. 5, p. 275, 1919.

electrical network theory to the design of electro-mechanical systems mentioned in the introduction.

Aside from this electro-mechanical field special applications have been made in acoustic and mechanical apparatus where problems occur similar to those solved by electrical means. In all of these applications it is the filter type structure that is applied.

One of the first of these applications was the acoustic filter. In ventilating ducts, automobile, and other types of engines, and for many other uses, it is desirable to pass a steady or slowly varying stream of air, and attenuate the more rapid vibrations which constitute the undesired noise. Furthermore, it is desirable to pass the low-frequency variations with little or no loss, since such loss increases the back pressure on the engine or blower and greatly decreases their efficiency. For this purpose the low-pass filter type structure is well suited since it passes a low-frequency band with little or no attenuation and strongly suppresses higher frequency components.

The rudimentary idea of the acoustic filter probably dates back to Herschel (1833) who suggested the use of combinations of tubes capable of suppressing certain frequencies. Following the development of electrical wave filters, Professor G. W. Stewart[9] showed that combinations of tubes and resonators could be devised which would give transmission characteristics at low frequencies similar to electrical filters. This theory worked well as long as the structure was small or the frequency low, but broke down for large structures and high frequencies due to the essentially distributed nature of the elements. A theory of acoustic filters was given by the writer in 1927,[10] which took account of wave motion in the elements, and this theory could account for the properties of the filters to much higher frequencies. Since then, Lindsay[11] and his collaborators have discussed a number of acoustic type filters with various types of side branches and obstructions.

Mufflers existed long before the theory of acoustic filters was worked out but they were designed as a series of baffles, which introduced considerable back pressure on the engine. Most recently designed mufflers have a straight conducting path with side-branches in conformance with acoustic filter theory and are proportioned to attenuate most of the frequencies above 100 cycles. As a result they are considerably more effective than

[9] *Phys. Rev.* 20, 528 (1922); 23, 520 (1924); 25, 590 (1925). See also Stewart and Lindsay "Acoustics," Chap. VII. D. Van Nostrand.

[10] A Study of the Regular Combination of Acoustic Elements, with Applications to Recurrent Acoustic Filters, Tapered Acoustic Filters, and Horns. *B.S.T.J.* Vol. VI, pp. 258–294, April 1927.

[11] An excellent review and resumé of the literature on gaseous and solid acoustic filters is given by Lindsay, "The Filtration of Sound I," *Jour. App. Phys.* 9, 612 (1938); "The Filtration of Sound II," *Jour. App. Phys.* 10, 620 (1939).

early mufflers and introduce considerably less back pressure on the engine or blower.

Other uses to which mechanical filters have been put are in obtaining shockproof mountings and vibration damping devices, in obtaining vibration and noiseproof walls and floors, and in obtaining constant speed motors in which the effects of gear irregularities are removed by the use of a low-pass mechanical filter.

## Mechanical and Electro-Mechanical Counterparts of Electrical Filters

Although combinations of electrical elements were first studied and applied in wave filters and other structures, it does not follow that they have any inherent advantages over analogous combinations of mechanical or electro-mechanical elements which can be used as filters. In fact, elements which depend on mechanical motion have the great advantage that they have very little energy dissipation associated with their motion and, hence, the equivalent mechanical elements have a higher ratio of reactance to resistance, or "Q," than do their electrical counterparts. The result is that considerably more selective filters can be made from mechanical or electro-mechanical elements than can be obtained by employing electrical coils and condensers.

The first attempts[12,13] along this line were made in substituting masses for coils and springs for condensers in standard electrical filter configurations. This work resulted in usable filters up to several thousand cycles in frequency, which have been used for certain special purposes.

More recently, electro-mechanical elements have been used to take the place of some or all of the electrical elements of a filter and this work has resulted in filters with markedly superior characteristics to those obtained with filters using only electrical elements. The type of electro-mechanical element which has been used most extensively in selective filters is the piezo-electric crystal and particularly the quartz crystal. This element has the advantage of an electro-mechanical converting system in the piezo-electric effect and a very high mechanical Q. Moreover, a quartz crystal is very stable mechanically and can be cut so that its frequency changes very little over a wide temperature range. For these reasons, quartz crystals have been applied extensively when it is desirable to obtain a narrow band filter or a very selective filter.

[12] This work was carried out principally by Messers. Hartley, Lane and Wegel.

[13] The use of a mechanical filter in visually studying the properties of a wave filter is described in a paper "A Mechanical Demonstration of the Properties of Wave Filters," C. E. Lane, *S.M.P.E. Jour.* Vol. 24, pp. 206–220, March 1935.

Such filters have received a wide variety of uses. Very narrow band filters have been used in carrier systems as pilot channel filters for separating out the pilot or control frequency from the other frequencies present; in radio systems for separating the carrier frequency from the sideband frequencies; and in heterodyne sound analyzing devices for analyzing the frequencies present in industrial noises, speech, and music. Wider band filters employing coils as well as crystals have provided very selective devices which are able to separate one band of speech frequencies from another band different by only a small percentage frequency range from the desired band. This property makes it possible to space channels close together with only a small frequency separation up to a high frequency, and such filters have had a wide use in the high frequency carrier systems and in the coaxial system which transmits up to 480 conversations over one pair of conductors. In radio systems such filters have been used extensively in separating one sideband from the other in single sideband systems.

# The History of Electrical Resonance

By JULIAN BLANCHARD

OUR earliest knowledge of electricity was of the static kind; later came the voltaic cell and the direct current. But not until the discovery of alternating or oscillating currents of electricity could the phenomenon of electrical resonance make its appearance. Today, as we turn the dials of our radio receivers and "tune in" on the station we want it is recognized how widespread its application has become. Nevertheless, it seems that few have given thought to how this important principle came to light and how and when it got into common use.

## THE OSCILLATORY NATURE OF THE LEYDEN JAR DISCHARGE

The Leyden jar, discovered in 1746, was for many years one of the most important instruments in the meager equipment of electrical experimenters. When the jar was charged by an electrical machine and the discharging knobs brought close enough together a spark would jump between them. The savants of those days reasoned that this doubly coated jar was a storer of electricity, a condenser; that before the spark passed there was an accumulation of positive charge on one coating and of negative on the other; and when the spark passed these charges neutralized each other and the jar was discharged. But they did not know or suspect that this discharge was oscillatory, that first one side and then the other became positively charged, until the motion gradually came to rest.

The view that such was the case seems first to have been put forward in 1826 by Felix Savary, in France. It had been observed by him, and very likely by others as well, that a steel needle magnetized by the discharge of a Leyden jar did not in all circumstances have the same polarity. In the following words he suggested the idea that the results were due to the oscillatory discharge of the jar:

"An electric discharge is a phenomenon of movement. Is this movement a continuous translation of matter in a determined direction? Then the opposite polarity of magnetism observed at different distances from a straight conductor, or in a helix with gradually increasing discharges, would be due entirely to the mutual reactions of the magnetic particles in the steel needles. The manner in which the action of a wire changes with its length appears to me to exclude this supposition.

"Is the electric movement during the discharge, on the other hand, a series of oscillations transmitted from the wire to the surrounding medium and soon attenuated by resistances which increase rapidly with the absolute velocity of the moving particles?

