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PROCEEDINGS

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TECHNICAL PAPERS AND DISCUSSIONS

EDITED BY
ALFRED N. GOLDSMITH, Ph.D.

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ANNUAL AWARD OF A MEDAL OF HONOR

By the Board of Direction of the Institute

By vote of the Board of Direction at the February 15, 1917, meeting, there will be awarded annually a Medal of Honor of The Institute of Radio Engineers under the conditions set forth below. The medal, which has been designed by the well-known sculptor, Mr. Edward Field Sanford, Junior, of New York, has on its face a symbolic representation of electromagnetic waves and the words: "Institute of Radio Engineers." On the reverse are a laurel wreath and the words: "To ———— In Recognition of Distinguished Service In Radio Communication ———— (date)."

The medal will be awarded annually by vote of the Board of Direction of the Institute at its April meeting to that person, who, during the previous year (January thru December) shall have made public the greatest advance in the science or art of radio communication. The advance may be an unpatented or patented invention, but it must be completely and adequately described in a scientific or engineering publication of recognized standing and must be in actual (tho not necessarily commercial) operation. However, preference will be given to widely used and widely useful inventions. The advance may also be a scientific analysis or explanation or hitherto unexplained phenomena of distinct importance to the radio art, tho the application thereof need not be immediate. Preference will be given analyses directly applicable in the art. In this case, also, publication must be in full and approved form. The advance may further be a new system of traffic regulation or control; a new system of administration of radio companies or of the radio service of steamship, railroad, or other companies; a legislative program beneficial to the radio art; or any portion of the operating or regulating features of the art. It must be publicly described in clear and approved form, and must, in general, be actually adopted in practice.

The method of awarding the medal is as follows:

1. At least 30 days before the April meeting, the Board of Direction shall call, from a number of Members and Fellows

of the Institute whom it may choose to consult, for suggested candidates.

2. At the April meeting of the Board, those actually present or voting by mail shall nominate at least one, but not more than three candidates for the award, in order of preference. The names of these candidates shall then be sent to each member of the Board, and each member of the Board shall have the privilege of returning a vote for one candidate. Four weeks after the April meeting, the ballots shall be read, and the candidate receiving the most votes shall be awarded the medal.

3. The official presentation of the medal to the successful candidate or his representative shall be at the May or June meeting immediately following. The person awarded the medal shall be privileged to indicate this fact in giving his titles and honors in the fashion customary in learned and artistic societies: thus—Mr. William Jones, E.E., Medal (or Award) of Honor, Institute of Radio Engineers, 191–.

For 1917, the medal was awarded to Mr. (now Captain) Edwin H. Armstrong in recognition of his work and publications dealing with the action of the oscillating and non-oscillating audion. For 1918, in view of the limitation of publication brought about by war conditions, no award was made.

THE MORRIS LIEBMANN MEMORIAL PRIZE

The Board of Direction of The Institute of Radio Engineers, at its regular meeting held on February 5, 1919, accepted a gift of $10,000 to The Institute of Radio Engineers, from an anonymous donor, himself a friend and member of the Institute, to "preserve the memory of our late friend and fellow member, Colonel Morris N. Liebmann, who has sacrificed his life in the cause of our country."

The principal of this fund will be preserved in perpetuity and the annual income derived therefrom only will be expended. The present amount of this income is $425.00 per annum, and is to be awarded each year on the first day of October (beginning October 1, 1919), by a special committee appointed annually by the Board of Direction, to that member of the Institute, who, in the opinion of this committee, shall have made the most important contribution to the radio art during the preceding calendar year.

This annual award is to be known as the "Morris Liebmann Memorial Prize" and it is hoped will act as an additional incentive to the further rapid development of radio communication.

THEORY AND OPERATING CHARACTERISTICS OF THE THERMIONIC AMPLIFIER*

By

H. J. van der Bijl, M.A., Ph.D.

(Western Electric Company, Incorporated, New York City)

I. Introduction

The three-electrode thermionic tube has been responsible for a great deal of the recent rather remarkable developments in the art of radio communication. In its most commonly known form it consists of an evacuated vessel containing a hot filament cathode, an anode placed at a convenient distance from the cathode and a third electrode in the form of a grid placed between cathode and anode. To discuss in detail the theory of operation of the device in its various applications, such as oscillation generator, radio detector, and amplifier would be beyond the scope of the present paper. What I intend to give here is merely its fundamental principles of operation, with particular reference to its application as an amplifier. The framework of this theory was worked out in the winter of 1913-14 and formed the basis of a considerable amount of research and development work that has since been done in this laboratory on the device and its various applications.

A condition which is assumed in the elaboration of the views expressed in the following is that the operation of the device is independent of any gas ionization, or in other words, that the current is carried almost entirely by the electrons emitted from the hot cathode. It is, of course, to be understood that it is at present impossible completely to eliminate ionization by collision of the electrons emitted from the cathode with the residual gas molecules. But the condition assumed can always be realized practically by evacuating the tube to such an extent that the number of positive ions formed by collision ionization is always small compared with the number of electrons moving from cathode to anode. This happens when the mean free path of the electrons in the residual gas becomes large compared with

* Received by the Editor, October 28, 1918.

the dimensions of the device. The pressure necessary for this is not very low, and were it not for the gases occluded in the electrodes and walls of the vessel, it would be a comparatively simple matter to make the tube operate independently of gas ionization. The energy liberated by the electrons striking the anode, however, usually causes a sufficient rise in the temperature of the device to liberate enough gas to increase the pressure unduly.[1]

This is especially marked in the case of tubes handling large amounts of power. It is, therefore, necessary to denude the electrodes and walls of the tube of gases during the process of evacuation. Furthermore, since the energy liberated at the anode increases with the applied voltage, it is seen that this voltage must be kept within limits depending upon the degree of evacuation obtained. This is very important when using the device as a telephone relay, as was recognized by Dr. Arnold of this laboratory in the early stages of his experiments with this type of device. As is well known to workers in this field, it is difficult to keep a discharge steady and reproducible when ionization by collision is appreciable, and steadiness and reproductibility are conditions which must be complied with by a telephone relay.

The success of the tubes developed by the Western Electric Company is mainly due to the extensive study that has been made of the bearing of the structural parameters of the device on its operation. It is hardly possible to meet the requirements of efficiency and satisfactory operation of the device without an explicit mathematical formulation of its operation. A satisfactory telephone relay must, for example, do more than merely utilize the direct current power in its local circuit to amplify alternating current power: it must faithfully reproduce the incoming speech currents, it must also be capable of handling sufficient power, and have a definite impedance that can conveniently be made to fit the impedance of the telephone line. Since all these conditions depend on the structural parameters of the amplifier, they will not be satisfied unless the amplifier be properly designed, and so much distortion may be produced as to make the device worthless as a telephone relay. On the other hand, it has been found that by properly designing the amplifier the above-named requirements can be met very satisfactorily.

[1] For a fuller explanation of the effect of gas, see H. J. van der Bijl, "Phys. Rev.", (2), 12, page 174, 1918.