"All the phenomena lead to this hypothesis which makes not only the intensity but the polarity of the magnetism depend on the laws in accordance with which the small movements die out in the wire, in the surrounding medium, and in the substance which receives and conserves the magnetism."[1]

Some fifteen years later Joseph Henry in America was experimenting with the Leyden jar and studying the currents induced in adjoining conductors by the discharge through another conductor. To determine the direction of the induced current he observed the polarity of a small steel needle magnetized by the current. In describing his experiments at a session of the American Philosophical Society he made reference to the work of Savary and stated that he had undertaken to repeat this investigator's experiments before attempting any new advances. He observed the same effect, the occasional reversal of the polarity of the needle after a discharge, and arrived at the same explanation:

"This anomaly which has remained so long unexplained, and which at first sight appears at variance with all our theoretical ideas of the connection of electricity and magnetism, was after considerable study satisfactorily referred by the author to an action of the discharge of the Leyden jar which had never before been recognized. The discharge, whatever may be its nature, is not correctly represented (employing for simplicity the theory of Franklin) by the single transfer of an imponderable fluid from one side of the jar to the other; the phenomena require us to admit *the existence of a principal discharge in one direction, and then several reflex actions backward and forward, each more feeble than the preceding, until the equilibrium is obtained.* All the facts are shown to be in accordance with this hypothesis, and a ready explanation is afforded by it of a number of phenomena which are to be found in the older works on electricity, but which have until this time remained unexplained."[2]

The published account of Henry's observations is not precisely in his own words but apparently in those of the reporter or secretary of the Society before which he spoke. It would seem from the above quotation, if it correctly represented the author, that Henry had overlooked the conclusions drawn by Savary, for they appear to be the same as his own.

This was in 1842. At a meeting of the Physical Society of Berlin in 1847 Helmholtz read his celebrated paper "On the Conservation of Force" (*Über die Erhaltung der Kraft*). Among the many illustrations of the conservation of energy principle in various branches of physics he discussed the case of the Leyden jar discharge, and incidentally noted another bit of evidence in favor of its oscillatory nature, an experiment by Wollaston in electrolysis. Commenting on the energy relations found to hold in this case he said:

"It is easy to explain this law if we assume that the discharge of a battery is not a simple motion of the electricity in one direction, but a backward and forward motion between the coatings, in oscillations which become continually smaller until the entire *vis viva* is destroyed by the sum of the resistances. The notion that the current of discharge consists of alternately opposed currents is favored by the alternately opposed magnetic actions of the same; and secondly by the phenomena observed by Wollaston while attempting to decompose water by electric shocks, that both descriptions of gases are exhibited at both electrodes."[3]

It may be interesting to note in passing that this now famous memoir by Helmholtz on the conservation of energy was considered so advanced and speculative as to be refused publication in the leading German scientific journal of the time. In it there was set forth, with far more thoroughness and generality than had been done before (by Mayer and Joule, for instance), the theorem that in any closed system the sum total of the energy is constant; a principle that at once denies the possibility of perpetual motion. It was privately published in pamphlet form in 1847. Its author, later to be recognized as the greatest German physicist of the century, was then an obscure young army surgeon, just twenty-six years of age.

By this time, it can be assumed from the foregoing, it was generally accepted by the learned in electrical science that the spark of a Leyden jar discharge was an oscillatory motion of electricity. This conclusion was arrived at as a logical and reasonable deduction from the results of various experiments, although the mode of action was still obscure. It was time now for a more analytical examination of the subject, and this was soon to appear.

In 1853 the British physicist Sir William Thomson, afterwards Lord Kelvin, published a paper with the title "On Transient Electric Currents,"[4] which, like that of Helmholtz, became in time a classic. In this paper the generalized problem of the discharge of a condenser through a conductor was treated mathematically. In addition to resistance and capacity he recognized the effect of inductance (called by him the "electrodynamic capacity") upon the discharge, and established an equation expressing the fact that the energy of the charged condenser at any instant during discharge is partly being dissipated as heat and partly conserved as current energy in the circuit; his equation being, in present day terminology,

$$-\frac{d}{dt}\left(\frac{1}{2}\frac{q^2}{C}\right) = \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + Ri^2,$$

or

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0,$$

an equation that is easily solved. He analyzed the various solutions, which depend upon the relative values of the constants, or their ratios, and showed that under certain conditions the discharge is unidirectional and under others it is oscillatory, but damped. This beautiful bit of mathematical analysis, exact and thorough as it was, and clarifying the entire phenomenon, passed almost unnoticed at the time; but it came into its own with the arrival of wireless telegraphy.

There followed a few years later a direct experimental verification of the theory of the oscillatory nature of the Leyden jar discharge. In 1858 Feddersen[5] examined the spark by means of a revolving mirror, and extended his researches during the following year by the use of photography. There was thus obtained visual evidence of the reversal of direction of the discharge, and it was even possible to determine the frequency; photographs of these oscillatory sparks were sent to Thomson, who had suggested in his paper this very possibility of proof. Other experimenters followed with variations of this method of investigation, and in 1890 Boys[6] improved upon it by photographing the spark by means of a series of rapidly revolving lenses. Shortly before this a very important discovery had been made in connection with the spark discharge, Hertz's discovery of electric waves, and as a consequence more physicists were turning to a study of its characteristics, chiefly with the aid of photography.

## The Effect of Capacity in an Alternating Current Circuit

It will be noted that the foregoing account is concerned with the very rapid, and transient, motion of electricity in open circuits. While knowledge of this sort of electric current was being advanced, Faraday's (and Henry's) discoveries in electromagnetic induction had made possible the invention of the dynamo and the production of a sustained alternating current—ordinarily of much slower motion. This generator, at first in the form of the feeble magneto, was for a long time not much more than a toy, and experience continued to be limited largely to the direct "galvanic" current. When eventually alternating currents began to be employed to an appreciable extent, in experiment and in industry, there were some new phenomena encountered, and those less theoretically grounded were slow to realize the peculiar effect of a condenser in the circuit, although the choking effect of an inductance alone was easily apparent.

The name of the great genius Maxwell now comes into our history. As we have seen, Lord Kelvin was the first to give a mathematical treatment of the oscillations of a Leyden jar discharge. So Clerk Maxwell was the first to publish an analysis of the effect of capacity in a circuit containing inductance and resistance and an impressed alternating electromotive force, and to show the conditions for resonance. The way in which he came to solve this problem makes an interesting story, and it was told in a characteristically interesting manner by the late Professor Pupin in the course of a discussion at a meeting of the A.I.E.E. Some preliminaries to this story may first be related.