II. Current-Voltage Characteristics of Simple Thermionic Devices

We shall not here enter into a discussion of the extensive investigations that have been carried out on thermionics, but merely, for the purpose of elucidation, touch upon those phases of the subject which have a direct bearing on the theory of operation of the thermionic amplifier.

Consider a structure consisting of a heated cathode and an anode, and contained in a vessel which is evacuated to such an extent that the residual gas does not play any part in the current convection from cathode to anode. The number of electrons emitted from the cathode is a function of its temperature. If all the electrons emitted from the cathode pass to the anode, the relation between the resulting current I and cathode temperature T is given by a curve of the nature shown in Figure 1. This curve is obtained provided the voltage between anode and cathode is always high enough to drag all the electrons to the anode as fast as they are emitted from the cathode; that is, I in Figure 1 represents the saturation current. The saturation

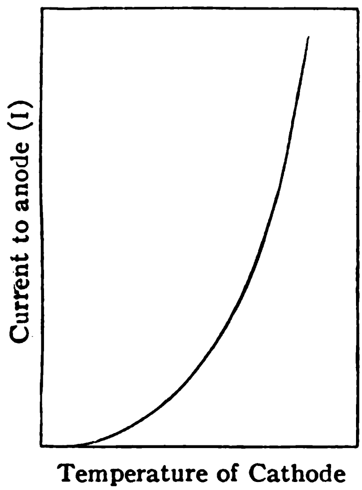

FIGURE 1

current is obtained in the following way: Suppose the cathode be maintained at a constant temperature T_1 and the voltage V between anode and cathode be varied. As this voltage V is raised from zero, the current I to the anode at first increases, the relation between V and I being represented by the curve OA_1 of Figure 2. Any increase in V beyond the value corresponding to A_1 causes no further increase in I, and we get the part A_1B_1 of the curve. Clearly this part of the curve corresponds

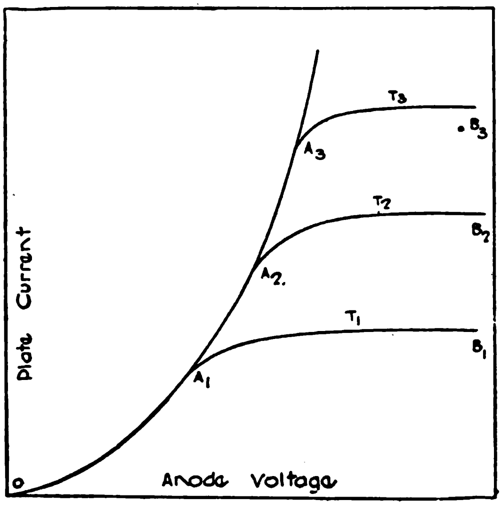

FIGURE 2

to the condition when all the emitted electrons are drawn to the anode as fast as they are emitted from the cathode. If the cathode temperature be increased to T_2, the number of emitted electrons is increased, and we get the curve OA_2B_2. When these values of the saturation current are plotted as a function of the temperature, we obtain the curve of Figure 1. This curve is represented very approximately by the equation

$$I = aT^{\frac{1}{2}} e^{\frac{b}{T}}, \qquad (1)$$

where a and b are constants. This equation was derived by O. W. Richardson in 1901[2] on the basis of the theory that the electrons are emitted from the hot cathode without the help of any gas, but solely in virtue of their kinetic energy. The formulation of this theory was the first definite expression of what may be termed a pure electron emission.

In the state in which Richardson's equation holds the cur-

[2] "Proc. Camb. Phil. Soc.," volume II, 285, 1901; "Phil. Trans. Roy. Soc.," A, 201, 1903.

rent is independent of the voltage. Under these conditions the device to be treated in the following does not function as amplifier or detector, since it depends for its operation on the variation of current produced by variation of the voltage. In this device the current established in the circuit connecting filament and anode (that is, the so-called output circuit) by the electrons flowing from filament to anode is varied by potential variations applied between the filament and grid. The condition under which the current is a function of the voltage is represented by the part *OA* of Figure 2. Here the voltage is not high enough to draw all the electrons to the anode as fast as they are emitted from the cathode; in other words, there are more electrons in the neighborhood of the cathode than can be drawn away by the applied voltage. It was first pointed out explicitly by C. D. Child in 1911 that this limitation to the current is due to the space charge effect of the electrons in the space between anode and cathode. The influence of space charge is something which must always be considered where conduction takes place by means of dislodged electrons or ions, such as the conduction thru gases at all pressures, liquids, and high vacua. Assuming that in the space only ions of one sign are present, Child deduced the equation:[3]

$$I = \frac{1}{9\pi}\sqrt{\frac{2e}{m}} \cdot \frac{V^{\frac{3}{2}}}{x^2} \qquad (2)$$

In this equation, which was deduced on the assumption that both cathode and anode are equipotential surfaces of infinite extent, I is the thermionic current per square centimeter of cathode surface, V the voltage between anode and cathode, x the distance between them, and e and m the charge and mass of the ion, respectively.

When the full space charge effect exists, the current is independent of the temperature of the cathode. This can be understood more easily with reference to Figure 3, which gives the current as a function of the temperature of the cathode for various values of the voltage between anode and cathode. Suppose a constant voltage V_1 be applied between anode and cathode, and the temperature of the cathode be gradually increased. At first when the temperature is still low, the voltage V_1 is large

[3] C. D. Child, "Phys. Rev.," 32, 498, 1911. The space effect has been fully studied by J. Lilienfeld ("Ann. d. Phys.," 32, 673, 1910); I. Langmuir ("Phys. Rev.", (2), 2, 450, 1913), who also independently derived the space charge equation (2) and published a clear explanation of the limitation of current by the space charge; and Schottky ("Jahrb. d. Rad. u. Elektronik," volume 12, 147, 1915).

enough to draw all the emitted electrons to the anode, and an increase in the temperature results in an increase in the current. This gives the part OC_1 of the curve of Figure 3. When the temperature corresponding to C_1 is reached, so many electrons are emitted that the resulting volume density of their charge causes all other emitted electrons to be repelled, and these return to the filament. Obviously any further increase in the tempera-

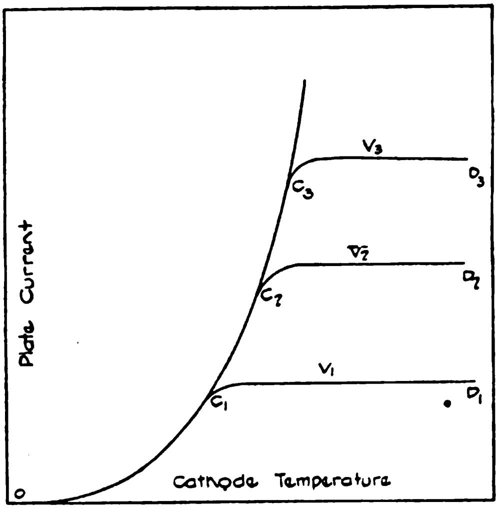

FIGURE 3

ture of the cathode beyond that given by C_1 causes no further increase in the current, and we obtain the horizontal part C_1D_1. If, however, the voltage be raised to V_2, the current increases, since more electrons are now drawn away from the supply at the filament, the full space charge effect being maintained by less emitted electrons being compelled to return to the filament. It is now clear that the part OC of Figure 3 corresponds to the part AB of Figure 2 and CD of Figure 3 to OA of Figure 2. The latter represents the condition under which the thermionic amplifier operates.

It is important to note that the thermionic amplifier operates under the condition characterized by the circumstances that the applied voltage is not sufficiently high to give the saturation current.

III. Action of the Auxiliary Electrode

So far we have considered the case of a simple thermionic device consisting of a cathode and anode. When a third electrode is added to the system, the matter becomes more complicated.

The insertion of a third electrode to control the current between cathode and anode is due to de Forest.[4] De Forest later gave this electrode the form of a grid placed between cathode and anode.[5] About the same time von Baeyer[6] used an auxiliary electrode in the form of a wire gauze to control thermionic discharge. The gauze was placed between the thermionic cathode and the anode.

The quantitative effect of the auxiliary electrode was first given by the present writer.[7]

To get an idea of the effect of the auxiliary electrode consider the circuit shown in Figure 4. *F* denotes the cathode, *P* the anode, and *G* the auxiliary electrode which is in the form of a grid between *F* and *P*. Let the potential of *F* be zero, and that of *P* be maintained positive by the battery *E*, and let E_c for the present be zero. Now, altho there is no potential difference between *F* and *G*, the electric field between *F* and *G* is not zero,

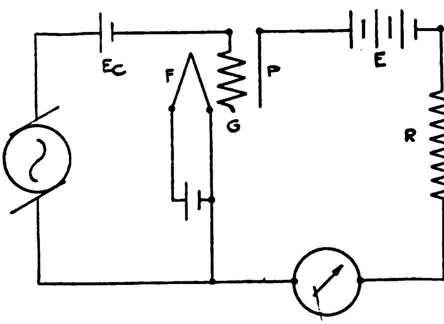

Figure 4

[4] De Forest, U. S. Patent number 841,387, 1907.

[5] De Forest, U. S. Patent number 879,532, 1908.

[6] von Baeyer, "Verh. d. D. Phys. Ges.," 7, 109, 1908.

[7] H. J. van der Bijl, "Verh. d. D. Phys. Ges.," May, 1913, page 338. In these experiments which were also performed under such conditions that the current was carried almost entirely by electrons, the source of electrons was a zinc plate subjected to the action of ultra-violet rays. It is obvious that the action of the auxiliary electrode is independent of the nature of the electron source. Hence the results then found apply also to the present case.

but has a finite value which depends upon the potential of P. This is due to the fact that the potential of P causes a stray field to act thru the openings of the grid. If the potential of P be E_B the field at a point near F is equal to the field which would be sustained at that point if a potential difference equal to γE_B were applied directly between F and an imaginary plane coincident with the plane of G, where γ is a constant which depends on the mesh and position of the grid. If the grid is of very fine mesh γ is nearly zero, and if the grid be removed—that is, if we have the case of a simple valve—γ is equal to unity.

These results can be expressed by the following equation:

$$E_s = \gamma E_B + \varepsilon. \tag{3}$$

Here ε is a small quantity which depends upon a number of factors, such as the contact potential difference between cathode and grid and the power developed in the filament, which is the usual form of cathode used. It is generally of the order of a volt and can be neglected when it is small compared with γE_B. Obviously the current between anode and cathode depends on the value of E_s.

Now, suppose a potential E_c be applied directly to the grid G, the cathode F remaining at zero potential. The current is now a function of both E_s and E_c:

$$I = \Phi\,(E_s,\ E_c). \tag{4}$$

Before determining the form of this function, let us consider in a general way how the current is affected by E_s and E_c. We have seen that E_s is due to the voltage E_B between anode and cathode, and is less than E_B if the grid is between anode and cathode, since in this case γ is always less than unity. Under the influence of E_s the electrons are drawn thru the openings of the grid and are thrown on to the anode by the strong field existing between grid and anode. The effect of E_c on the motion of the electrons between F and G is similar to that of E_s. Whether or not electrons will be drawn away from the cathode depends on the resultant value of E_s and E_c. If $E_s + E_c$ is positive, electrons will flow away from the cathode, and if $E_s + E_c$ is zero or negative, all the emitted electrons will be returned to the cathode, and the current thru the tube will be reduced to zero. Now, $E_s + E_c$ will be positive: (1) when E_c is positive (E_s is always positive), and (2) if E_c is negative and less than E_s.

1. When E_c is positive, some of the electrons moving toward the grid are drawn to the grid, while the rest are drawn thru the openings of the grid to the anode under the influence of E_s.

The relative number of electrons going thru and to the grid depends upon the mesh of the grid, diameter of the grid wire, and the relative values of E_s and E_c. When, for example, E_s is large compared with E_c, the number of electrons going to the grid is comparatively small, but for any fixed value of E_s the current to the grid increases rapidly with increase in E_c. Hence, for positive values of E_c, current will be established in the circuit $F\,G\,E_c$, Figure 4.

2. If, however, E_c is negative and less than E_s, as was the case in the above named experiments of the writer, nearly all the electrons drawn away from the filament pass to the plate, practically none going to the grid. In this case the resistance of the circuit $F\,G\,E_c$ is infinite.

If, now, an alternating emf. be impressed upon the grid so that the grid becomes alternately positive and negative with respect to the cathode, the resistance of the circuit $F\,G\,E_c$, which may be referred to as the input circuit, will be infinite for the negative half cycle and finite and variable for the positive half cycle. If, on the other hand, the alternating emf. be superimposed upon the negative value, E_c, the values of these voltages being so chosen that the resultant potential of the grid is always negative with respect to the cathode, the impedance of the input circuit is always infinite.

Broadly speaking, the operation of the thermionic amplifier is as follows: The current to the anode we have seen is a function of E_s and E_c, or keeping the potential E_B of the anode constant, the current for any particular structure of the device is a function only of the potential on the grid. Hence, if the oscillations to be repeated are impressed upon the input circuit, variations in potential difference are set up between cathode and grid, and these cause variations in the current in the circuit FPR, the power developed in the load R being greater than that fed into the input circuit. It is seen then that the device functions broadly as a relay in that variations in one circuit set up amplified variations in another circuit unilaterally coupled with the former.