For long the Ruhmkorff induction coil and the magneto-electric machine had been familiar objects in physical laboratories. In 1866 there appeared a description of Henry Wilde's striking experiments in which he virtually reinvented and introduced the separately excited dynamo, passing the small (commutated) current from a magneto-electric generator through the field magnet windings of another machine, from the armature of which a very much larger current was obtained. Sir William Grove, reading of these experiments, had the idea that a magneto might also be used advantageously to operate a Ruhmkorff coil with an alternating current. Induction coils had always been excited by means of a battery with a self-acting circuit breaker to interrupt the primary current, and in order to prevent sparking at the contacts and to stop the current more abruptly a condenser was connected across the contact terminals. Grove screwed up and kept closed the contact breaker, thus short-circuiting the condenser, and applied an ordinary medical magneto-electric machine to the primary terminals of his induction coil. To his surprise he found that he could get no secondary discharge at all; but by holding open the contact breaker, and so putting the condenser permanently in series with the primary coil and the armature of the magneto-electric machine, he obtained sparks nearly a third of an inch in length between the ends of the secondary. He saw that the effect was dependent upon the presence of the condenser in the circuit; "But why there should be no effect, or an appreciable one, when the primary circuit is completed, the current being alternated by the rotation of the coils of the magneto-electric machine, I cannot satisfactorily explain," he said.[7]

And now Professor Pupin's story:

". . . Maxwell, I think, was the first to show the effect of introducing a condenser into an alternating current circuit, and it is very interesting to observe this circumstance. Maxwell was spending an evening with Sir William Grove who was then engaged in experiments on vacuum tube discharges. He used an induction coil for this purpose, and found that if he put a condenser in parallel [it was in series, rather] with the primary circuit of his induction coil he could get very much larger sparks, which meant, of course, that he got a very much larger current through his primary coil, an alternating current generator being used to feed the primary. He could not see why. Maxwell, at that time, was a young man. That was about 1865, if I do not err. [It was 1868.] Grove knew that Maxwell was a splendid mathematician, and that he also had mastered the science of electricity as very few men had, especially the theoretical part of it, and so he thought he would ask this young man how it was possible to obtain such powerful currents in the primary circuit by adding a condenser. Maxwell, who had not had very much experience in experimental electricity at that time, was at a loss. But he spent that night in working over his problem, and the next morning he wrote a letter to Sir William Grove explaining the whole theory of the condenser in multiple [series] connection with a coil. It is wonderful what a genius can do in one night! He pointed out the exact relations between the condenser, the self-induction and the frequency which would give the largest current, and he was the first to do this, so far as I know . . ."[8]

Maxwell's letter, which began with the sentence, "Since our conversation yesterday on your experiment on magneto-electric induction, I have considered it mathematically, and now send you the result," was dated March 27, 1868; it was sent by Grove to the Philosophical Magazine, where it was published in May. Preliminary to the mathematical treatment Maxwell gave in this letter an unusually clear exposition of the analogy existing between certain electrical and mechanical effects, and from the standpoint of pedagogy as well as physics it will be interesting to see the language he used. He expressed himself thus:

"The machine produces in the primary wire an alternating electromotive force, which we may compare to a mechanical force alternately pushing and pulling at a body.

"The resistance of the primary wire we may compare to the effect of a viscous fluid in which the body is made to move backwards and forwards.

"The electromagnetic coil, on account of its self-induction, resists the starting and stopping of the current, just as the mass of a large boat resists the efforts of a man trying to move it backwards and forwards.

"The condenser resists the accumulation of electricity on its surface, just as a railway buffer resists the motion of a carriage towards a fixed obstacle."[9]

Using such concepts as these he gave a simple and lucid explanation of the problem without resort to mathematics; and then in a postscript, or appendix, he gave the mathematical theory of the experiment, employing a schematic diagram of the apparatus. Using different, but equivalent, symbols, he derived and solved the now familiar expression for the current $i$ in such a circuit,

$$E \sin \omega t = L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt$$

This is recognizable as similar to that set up by Lord Kelvin for the discharge of a condenser $\left( E \text{ being zero in that case, and } i \text{ being equal to } \frac{dq}{dt} \right)$.

The solution of this equation is

$$i = \frac{E}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}},$$

from which Maxwell pointed out that the current would be a maximum when $\omega L = \frac{1}{\omega C}$ ($\omega$ being proportional to the frequency and $L$ and $C$ being the inductance and capacity, all in their proper units). Effecting this condition would of course be a case of "electrical resonance", brought about by electrical "tuning", though Maxwell himself did not specifically make use of these terms.

It is apparent, therefore, that all the knowledge necessary for the realization of electrical tuning was available in 1868 when this little communication of Maxwell's was published. Yet it was some years before electrical resonance was encountered to any extent in practice, or any practical use deliberately made of it. It was something known about by advanced physicists, perhaps, but outside the ken of most of those experimenting with electricity at that period; it must be remembered that there was no profession of electrical engineering as early as this. Not until 1884 do we find any other published discussion of the effect of a condenser in an alternating current circuit. This was in a technical paper by John Hopkinson, Cambridge trained physicist and later to become one of the foremost electrical engineers of his day. The paper had to do with alternating current theory and the operation of alternating current machines, and in it occurs the following:

> "Some time ago Dr. Muirhead told me that the effect of an alternating-current machine could be increased by connecting it to a condenser. This is not difficult to explain: it is a case of resonance analogous to those which are so familiar in the theory of sound and in many other branches of physics.
>
> "Take the simplest case, though some others are as easy to treat. Imagine an alternating-current machine with its terminals connected to a condenser; it is required to find the amplitude of oscillation of potential between the two sides of the condenser. . ."[10]

By setting up an equation similar to that used by Maxwell the required expression was found; and by assuming certain reasonable values of the frequency, resistance, inductance and capacity, he calculated that the amplitude of the potential difference across the condenser might be many times the voltage of the generator. It is apparent from the language he used that he had a perfectly clear understanding of electrical resonance.

## The First Electrical Resonance Curve

Following Maxwell, there was another brilliant young physicist destined to become famous who showed a thorough acquaintance with electrical resonance and who made good use of it in his celebrated researches. This was Heinrich Hertz, in Germany, who applied it in the detection of electric waves produced by a spark discharge, the oscillatory nature of which had already been well investigated, as we have seen. The simple device he used for exploring the field in the vicinity of the discharge was a rectangle or circle of wire containing a minute spark gap, the loop being of such dimensions as to be in resonance with his high frequency oscillator. It was by this careful exploration that Hertz demonstrated, for the first time, the existence of electromagnetic waves in space. In the first of his series of papers describing these experiments, "On Very Rapid Electric Oscillations," published in 1887, he devotes one section to a discussion of "Resonance Phenomena." An extract from this will show how he was thinking:

"But it seemed to me that the existence of such oscillations might be proved by showing, if possible, symphonic relations between the mutually reacting circuits. According to the principle of resonance, a regularly alternating current must (other things being similar) act with much stronger inductive effect upon a circuit having the same period of oscillation than upon one of only slightly different period. If, therefore, we allow two circuits, which may be assumed to have approximately the same period of vibration, to react on one another, and if we vary continuously the capacity or coefficient of self-induction of one of them, the resonance should show that for certain values of these quantities the induction is perceptibly stronger than for neighbouring values on either side."[11]

A series of experiments along these lines demonstrated the effect conclusively. In the secondary circuit the length of spark that could be obtained across the adjustable gap increased to a maximum when the two circuits were in tune. The first electrical resonance curve ever published is given in the above mentioned paper,[12] a relation between the length of wire in the detecting loop and the greatest length of spark obtainable for each length of wire, all other conditions remaining unchanged. The curve shows the familiar sharp peak at the point of resonance. In all his succeeding researches on electric waves Hertz used this simple tuned circuit as a detector. It was the forerunner of the resonance type of wave-meter to be used later in the yet unborn art of radio.