IV. Current-Voltage Characteristic of the Thermionic Amplifier

Equation (2) which gives the current to the anode as a function of the applied voltage in the case of a simple device containing equipotential electrodes of infinite extent is of little use in deriving the amplification equations of the thermionic amplifier. In the first place, the cathode in this device is not an equipotential surface, but a filament which is heated by passing a current thru it. Secondly, the insertion of a grid between the filament and the anode so complicates the electric field distribution that a theoretical deduction of the relation between the current to the anode and the applied voltages between filament and grid and filament and anode is difficult and leads to expressions that are too complicated for practical use. I have, therefore, found it more practical to determine the characteristic of the tube empirically, and found as the result of a large number of experiments that the characteristic can be represented with sufficient accuracy by the following equation:

$$I = a\,(E_s + E_c)^2, \tag{5}$$

where a is a constant depending on the structure of the device.[8]

With the help of equation (3) this becomes

$$I = a\,(\gamma E_B + E_c + \varepsilon)^2. \tag{6}$$

This gives the current to the anode as a function of the anode and grid potentials, the potential of the filament being zero. If a number of voltages be impressed upon the grid and anode, we have generally

$$I = a\,(\gamma \Sigma E_B + \Sigma E_c + \varepsilon)^2. \tag{7}$$

If, for example, an alternating emf., $e \sin pt$ be superimposed upon the grid-voltage, E_c, the equation becomes:

$$I = a\,(\gamma E_B + E_c + e \sin p\,t + \varepsilon)^2. \tag{8}$$

It must be understood that equation (6) gives the direct characteristic of the device itself; that is, E_B in equation (6) is the voltage directly between the filament and the anode P (Figure 4). If the resistance R be zero, E_B is always equal to E, the voltage of the battery in the circuit $EPRE$, which is constant. If R be not zero, the potential difference established between the ends of R by the current flowing in it makes E_B a function of the current. The effect of the resistance R on the characteristic will be explained later. For the present we shall confine ourselves to a discussion of the characteristic of the amplifier itself. This characteristic can always be obtained experimentally by making R equal to zero and using an ammeter in the circuit $FPER$ (Figure 4), the resistance of which is small compared with the internal output resistance of the amplifier itself.

[8] Altho this equation is sufficiently accurate when using the device as an amplifier, its accuracy does not suffice for purposes of detection, since the detection action is a function of the second derivative of the characteristic.

A graphical representation of equation (6) is given in Figure 5. The curves give the current to the anode as a function of the grid voltage E_c for different values of the parameter, E_B.

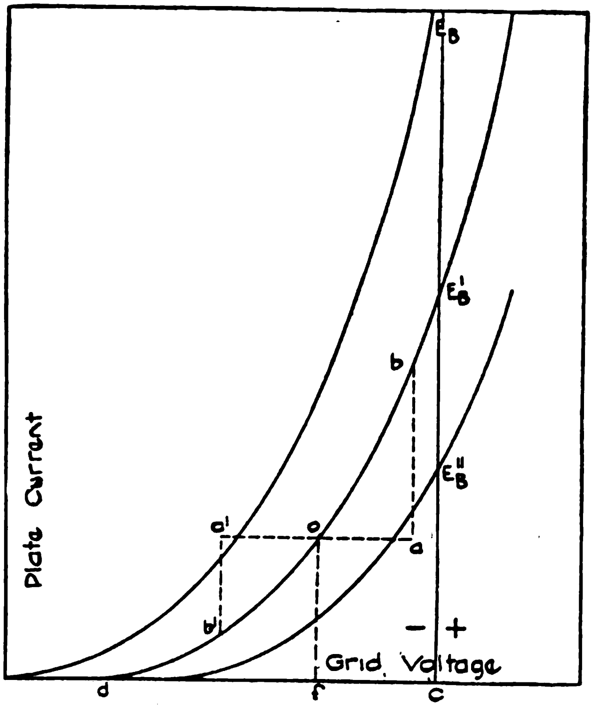

FIGURE 5

Referring to equation (6) and Figure 5, we see that the current is finite for negative values of the grid voltage E_c, and is only reduced to zero when

$$E_c = -(\gamma E_B + \varepsilon).$$

Differentiating I (equation 6), first with respect to E_B, keeping E_c constant, and then with respect to E_c, keeping E_B constant, we get:

$$\frac{\partial I}{\partial E_B} = 2\alpha\gamma(\gamma E_B + E_c + \varepsilon) = Q, \tag{9}$$

$$\frac{\partial I}{\partial E_c} = 2\alpha(\gamma E_B + E_c + \varepsilon) = S. \tag{10}$$

Hence

$$\frac{Q}{S}=\gamma=\text{constant}, \tag{11}$$

from which it follows that for equivalent values of E_B and E_c, a change in the anode voltage E_B produces γ times as great a change in the current to the anode as an equal change in the grid voltage E_c.

The output impedance of the tube is obtained from the admittance K which is given by

$$K=\frac{1}{2\pi}\int_o^{2\pi}\frac{\partial I}{\partial E_B}\,dt,$$

or putting in the value of $\frac{\partial I}{\partial E_B}$ from (9):

$$K=\frac{1}{2\pi}\int_o^{2\pi}2\,a\,\gamma\,(\gamma E_B+E_c+\varepsilon+e\sin pt)\,dt.$$

It is seen that $\frac{\partial I}{\partial E_B}$ is not constant but depends upon the instantaneous value of the input voltage $e\sin pt$. This is also obvious since the characteristic is curved. The admittance and impedance, however, are independent of the input voltage, as is seen readily by integrating the expression for K:

$$R_o=\frac{1}{K}=\frac{1}{2\,a\,\gamma\,(\gamma E_B+E_c+\varepsilon)}. \tag{12}$$

Comparing this with equation (9), it is seen that the impedance can readily be obtained by taking the slope of the characteristic at a point corresponding to the direct current values E_B and E_c at which it is desired to operate the tube.

Equation (12) can be expressed in a more convenient form by multiplying its numerator and denominator by $(\gamma E_B+E_c+\varepsilon)$:

$$R_o=\frac{\gamma E_B+E_c+\varepsilon}{2\,a\,\gamma\,(\gamma E_B+E_c+\varepsilon)^2},$$

which, with the help of equation (6) becomes

$$R_o=\frac{E_B+\mu_o(E_c+\varepsilon)}{2I}, \tag{13}$$

where

$$\mu_o=\frac{1}{\gamma}. \tag{14}$$

We shall see that μ_o is the maximum voltage amplification obtainable from the device.

Comparing (12) with (9) it is seen that $R_o = 1/Q$ and therefore from (11) and (16) the slope of the I, E_c-curve is given by

$$S = \frac{\mu_o}{R_o}. \tag{15}$$

This constant is very important. It will be shown later that the quality of the device is determined by the value of S, that is, the slope of the curve giving the current to the plate as a function of the grid voltage.

V. Experimental Determination of the Constants of the Tube and Verification of the Characteristic Equation

In order experimentally to verify equation (6) it is necessary to know the value of the constants γ and ε. Both these constants can be determined by methods which do not depend on the exponent of the equation. The linear stray field relation

$$E_s = \gamma E_B + \varepsilon, \tag{3}$$

which is involved in equation (6) is also independent of the exponent. The constants γ and ε can be determined and the relation (3) tested as follows:

Let us assume an arbitrary exponent β for equation (6):

$$I = \alpha\,(\gamma E_B + E_c + \varepsilon)^\beta. \tag{16}$$

Takine the general case in which both E_B and E_c are variable, we have:

$$\frac{dI}{dE_c} = \frac{\partial I}{\partial E_B}\frac{dE_B}{dE_c} + \frac{\partial I}{\partial E_c}.$$

Now

$$\frac{\partial I}{\partial E_B} = \alpha\beta\gamma\,(\gamma E_B + E_c + \varepsilon)^{\beta-1},$$

$$\frac{\partial I}{\partial E_c} = \alpha\beta\,(\gamma E_B + E_c + \varepsilon)^{\beta-1}.$$

Hence

$$\frac{dI}{dE_c} = \alpha\beta\,(\gamma E_B + E_c + \varepsilon)^{\beta-1}\left(\gamma\frac{dE_B}{dE_c} + 1\right). \tag{17}$$

Now, let I be constant, then

$$\gamma E_B + E_c + \varepsilon = 0,$$

that is,

$$-E_c = \gamma E_B + \varepsilon = E_s, \tag{18}$$

or

$$\frac{dE_B}{dE_c} = -\frac{1}{\gamma}.$$

Integrating and putting $\frac{1}{\gamma}$ equal to μ_o, we get

$$E_B' = E_B + \mu_o E_c. \qquad (19)$$

Equations (18) and (19) are, therefore, independent of the exponent of (6). Equation (18) gives the case in which the current has the constant value zero. It shows that E_s in equation (3) is simply the absolute value of the grid potential E_c, which suffices to reduce the current to the anode to zero when the anode has a potential E_B. (The potentials are referred to that of the filament which is supposed to be grounded.) Referring to Figure 5, we see that equation (18) gives the relation between the intercepts of the curves on the axis of grid potential E_c and the corresponding values of anode potentials E_B. This is the method which I used several years ago to test the linear stray field relation (3). The accuracy with which this relation is obeyed is seen from Figures 3 and 5 of my above mentioned publication.[9]

The factor μ_o, which plays a very important part in the theory of operation of the thermionic amplifier, can be obtained by taking the slope of the curve giving the relation between E_B and E_c in accordance with equation (18). It can be more easily determined with the help of equation (19), which gives the relation between the anode and grid potentials necessary to maintain the current at some convenient constant value. Figure 6 gives results obtained in this manner. The linear relation obtained between E_B and E_c verifies equation (19).

Another method of determining μ_o is with the help of equation (11):

$$\frac{Q}{S} = \gamma = \frac{1}{\mu_o}. \qquad (11)$$

S is the slope of the curve giving the current to the anode as a function of the grid potential, and Q the slope of the curve which gives the current as a function of the anode potential. Since both these slopes depend upon the anode and grid potentials E_B and E_c, they must be measured for the same values of E_B and E_c. This method gives quite reliable results but is not as convenient as the one explained above.

[9] "Vehr. d. D. Phys. Ges.," above. For further experimental verification of this relation when applied to the case in which the cathode consists of a hot filament, see H. J. van der Bijl, "Phys. Rev.", (2), 12, 184, 1918.

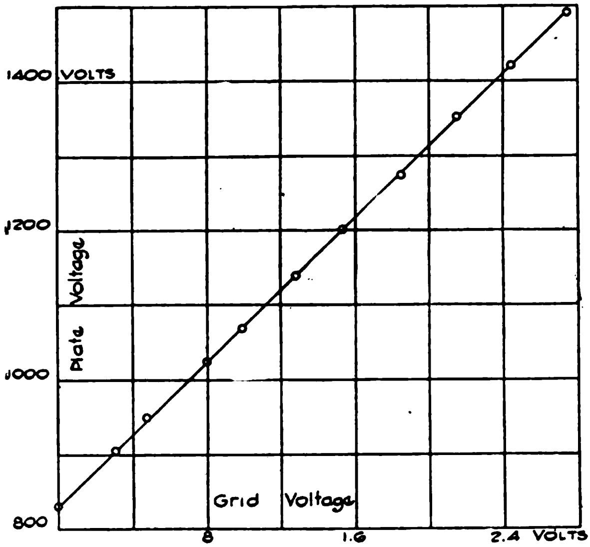

FIGURE 6

The most convenient method of measuring the amplification constant μ_o is that recently given by Miller.[10] The principle of this method is the same as one that has been frequently used in this laboratory where it is necessary to determine μ_o for a large number of tubes. The circuit shown in Figure 7 is contained in a box with terminals for the ammeter A and batteries E_B and E_A. The tube is plugged into a socket provided for it in the box. It is seen from the previous paragraph that a voltage in the grid circuit is equivalent to μ_o-times that voltage in the plate circuit. Hence, referring to Figure 7, it is evident that no change will be produced in the ammeter A on closing the key K, if $\frac{r_1}{r_2} = \mu_o$. For convenience in measurement r_2 is a fixed value of 10 ohms, and r_1 consists of three dial rheostats of 1,000, 100, and 10 ohms arranged in steps of 100, 10, and 1 ohms each. The rheostats are marked in tenths of the actual resistances, so that the setting of the dials read the μ_o directly. The drain of the battery E_1 is very small because the circuit is only closed momentarily by the push button K. This battery, therefore, consists of small dry cells inclosed in the box. Instead

[10] J. M. Miller, PROCEEDING OF THE INSTITUTE OF RADIO ENGINEERS, volume 6, page 141, 1918.

of using a direct current supplied by the battery E_1, an alternating current can be used, in which case the ammeter A must be replaced by a telephone receiver. The use of an alternating

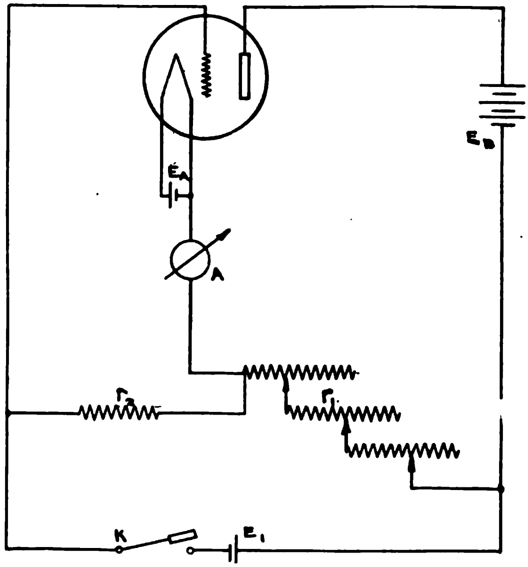

FIGURE 7

current has the advantage that it allows a simple determination of the impedance of the tube according to the method given by Miller. Figure 8 shows a photograph of the μ_o-meter with a tube inserted in its socket. The rheostat R enables the filament current to be adjusted to the desired value.