Among the prominent British physicists Oliver Lodge at this time was also experimenting with electrical resonance and writing and lecturing about it. He had from the first taken a keen interest in the work of Hertz and in fact came close to anticipating Hertz in the discovery of electric waves through his notable work on lightning conductors. In a brief article published in 1890 he described a method, which he had used a year before in a lecture, of displaying the spark producing power of electric radiation by tuning the circuit of one Leyden jar to that of another containing a spark gap and excited in the usual way.[13] When the secondary circuit was in resonance with the first its Leyden jar would "overflow." But Lodge objected to the use of the term "resonance" and preferred the term "syntony"; "the name 'resonance' is too suggestive of some acoustic reverberation phenomenon to be very expressive," he maintained.[14] Although he and some of the other English writers continued to say "syntony" and "syntonic", this terminology did not permanently stick.

During the next decade, as electrical engineering developed somewhat, especially in alternating currents, we find more attention being paid to this subject, both by physicists and engineers. Among those interesting them-

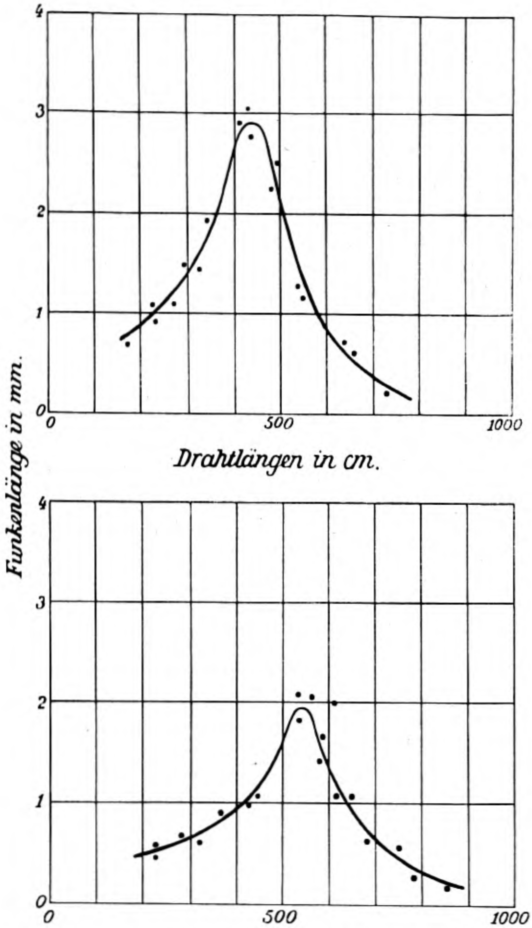



Fig. 1—The first electrical resonance curves published, by Hertz, 1887; showing the greatest length of spark obtainable in his detecting loop for various lengths of wire in the loop.

selves in the matter may be mentioned such men as T. H. Blakesley, Gisbert Kapp, J. A. Fleming, R. T. Glazebrook, James Swinburne, Maurice Hutin and Maurice Leblanc, Frederick Bedell and A. C. Crehore, Nikola Tesla, M. I. Pupin, and John Stone Stone; some of these being concerned with high frequencies, some with low. As indicating the general state of knowledge at this time concerning alternating current theory, a statement in a textbook by Blakesley, with preface dated May, 1889, is illuminating. This author says:

"It is often taken for granted that the simple form of Ohm's Law, total E.M.F. ÷ total resistance = total current, is true for alternating currents. That is to say, the E.M.F. employed in the formula is taken to be the sum of the impressed E.M.F.'s alone. That there are causes which modify the value of the current as deduced from this simple equation, such as mutual or self-induction, or the action of condensers, is often acknowledged in textbooks, and the values and laws of variation of the current are correctly stated for certain cases of instantaneous contact and breaking of circuit. But the effect of an alternating E.M.F. upon a circuit affected by self-induction, mutual induction, and condensing action, has not been, so far as I know, put into a tangible working form."[15]

Somewhat similar observations were expressed by Kapp in an article in the *Electrician* a year or so later. Referring to the paper by Hopkinson mentioned above, he commented as follows:

". . . he showed that with a certain capacity, periodic time, self-induction and resistance in circuit, the potential difference between the plates of the condenser may be 80 times the E.M.F. of the alternator. Startling as such a result must naturally appear, it failed at the time to attract much attention from practical engineers who, no doubt, preoccupied with the problems relating to continuous-current work, were content to let such an intricate and apparently abstruse problem lie at rest until such time as its consideration should be forced upon them. This time has now come, and what in 1884 was merely an interesting laboratory experiment, having no further application than perhaps the breaking down of a condenser, is at present an interesting practical problem, which the electrical engineer has to face. Phenomena arising from the effects of capacity in alternate-current circuits are forcing those who have to do with such circuits to give attention to the problems connected with the phenomena."[16]

This quotation gives a fair picture of the situation with respect to electrical engineering around 1890. Familiarity with capacity and resonant effects, it appears, was beginning to grow with the enlargement of professional experience.

## Resonance in Electric Communication

While resonance, or an approximation thereto, is occasionally encountered in ordinary power engineering and electric lighting, here it is generally a case of something to be avoided, evidence of something gone wrong. An unintentional resonant condition in a power circuit could result in considerable damage due to excessive current flow. In electric communication, on the other hand, where frequencies are higher, and where *frequency* itself is one of the fundamental elements, and currents comparatively small, resonance is of prime importance and may be of great practical value.

As we consider the use of resonance in electric communication there may occur to some readers a recollection of a very early system of multiplex signaling known as the "harmonic telegraph", representing the attempts of Elisha Gray, Alexander Graham Bell, E. Mercadier and others to transmit simultaneously a number of telegraph messages over the same line; experiments which, in the case of Bell, led to the invention of the telephone. These various schemes, however, were all based on the principle of mechanical resonance; electromagnetically driven tuned reeds at the receiving end were set to vibrating by signaling currents generated by corresponding reeds at the transmitting end. The principle of electrical resonance was not involved in such methods.

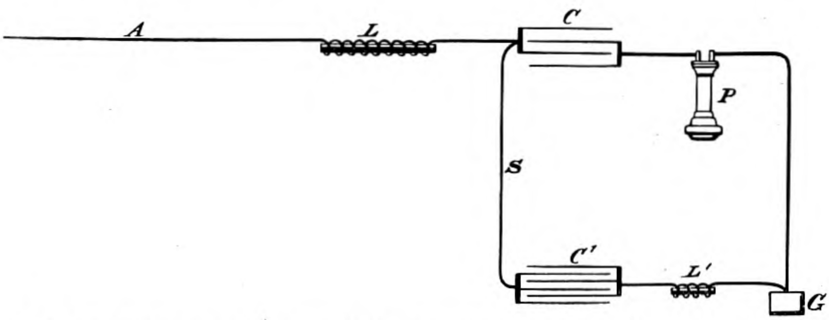



Fig. 2—Combination of series and shunt resonant elements to lessen interference of power source low frequencies with higher telephone frequencies; from U. S. patent of Stanley and Kelly, 1891.

Suggestions for the practical application of electrical resonance began to appear in the early 1890's. By this time, as our history shows, the phenomenon was generally understood by the technically trained and the well informed; it was one of the facts of science open to all. Henceforth, progress in the putting to use of it was largely in the hands of inventors and its history is to be found in the study of patents.

In telephony, one of the earliest proposals is illustrated by a United States patent issued to Stanley and Kelly in 1891,[17] showing methods for preventing interference with telephone currents by lower frequency currents induced in the line by power sources. One of the methods described was the insertion in series with the receiver of a capacity making a combination resonant to the mean speech frequency, supplemented by a shunt combination of capacity and inductance resonant to the interfering frequency. It need hardly be said that such an arrangement, favoring only a narrow band of the speech frequencies, would greatly promote distortion and would find little favor with telephone engineers.