In order to test the characteristic equation (6) it is still necessary to know the value of ε. This can be obtained by applying a convenient negative potential to the grid and keeping it constant while observing the current to the anode for various values of the anode potential. The grid being negative with respect to the filament, no current could be established in the filament-grid circuit. There should be no resistance in the circuit *FPE* except that of the ammeter, and this should be small compared with the internal output resistance of the amplifier itself. Under these conditions the voltage of the battery E is always equal to E_B, the voltage between filament and anode, so that the observed values of current and voltage give the true characteristic of the amplifier. From the curve giving the cur-

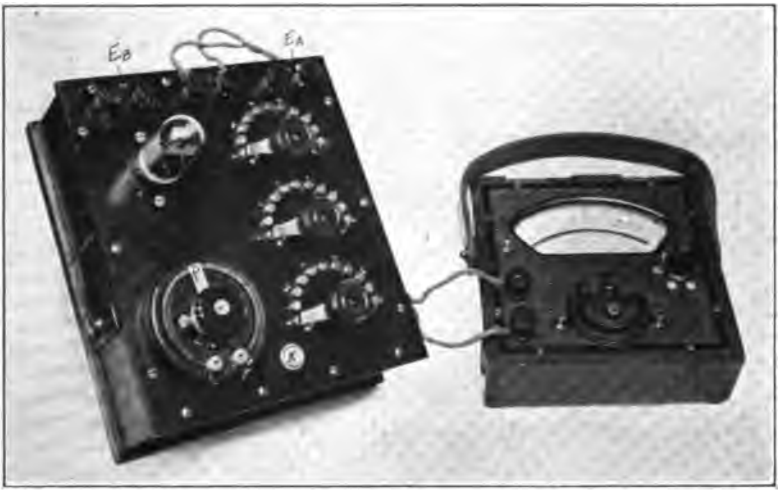

FIGURE 8

rent as a function of the anode potential the value of the anode potential can be determined for which the current is reduced to zero. This potential, of course, depends upon that of the grid. By putting I equal to zero in equation (6), and $\gamma = \frac{1}{\mu_o}$ we get

$$\varepsilon = -\left(\frac{E_B}{\mu_o} + E_c\right).$$

Once μ_o and ε are known, the current can be plotted against the expression

$$\left(\frac{E_B}{\mu_o} + E_c + \varepsilon\right)^2 \qquad (20)$$

Figure 9 shows the results for one particular type of tube.

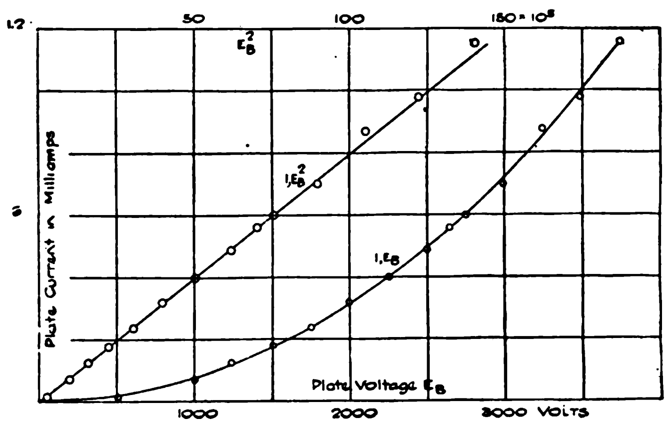

FIGURE 9

In this case the grid had a constant negative potential equal to ε. Hence the current was simply plotted as a function of E_B. The straight line gives the relation between E_B^2 and the anode current. It is seen that the parabolic relation of the fundamental equation (6) is obeyed with sufficient accuracy.[11]

VI. Characteristic of Circuit Containing Thermionic Amplifier and Ohmic Resistance in Series

In discussing the behavior of the thermionic amplifier in an alternating current circuit, we shall make two assumptions:

First. The alternating current established in the circuit *FPER* (Figure 4) is a linear function of the voltage impressed upon the input circuit FGE_c. This implies that the power amplification is independent of the input. This is the condition for an ideal amplifier.

Second. The thermionic amplifier shows no reactance effect. This implies that if the amplifier be inserted in a non-inductive circuit, the power amplification produced is independent of the frequency.

No proof is needed to establish the validity of the second assumption. The first is, however, not true except under certain conditions, and it remains to determine these conditions and operate the amplifier so that they are satisfied. Let the external resistance of the output circuit *FPE* (Figure 4) be zero, and the resistance of the current-measuring device negligibly small compared with the internal output impedance of the amplifier itself. Under these circumstances E_B, which denotes the potential difference between filament and anode, is independent of the current in the circuit *FPE*, and always equal to the voltage E of the battery in the circuit. Hence, if the current be plotted as a function of E_c, the potential difference between filament and grid, the parabola given by equation (6) and Figure 5 is obtained. If, now, the potential of the grid be varied about the value E_c equal to *cf* (Figure 5), it is obvious from the curve that the increase, *ab*, in current to the anode due to a decrease, *oa*, in the negative potential of the grid is greater than the decrease $a'b'$ in current caused by an equal increase, oa', in the negative grid potential. In this case, the output current consists of the following parts: Let the alternating input voltage superimposed upon E_c be $e \sin pt$ then

$$I_o = a(\gamma E_B + E_c + \varepsilon + e \sin pt)^2. \tag{8}$$

[11] For further results of these experiments, see "Phys. Rev.", (2), 12, 186, 1918.

Expanding this we get

$$I_o = a(\gamma E_B + E_c + \varepsilon)^2 + 2a(\gamma E_B + E_c + \varepsilon)\, e \sin pt + \frac{a e^2}{2} \cos(2pt + \pi) + \frac{a e^2}{2}. \qquad (21)$$

The first term represents the steady direct current maintained by the constant voltages E_B and E_c when the input voltage e is zero (equation 6). The second term gives the alternating output current oscillating about the value of direct current given by (6). It is in phase with and has the same frequency as the input voltage. When using the device as an amplifier, this is the only useful current we need to consider. The first harmonic represented by the third term is present, as was to be expected in virtue of the parabolic characteristic. The last term, which is proportional to the square of the input voltage, represents the change in the direct current component due to the alternating input voltage, and is the only effective current when using the device as a radio wave detector. If a direct current meter were inserted in the output circuit, it would show a current which is greater than that given by equation (6) by an amount equal to $\frac{a e^2}{2}$, the last term of equation (21). This is the state of matters when the device works into a negligibly small resistance.

If, on the other hand, the output circuit contains an appreciable resistance,[12] R, the voltage E_B between filament and plate is not constant, but is a function of the current, and an increase in the current due to an increase in the grid potential sets up a potential drop in the resistance R, with the result that E_B decreases, since the battery voltage E is constant. E_B is now given by

$$E_B = E - RI. \qquad (22)$$

In order to obtain the characteristic of the circuit containing the tube and a resistance R, let us substitute (22) in (8):

$$I_o = a[\gamma(E - RI) + E_c + \varepsilon + e \sin pt]^2$$

and put $\gamma E + E_c + \varepsilon = V$. This gives

$$I_o = \frac{1 + 2a\gamma R(V + e \sin pt) - \sqrt{1 + 4a\gamma R(V + e \sin pt)}}{2a\gamma^2 R^2}. \qquad (23)$$

---

[12] The insertion of a suitable resistance in the output circuit to straighten out the characteristic and so reduce distortion was, I believe, first suggested by Dr. Arnold, who also showed experimentally that distortion is almost negligible when the external resistance is equal to the impedance of the tube.