Another application, for a different purpose, appeared in a French patent issued to Hutin and Leblanc during the same year.[18] These engineers were pioneers in the attempt at multiplex telephony by means of high-frequency carrier currents, a method now so greatly extended. In May, 1891, several months before their patent papers were filed, they had reviewed in a French journal the theory of resonance in an inductively coupled circuit in the course of a general article on alternating currents, and had briefly suggested therein its application to multiplex signaling.[19] The scheme disclosed in their patent comprised the transmission over the line of a number of super-audible frequencies, now called carrier currents, the modulation of each by a

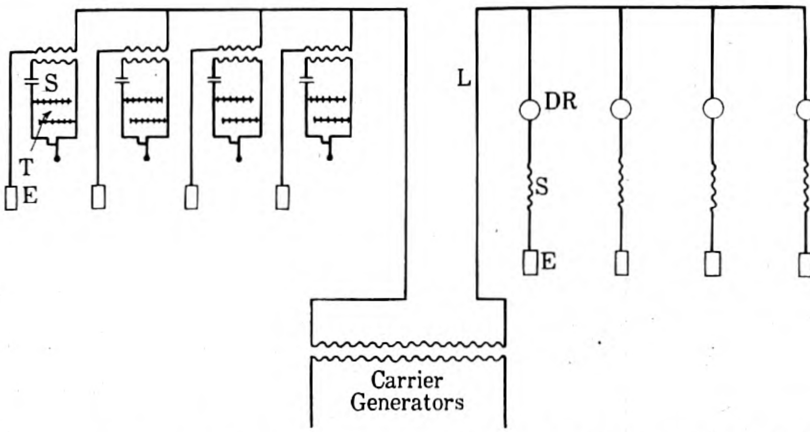



Fig. 3—Multiplex carrier telephone circuit of Hutin and Leblanc, as in their French patent of 1891. At each terminal are shown four branches, each of these branch circuits being tuned to one of the carrier frequencies.

separate telephone transmitter, and means for separating and detecting the individual messages. At both the transmitting and receiving ends, in their plan, there were branches from the connecting line, each of these branch circuits being tuned by means of a capacity which balanced the inductance in the circuit, so that each responded by resonance to its own carrier frequency to the exclusion of the others.* Here was a substitution of electrical resonance for the mechanical resonance of the older harmonic telegraph.

At about this same time Professor Pupin in his early research work at Columbia had developed a method of analyzing a complex current by picking out its components in an inductively coupled resonant circuit—a

* While their patent drawing fails to show tuning condensers in the receiving branches, without which the scheme would be inoperative, and the description on this point is vague, in the interference cases that later developed in the U. S. Patent Office (reference 24) it was claimed that this omission was an oversight. In a later French patent, No. 234,785, granted March 5, 1894, as in their U. S. patent, No. 838,545, this defect was corrected.

tool, or technique, that is now quite familiar in electrical laboratories. He made use of this in the study of the harmonics generated in a circuit by the magnetic reactions of an iron core upon the magnetizing current, an effect that had been observed and for the first time correctly explained by Rowland at Johns Hopkins, and a description of his method was published in 1893.[20] The following year, yielding to the suggestions of his scientific friends, according to the account he has written ("I often regretted it, because it involved me in a most expensive and otherwise annoying legal contest"[21]), he made application for a patent on "Multiple Telegraphy", applying this idea of selecting by resonance to the problem of separating the signals.[22] Very soon afterwards another inventor, John Stone Stone, appeared upon the scene with practically the same idea,[23] and interference cases thereupon resulted in the U. S. Patent Office and the courts involving these two and the French inventors Hutin and Leblanc, who had also filed in the United States.[24] Upon the claims of the contestants and the differences that characterized their schemes for multiplex signaling we need not dwell; suffice it to say that in the matter of priority Pupin was adjudged the winner.* It appears that this distinguished scientist was not unimpressed with what he considered the originality of his ideas about the practical use of resonance. In the inimitable story of his life, "From Immigrant to Inventor," he refers to this as "my invention of electrical tuning,"[25] and says again, "I called it electrical tuning, a term which has been generally adopted in wireless telegraphy."[25] In another place, and on another occasion, he said, "It was badly needed and I had it developed several years before Marconi had made his invention. . ."[26]

Before passing to other applications in the field of electric communication, chiefly in the radio art, it might be said that in these early proposals for multiplex operation the separation of the carrier frequencies could not be successfully achieved by so simple a means as an ordinary resonant circuit. For one thing, the distortion introduced would be prohibitive, unless the carrier channels were placed so far apart as to be uneconomic. It remained for the Campbell band filter, invented about twenty years later, to enable the frequencies to be squeezed close together and distortion and other difficulties to be overcome. Furthermore, the whole art had to wait for the invention of the vacuum tube as the perfect generator of the kind of currents required, as well as modulator, amplifier and demodulator of these currents.

Later developments in the intricate and complex technique of wire

* An examination of the report of the interference hearings (reference 24) shows that Pupin claimed to have conceived the idea of using electrical resonance in multiplex telegraphy in the summer of 1890, following a careful study he had made of the investigations of Hertz, and to have begun experimental work on it in October of that year, thus antedating Hutin and Leblanc. Upon the adjudication of this contest in favor of Pupin on the main issues, patents on some of their claims were also allowed Stone and the French inventors.

communication have brought forth more useful applications of the principle of electrical resonance than the examples cited above. But let us now turn to radio. Here the application of resonance is elemental and fundamental. But not so in the beginning, however. When Marconi brought his embryonic outfit to England in 1896 and demonstrated his best accomplishments over the next three or four years, the problem of selectivity was non-existent. Further, the type of detector then available, the Branly

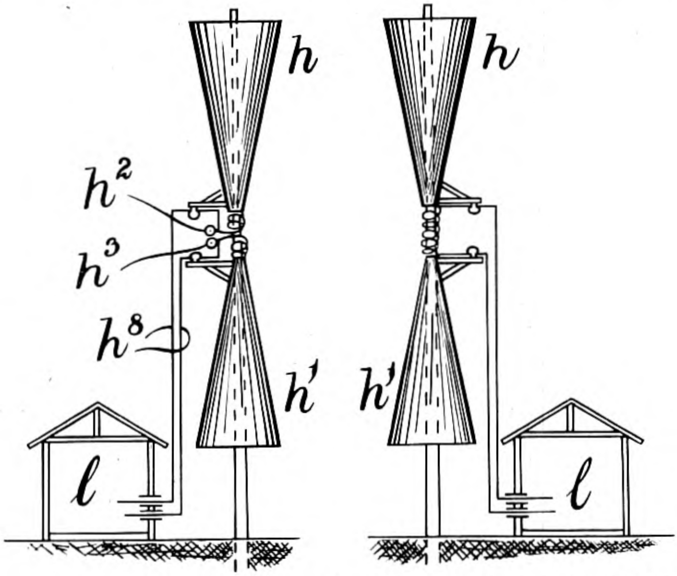



Fig. 4—First tuned radio transmitting and receiving antennas, as proposed in Lodge's British patent of 1897; tuning accomplished by inductance coils between the capacity areas *h* and *h'*.

metal filings coherer or modifications thereof, was responsive to electric waves varying considerably in frequency, so the need of tuning to obtain sensitivity was at first not actually imperative. This detector was connected directly in the untuned antenna circuit. Then the ambition to increase the distance of reception led to a search for greater sensitivity, and as a first step (1898) Marconi introduced into the receiving circuit his "jigger", or oscillation transformer.[27] The primary, of few turns, was in the antenna circuit; the tuned secondary, wound with an eye to the reduction of capacity, stepped up the voltage and applied it to the coherer. Here no adjustable tuning was provided, but instead there were different jiggers wound to suit the transmitted wave-lengths employed and thus secure the maximum effect.

It was foreseen in the early days of radio that if it were ever to become a commercial practicability it would be necessary to provide means for receiving one wave-length to the exclusion of others—to provide selectivity. Crookes in his prophetic Fortnightly Review article of 1892 had clearly envisaged this.[28] As a solution of this problem Lodge in 1897 applied for a British patent on "Improvements in Syntonized Telegraphy without Line Wires,"[29] the stated object of his invention being "to enable an operator by means of what is known as Hertzian wave telegraphy to transmit messages

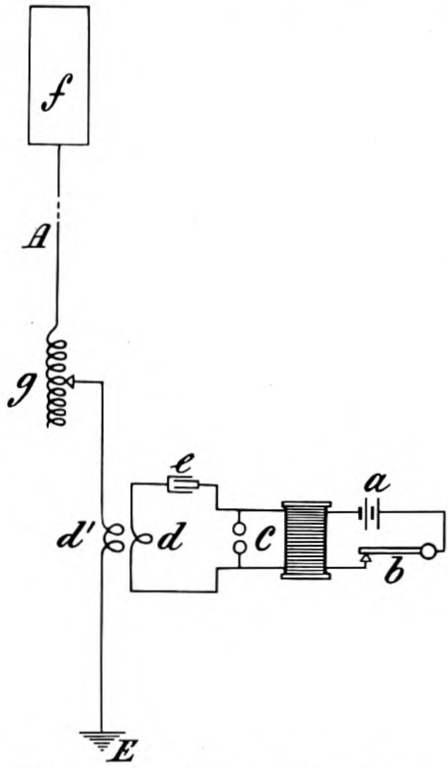



Fig. 5—The tuned inductively coupled two-circuit radio transmitter adopted by Marconi in 1900.

across space to any selected one or more of a number of different individuals in various localities each of whom is provided with a suitably arranged receiver." His radiator, modeled after Hertz, was a pair of "capacity areas", or triangular shaped metal plates (one of them preferably grounded), separated by a spark gap and having interposed an inductance coil of a few turns, for the purpose of tuning. This coil was not continuously adjustable but was to be replaced by others for changes of wave-length. The receiving station was provided with a similar arrangement except that in place of the spark gap there was connected a Branly type coherer as a wave detector.

While the particular forms of apparatus shown were never adopted in practice, nevertheless Lodge's tuning patent was upheld as valid in a legal contest for priority later on (1911) and it was purchased by the Marconi

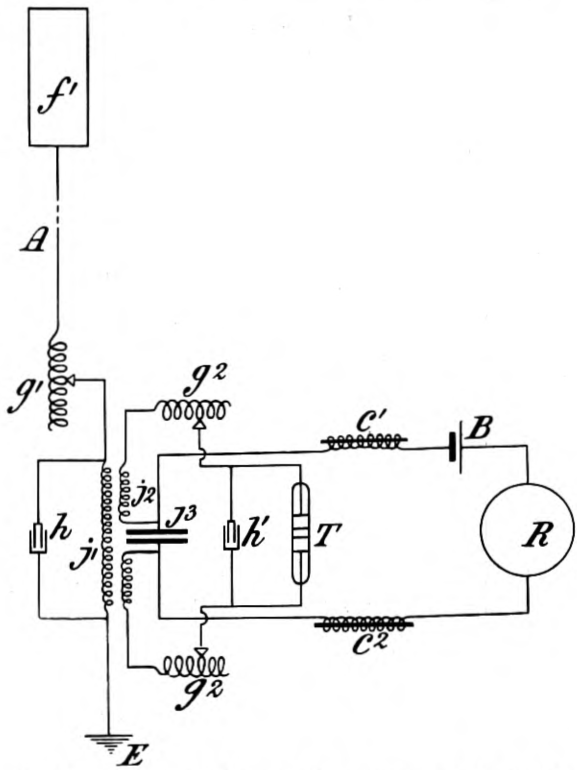



Fig. 6—Marconi's tuned radio receiving antenna and circuit of 1900.

Company as a result of the litigation.[30] Lodge, in his autobiography "Past Years", speaking with the usual modesty of the inventor rather than the scientist that he was, says, "Real selective tuning became possible through my patent of 1897. . ."[31]

In Marconi's first patent on wireless telegraphy, applied for in 1896 (British patent No. 12,039, the very first patent in radio), there was no reference whatever to the matter of tuning. But his well-famed patent No. 7777 of 1900[32] was primarily concerned with this object and went much ahead of his jigger ideas of 1898. Here there were four tuned circuits. At the sending station the spark gap circuit was inductively coupled to the transmitting aerial (an improvement credited to Ferdinand Braun of Germany), and by means of a variable condenser in the former and a variable inductance in the latter these two circuits could be tuned and brought into resonance with each other. This accomplished the production of a much more persistent train of oscillations in the aerial and a more efficient radiation of energy. At the receiving end the aerial was tuned to the incoming waves by means of a variable inductance, and the inductively coupled detector circuit was in turn tuned to resonance, likewise by means of a variable inductance. It was partly through such steps in the realization of greater sensitivity as well as selectivity that Marconi eventually succeeded with transoceanic telegraphy.*

In this patent the inventor gave the specifications for nine different *tunes*, as he called the different frequencies intended for different stations, or for different distances; that is to say, the details of design of the aerials, transformers, inductances and capacities of the transmitting and receiving circuits for each tune. Thus interference between one station and another might be avoided by using different frequencies. It may be observed here, however, that the matter of selectivity was not so easy at that time when the rather broad-spectrum spark transmitter was the only kind available. Very sharp tuning had to wait upon the advent of continuous waves, supplied first by the Poulsen arc or the high-frequency alternator and then by the vacuum tube. But many other improvements, and new wonders besides, were waiting on the vacuum tube.

It hardly seems necessary to pursue our subject further than this point, considering how it so quickly thereafter became a commonplace item in our electrical storehouse. Our interest was chiefly in how it got started. We have seen how it had its roots in certain experiments with the Leyden jar; how the results of experiments were clarified by mathematical analysis and a correct theory formulated; and then, as the need and opportunity arose, how the principle was applied and made use of by inventive minds: the wilderness first entered, then surveyed, and at last inhabited. So it is, we find, with most new ideas in the scientific world.

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# A Rapid Visual Test for the Quantitative Determination of Small Concentrations of Calcium in Lead*

By EARLE E. SCHUMACHER and G. M. BOUTON

A method is described for estimating calcium in lead which consists in casting a test ingot in a prescribed manner and comparing its surface appearance with the surface appearance of standards. The calcium content can be determined by inspection.