This expression can be expanded into a Fourier series:

$$I_o = \begin{cases} \left(\dfrac{1+2\alpha\gamma R V-\sqrt{1+4\alpha\gamma R V}}{2\alpha\gamma^2 R^2}\right) \\ +\left(1-\dfrac{1}{\sqrt{1+4\alpha\gamma R V}}\right)\dfrac{e}{\gamma R}\sin pt \\ +\dfrac{(-1)^{n+1}2^n \,|(2n-1)\, \alpha^n \gamma^{n-1} R^{n-1} e^{n+1} \sin^{n+1} pt}{\underline{n+1}\,(1+4\alpha\gamma R V)^{2n+\frac{1}{2}}}. \end{cases} \qquad (24)$$

From this it is seen that the rate of convergence of the series increases as R is increased. Actual computations show that when the tube is made to work into an impedance equal to or greater than that of the tube the harmonic terms become negligibly small compared with the second term of (24), which is the only useful term when using the tube as an amplifier, so that we can assume that the amplification is independent of the input voltage.[13]

When the tube works into a large external resistance it can show a blocking or choking effect on the current. This is seen from the following: The voltage E_B which is effective in drawing electrons thru the grid to the anode is given by equation (22). If now the current be increased, not by increasing the electromotive force in the circuit *FPR* (Figure 5), but by increasing the potential difference between filament and grid, the current I increases, while E, the electromotive force in the plate circuit, remains constant, from which it follows that E_B must decrease while E_c, the grid voltage, increases. The result of this is that more electrons that otherwise would have come thru the grid to the anode are now drawn to the grid. If the input voltage becomes large enough the anode circuit *FPR* may be robbed of so many of its electrons that no further increase in current in the anode circuit results no matter how much the grid voltage is increased. Under these conditions equation (24) does not apply. Its application is limited to the conditions stated by equations (25) and (26).

Even if the series represented by the last term of equation

[13] In this connection I want to point out that altho the parabolic relation used here represents the characteristic of the tube with sufficient accuracy when using the tube as an amplifier, and indeed with quite a good degree of accuracy, as shown by the experimental curves, yet the approximation is not close enough to represent accurately the second and higher derivatives of the characteristic, and therefore, too much reliance should not be placed on the actual values of the several harmonic terms represented by the last term of equation (24). This equation is merely intended to show, as it does, in a general way how the insertion of a resistance in the output circuit of the tube tends to straighten out the characteristic.

(24) were zero, distortionless transmission can only be obtained if the input voltage is kept within certain limits.

Let the input voltage, $e \sin pt$, be superimposed upon the negative grid voltage, E_c (Figure 5). Theoretically speaking, one condition of operation is that the grid should never become so much positive with respect to the filament that it takes appreciable current, for if this happens the current established in the grid circuit would lower the input voltage, and therefore the amplification. In actual practice the extent to which the grid can become positive before taking appreciable current depends upon the value of the plate voltage and the structure of the tube. We can therefore state that a condition for distortionless transmission is $e \leqq |E_c| + |g|$, where g is the positive voltage which the grid can acquire without taking enough current to cause distortion. Another condition is that the input voltage must not exceed the value given by df (Figure 5); otherwise the negative peaks of the output current wave will be chopped off. Now cd is given by $\gamma E_B + \varepsilon$. This is obtained by equating the current I to zero in equation (6). We therefore have the conditions

$$\begin{aligned} |e &\leqq |E_c| + |g|, \\ |e &\leqq |\gamma E_B + \varepsilon| - |E_c|, \end{aligned} \tag{25}$$

or when the tube is working at full capacity—that is, when operating over the whole curve,

$$e = |E_c| + |g| = |\gamma E_B + \varepsilon| - |E_c|. \tag{26}$$

It may be remarked here that when using the tube as an oscillation generator, these limits are not obeyed. From equation (12), it is seen that the impedance R_o of the tube is independent of the input voltage $e \sin pt$. This is, however, true only as long as the characteristic is parabolic. Referring to Figure 5, if the input voltage oscillates about the value f, the slope of the tangent at o is a measure of the impedance, in fact, it is μ_o divided by the impedance (equation 15). Since the characteristic is parabolic, it follows that the secant thru $b\,b'$ is always parallel to the tangent at o as long as $oa = oa'$ (equal to the input voltage e). The impedance can, therefore, be obtained by taking the slope of the secant thru the maximum and minimum current values. If now the tube works beyond the limits of the parabolic characteristic, such as along the curve OAB (Figure 2) the slope of this secant does not remain constant for all values of the input voltage, so that the impedance of the tube is not independent of the strength of the oscillations but increases with it, the minimum impedance being obtained when the oscillations are infinitely small. This is what happens when the tube operates as an oscillation generator. Part of the energy in the output circuit is fed back to the input circuit, thus increasing the strength of the oscillations in the output until the tube works beyond the limits of the parabolic characteristic. The impedance of the tube is thereby increased, and this increase continues until the impedance acquires the maximum value capable of sustaining the oscillations, consistent with the degree of coupling used between the output and input circuits. It is readily seen that the current obtained in the output is not a single pure sine wave, but contains a number of harmonics as well. The extent to which these harmonics influence the current values obtained, when the latter is measured simply by the insertion of a hot wire meter in the output circuit, which, of course, measures the total current, depends on the degree of coupling as well as the constants of the output circuit. These considerations must be borne in mind when dealing with the alternating current output power obtainable from an oscillation tube. The only useful power is, of course, that which is due to the fundamental.

VII. Amplification Equations of the Thermionic Amplifier

On the strength of the two assumptions discussed in the previous paragraph, namely, that the amplification is independent of the input and the frequency, it is possible to derive the equations of amplification in a very simple way. Referring to Figure 4, let the current in the external resistance R be varied by variations produced in the grid potential, E_c. Then, as was shown in the last paragraph, E_B is also a variable depending on the current I, as shown by

$$E_B = E - R\,I, \tag{22}$$

where E is the constant voltage of the battery in the output circuit $FPER$. Hence

$$I = \Phi\,(E_B, E_c),$$

from which

$$\frac{d\,I}{d\,E_c} = \frac{\partial\,I}{\partial\,E_B} \cdot \frac{d\,E_B}{d\,E_c} + \frac{\partial\,1}{\partial\,E_c}.$$

This gives the variation of current in R as a function of the variation in the grid voltage.