IN THE manufacture of lead-calcium sheath it is desirable to control the calcium content to 0.028 ± .005 per cent in order to obtain the most desirable combination of properties. Since calcium is a very active element chemically, special manufacturing procedures were developed to minimize the contact of molten lead-calcium alloy with the air. Despite the improved techniques, some calcium is always lost and must be replaced. Before this can be accomplished satisfactorily, obviously, the calcium content of the alloy must be determined. Conventional chemical procedures are accurate but not entirely satisfactory for plant control use because they are time consuming and too costly. The best of the chemical methods introduces a lag of at least two hours in melting kettle control. Quantitative spectrographic analysis methods were carefully tested, and while they showed some advantage over conventional methods, they were still unduly time consuming.

With the ever increasing interest in lead-calcium alloys for cable sheath, storage-battery grids, and other applications, it became desirable that a rapid, reliable and not too costly method be developed for determining their calcium content. In approaching the problem, several methods of attack involving physical, chemical, or electrical properties suggested themselves. A few of the methods investigated were:

1. Observations of the rate of oxidation or tarnish of freshly cut surfaces using a variety of atmospheres and temperatures.
2. Thermal EMF measurements against pure lead.
3. Measurements of hardness or strength of samples after various heat treatments.
4. Measurements of electrode potentials in various solutions.
5. Use of various metallographic techniques.
6. Observation of recrystallization tendencies after the samples had been deformed.

* This article is being published in *Metals and Alloys*.

Unfortunately, none of these methods proved adequate. Either the properties involved were insufficiently sensitive to changes in calcium content, or other factors masked the effect of calcium.

Early in the study of lead-calcium it was noticed that the molten alloys quickly filmed over with oxide. Careful observation of the characteristics of these molten alloys revealed no phenomena that varied sufficiently with calcium content to serve as a clue to the composition. However, when these alloys were chill cast with as little agitation as possible, the surface of the ingots became progressively duller with increasing calcium content to a certain value. Further increase in calcium content resulted in the fissuring of the surface oxide leaving bright metallic areas exposed. This fissuring phenomenon, which is illustrated in Fig. 1, is the type of composition-sensitive indicator desired. When samples of lead-calcium are melted and cast under controlled conditions the surface markings are reproduced with considerable fidelity in respect to areas of dull and bright surface. The ratio of these areas is dependent on the calcium content. For clearness of illustration the samples were photographed under lighting conditions that made the bright highly reflecting surfaces appear dark in the photograph.

The success of the method is dependent to a large extent on the details of procedure that are given below. Since calcium is readily removed from lead by oxidation, a melting and casting procedure for the test specimens was adopted that resulted in a minimum loss of calcium. Fluxes and inert atmospheres, which normally provide adequate protection against oxidation, could not be used here since they seriously interfere with the fissuring phenomenon that is the basis of the method. The means finally adopted consists in melting a strip of the cable sheath to be tested in a hemispherical sheet iron crucible about two inches in diameter. A Bunsen burner flame of sufficient intensity to melt a 100-gram sample in about two minutes is applied to the bottom of the crucible. The bottom edge of the sample melts first and the balance of the sample slides smoothly into the pool of metal first formed with a minimum of rupture of the surface. The broad round shape of the crucible permits it to be tilted until the lip is but a fraction of an inch from the surface of the mold before the metal starts to pour, thus subjecting the stream of molten alloy to only a brief exposure to the atmosphere during pouring. The molten alloy should never be stirred nor should the crucible be shaken unnecessarily during the casting operation. Under the melting conditions described, the casting temperature of the melt is controlled sufficiently if the crucible is removed from the flame three or four seconds after the last portion of the sample has melted. By slight modifications in technique, samples for analysis may be taken directly from the commercial melting kettles. The mold used is probably

Per Cent Calcium

.021 .024 .026 .028 .030

Fig. 1—Surface appearance of test ingots of chemical lead-calcium alloys cast in atmosphere containing 0.02% carbon dioxide and having 50% relative humidity. The dark areas are very bright when viewed with directly reflected light. Approximately full size

not critical in shape or size. The one used in this development is an iron plate $\frac{3}{4}$ in. x 4 in. x 8 in. with a tapered depression milled in its surface. The test ingot is about 4 in. long, $\frac{5}{8}$ in. wide, $\frac{1}{32}$ in. thick at the casting end and $\frac{3}{8}$ in. thick at the other end. It weighs 80 to 100 grams. The effect of mold temperature has been studied and is not critical in the range from room temperature to that reached by the mold as a result of casting into it at intervals of a few minutes.

Control of the atmosphere over the surface has been found necessary to insure reproducible results. For this reason the cellophane enclosed casting chamber shown in Fig. 2 was devised. There is a door on the right hand side for insertion of the crucible and a rubber inlet tube on the left for entry of the gas mixture. It has been found by experiment that both moisture and the carbon dioxide content of the air influence the results obtained. Therefore, both are removed chemically and then re-introduced

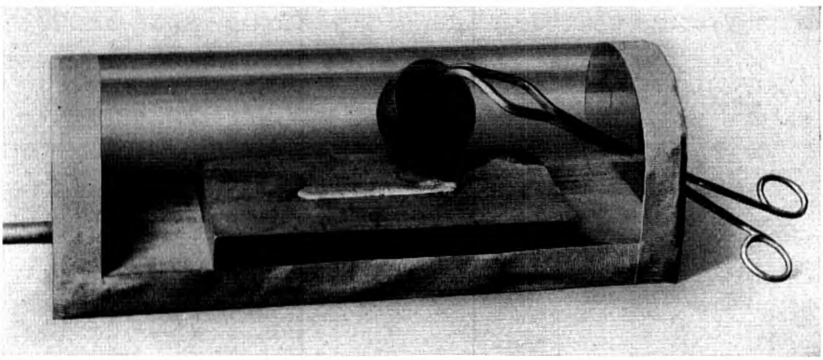

Fig. 2—Apparatus used in casting lead-calcium ingots under controlled atmospheres

in metered quantities to the air passing into the casting chamber. By means of appropriate flow meters, dry air could be mixed in definite proportions with air saturated with moisture to produce the desired humidity. Another and perhaps simpler system is to pass the dry air over certain salt solutions of known vapor pressure. Carbon dioxide is conveniently made available by placing solid carbon dioxide (dry ice) in a Dewar flask having a stopper with two exit tubes. By means of an escape valve on one tube any desired $CO_2$ pressure can be built up in the flask to force the gas through a flow meter and into the air line leading to the casting chamber. One satisfactory arrangement of apparatus for controlling the composition of the atmosphere is shown in Fig. 3.

The surface appearances shown in Fig. 1 were obtained by casting in an atmosphere having 50 per cent relative humidity and 0.02 per cent carbon dioxide. Increasing the carbon dioxide or decreasing the moisture content causes the surfaces to become brighter and vice versa. This provides a few thousandths of a per cent latitude in adjusting the sensitive range of the method to the median calcium content desired. Alloys having calcium contents outside the range of the method may be estimated by admixture with known amounts of lead-calcium alloys of known calcium content. Pure lead, in general, cannot be used for dilution because the oxide and possibly traces of impurities present in it cause the loss of some calcium from the mixture.

To date most experience with the use of the visual test for calcium has been on alloys made with chemical lead. This grade of lead is substantially free from As, Sn, Bi, Fe, Sb and Zn, and contains about 0.004 per cent Ni, 0.06 per cent Cu and 0.007 per cent Ag. The variation in concentration of these elements in the commercial supply has not been found to be great enough to cause serious interference with the indications given by the casting test. However, the method is influenced by certain variations in impurity content which are in excess of those normally encountered in the

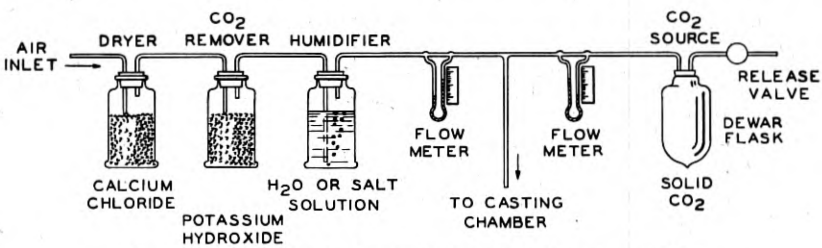


Fig. 3—Apparatus used for regulating atmosphere composition

usual supply. The presence of tin in the order of a few thousandths of one per cent causes low indications of calcium content. The use of high purity lead in place of "chemical" lead has a similar effect. Bismuth additions up to 0.1 per cent are inconsequential. Arsenic and antimony form compounds with calcium which dross off in the mixing kettle so that these elements would not be found in the finished sheath. The effects of the interfering impurities referred to above can be corrected for, when their presence is known, by varying the atmosphere in the casting chamber or by admixture with a known proportion of alloy having a pre-determined higher calcium content. By these procedures fissuring can be made to occur in alloys that otherwise do not give this manifestation. In practical operation the brand of lead being used will be known and the necessary adjustments can be made in the conditions of the test.

Sufficient analytical data have been collected to establish the fact that the method here presented is rapid, reliable, and extremely helpful for plant control application.

## Abstracts of Technical Articles by Bell System Authors

*Electron Microscopes and their Uses.*[1] JOSEPH A. BECKER and ARTHUR J. AHEARN. Three and a half centuries have passed since Zacharias Janssen, a spectacles maker of Middleburg, Holland, put two lenses in a six-feet-long tube and thereby made the first known compound microscope. In the years since then, the microscope, now grown into a powerful and intricate instrument, has played an important role in the discovery of much of man's knowledge of the physical world. There is, however, much that the microscope has been unable to reveal because of its limited range of useful magnification. Today a new type of magnifying instrument, the electron microscope, is extending the range of useful magnification far beyond its old limits and promises to supplement the traditional microscope in many fields of scientific research. In this article the authors describe types of electron microscopes, tell how they function, and outline how they are being used in physics, chemistry, metallurgy and the biological sciences. A number of pictures are shown to illustrate these uses.

*Recent Developments in Protective Metallic Coatings.*[2] R. M. BURNS. The prevention of corrosion is accomplished by two general methods: (1) the provision of a non-corrosive environment, and (2) the interposition of a protective film to exclude the corrosive environment from the metal. As an example of the first method one may cite the de-aeration of boiler feed waters and air conditioning in which moisture is controlled and dust, sulphur gases, etc. eliminated.

Referring to the second method, corrosion protective films may be divided into two main classes: the first consisting of those films formed naturally through the production of corrosion products on the surface of the metal to give a thin protective coating; and second, comprising films of paints, varnish, ceramic products or metals which themselves develop protective films. One natural type of protective film is the chemical conversion coating produced by various treatments, such as phosphate or chromate dipping or anodic oxidation.

Zinc is the most important of metallic coatings, 45% of the metal consumed in the United States being used in this manner. Hot galvanized coatings on steel have been improved by suitable pre-treatment of the

[1] *The Scientific Monthly*, October 1941.
[2] *The Monthly Review of the American Electroplaters' Society*, September 1941.

steel surface, such as results from an alternate oxidation and reduction and by the addition of small amounts of aluminum to the zinc bath.

Electroplated zinc deposits have the advantage of being applicable in greater thicknesses than hot-dipped coatings. Electroplating methods have made considerable progress, particularly in the wire field, with the speeding up of plating rates as much as twenty-five fold.

Bright zinc coatings have been developed in response to the demand for improved appearance and this finish is gradually replacing the older dull type.

The protective value of zinc depends directly upon the thickness of the coating. Experiments have listed environments in the order of increasing attack as follows: rural, tropic marine, temperate marine, suburban, urban and highly industrial. The resistance of zinc coatings to corrosion under water depends largely upon the degree of circulation of the water and its oxygen content. When a submerged zinc-coated armored cable is lapped with jute, thereby stagnating the water, the capacity of zinc to resist corrosion is increased.

Cadmium plate has good color and is very satisfactory for indoor use. It does not possess corrosion resistance equal to zinc under conditions of outdoor exposure. Bright nickel coatings or semi-bright coatings requiring mild buffing have largely replaced the older type of nickel coatings.

A very promising process of protecting steel, known as "Corronizing" consists in the application of a layer of nickel plate followed by either zinc or tin. The duplex coating is heated to 700–1000°F. yielding alloys practically free from pores which show high resistance to the salt spray test.

Seventy per cent of the production of tin plate is used in cans. The hot-dipped process is old and well established but is being challenged by continuous rolling processes involving electroplating methods of application.

Recent progress in the protection of metals by coatings of other metals is largely in the direction of electroplating and continuous processes.

*Measurements of the Delay and Direction of Arrival of Echoes from Near-By Short-Wave Transmitters.*[3] C. F. Edwards and Karl G. Jansky. Observations on pulses radiated by a high-power beam transmitter operating in the short-wave range show that when the receiver is located within the skip zone, echoes are observed having delays of from 1 to 50 milliseconds. These echoes are the result of scattering and three different types may be recognized, each arising from a different source.

Echoes of the multiple type were found to occur the most frequently and to have many of the characteristics of signals transmitted over long

[3] *Proc. I.R.E.*, June 1941.

distances. Components were observed from regions up to 4000 miles distant. Direction-of-arrival measurements using steerable arrays operating on the musa principle indicate that these multiple echoes are scattered from regions along the transmitted beam. Vertical angle-of-arrival measurements using a musa receiving system indicate that the surface of the earth may be the source of scattering.

Similarities between multiple echoes and southerly deviated waves from European transmitters have been found which indicate that the same phenomena may be responsible for both.

*Evolution by Design.*[4] REGINALD L. JONES. The evolution of the telephone plant is characterized by planning and invention. This article describes the Bell Telephone Laboratories' method of developing new and improved telephone apparatus by integrating the creative efforts of technicians in various fields of research and engineering. For telephone readers the recent development histories of telephone drop wire, relay, and station receiver are chosen to illustrate the problems encountered. Other apparatus, some of which is shown by the figures, follows a similar development pattern. New materials, improved processes of manufacture, economy in maintenance, and a better understanding of convenient use—all play continuing parts in the forward march of telephone design.

*Television Transmission.*[5] M. E. STRIEBY and C. L. WEIS. Experiments in the transmission of television signals over wire lines have been made from time to time as the television art has developed. The present paper discusses experiments made during the summer of 1940 with 441-line, 30-frame interlaced signals transmitted over coaxial cable and other telephone facilities. Some of the general problems of wire transmission have been included. In particular, the results of transmission studies on a system linking New York and Philadelphia are reported.

[4] *Bell Telephone Magazine*, August 1941.
[5] *Proc. I.R.E.*, July 1941.

## Contributors to this Issue

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