Substituting from (9) and (10),

$$\frac{d\,I}{d\,E_c} = 2\,a\,(\gamma\,E_B + E_c + \varepsilon)\left(\gamma\,\frac{d(E-RI)}{d\,E_c} + 1\right),$$

that is,

$$\frac{dI}{dE_c} = \frac{2\,a\,(\gamma E_B + E_c + \varepsilon)}{1 + 2\,a\,\gamma\,R\,(\gamma E + E_c + \varepsilon)}.$$

Multiplying thruout by R, and putting $\gamma = \frac{1}{\mu_o}$ we obtain by a simple transformation

$$R\frac{dI}{dE_c} = \frac{\mu_o R}{R + \frac{E_B + \mu_o(E_c + \varepsilon)}{2I}}. \tag{27}$$

Now, $R \cdot dI$ is the voltage change set up in the resistance R, and dE_c is the change in the input voltage. Hence equation (27) gives the voltage amplification produced by the device, which we shall call μ. Furthermore, it follows from paragraph IV. that the output impedance of the amplifier is given by

$$R_o = \frac{E_B + \mu_o\,(E_c + \varepsilon)}{2I}. \tag{13}$$

Hence the voltage amplification μ is given by

$$\mu = \frac{\mu_o R}{R + R_o}. \tag{28}$$

From this equation it is seen that the voltage amplification asymptotically approaches a finite value μ_o, which is attained when the external resistance R becomes infinitely large compared with the output impedance of the amplifier.

In order to find the power amplification it is necessary to know the input impedance of the amplifier; that is, the impedance of the circuit $F\,G\,E_c$ (Figure 4). Now, the amplifier is operated, as was stated above, under such conditions that no current is established in the circuit $F\,G\,E_c$. The impedance of this circuit is, therefore, infinite, and the power developed in it is indeterminate.

In order to give the input circuit a definite constant resistance, Mr. Arnold suggested shunting the filament and grid with a high resistance. This can be considered as the input resistance R_i, of the amplifier. The input voltage is that developed between the ends of this shunt resistance.

If, now, e and e_i represent the voltages established between the ends of the output and input resistances R and R_i respectively, the power developed in R and R_i is $\frac{e^2}{R}$ and $\frac{e_i^2}{R_i}$. Hence the power amplification is

$$\eta = \frac{e^2 R_i}{e_i^2 R} = \mu^2 \frac{R_i}{R},$$

which, with the help of (28), becomes

$$\eta = \frac{\mu_o^2 R_i R}{(R+R_o)^2}, \tag{29}$$

The amplification is, therefore, a maximum when R is equal to R_o.[14]

The power developed in R is

$$P = \frac{\mu_o^2 e_i^2 R}{(R+R_o)^2}, \tag{30}$$

from which it follows, as was to be expected, that the power in R is a maximum when the external output resistance R is equal to the output impedance R_o of the tube.

It is readily seen that the current amplification is given by

$$\xi = \frac{\mu_o R_i}{R+R_o}, \tag{31}$$

from which it follows that the current amplification asymptotically approaches zero as R is increased, the maximum current amplification being obtained when R becomes infinitely small compared with R_o.

Putting $R=R_o$ in (29) and $R=0$ in (31) and remembering that the slope of the curve giving the relation between plate current and grid voltage is given by

$$S = \frac{\mu_o}{R_o}, \tag{15}$$

we get for the maximum power amplification

$$\eta' = \frac{\mu_o R_i}{4} \cdot S, \tag{29a}$$

and for the maximum current amplification

$$\xi' = R_i \cdot S. \tag{31a}$$

These equations show the important part played by the slope S of the curve giving the plate current as a function of the grid voltage. The factor S is equally important in the operation of tube as an oscillation generator and detector. The slope S is what was called by Hazeltine the "mutual conductance" of the tube.[15]

---

[14] The amplification equations are derived on the assumption that the external output circuit of the tube contains only pure resistance. When the circuit is reactive, as is common in practice, we can, to a first approximation, substitute the effective impedance for R in the equations. Mathematical proof for this is given by J. R. Carson, PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 7, number 2, April, 1919.

[15] L. A. Hazeltine, PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 6, page 63, 1918.

It is seen here that this quantity can be expressed in terms of the two most important and easily determined constants of the tube, namely, the amplification constant μ_o and the impedance R_o.

While the amplification constant depends only upon the structure of the tube, the impedance, besides being a function of the structure, depends also upon the values of the applied voltages between filament and grid and filament and plate, as is readily seen from the above equation. If the impedance is determined as a function of the plate voltage, the grid voltage being, let us say, zero, a curve is obtained somewhat like that shown in Figure 10. If now it is desired to operate the tube

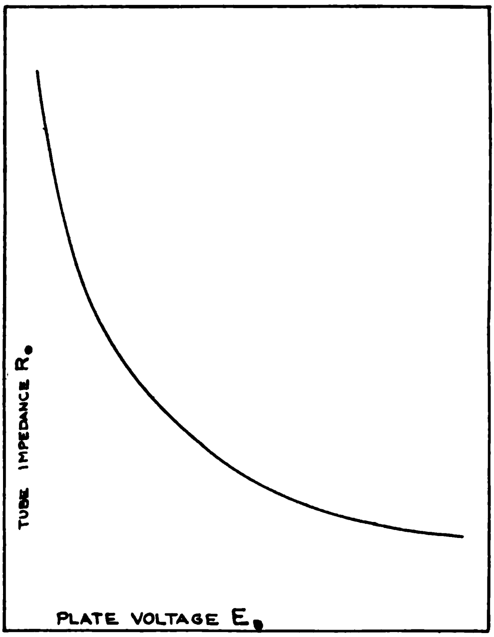

FIGURE 10

with a definite plate voltage E_B and a grid voltage E_c other than zero, the impedance under these conditions can be obtained from such a curve by adding $\pm \mu_o E_c$ to E_B and reading off the impedance from the curve at a value of the plate voltage equal to $E_B \pm \mu_o E_c$.

In designing a tube, the structural parameters are so chosen that the tube constants have definite values depending upon the purpose for which the tube is to be used. Suppose it is desired to design a two-stage amplifier set. Since the tube is a potential-operated device, the voltage impressed on the input of the tube must be made as high as possible, irrespective of the value of the current in the input circuit. This is usually done by stepping up the incoming voltage by means of a transformer, the secondary of which is wound to have as high an impedance as possible. For the same reason when the output current of one tube is to be amplified by another, the first tube is made to work into an impedance or resistance which is large compared with its internal output impedance. Such an arrangement allows of a large voltage amplification being obtained from the first tube. This follows from equation (28) which also shows that when used as a voltage amplifier, the tube must have a large amplification constant μ_o. Referring to equation (26), however, it is seen that the larger μ_o is, the smaller is the input voltage e that can be impressed on the tube without producing distortion, provided that E_B is fixed. When it is necessary to use a two- or three-stage amplifier set, the incoming voltage is generally so small that μ_o for the first tube can be quite large and the plate voltage still not excessively high. But then the voltage impressed on the second tube is much larger than that impressed on the first, and the second tube must be so designed as to be capable of handling this voltage. If the first tube, for example, has an amplification constant equal to 40 and works into a resistance four times its own impedance, the voltage on the second tube is 32 times that impressed on the first. It is seen, therefore, that unless the plate voltage on the second tube be made very much higher than that on the first, the two tubes must have entirely different structural parameters, if equation (26) is to be satisfied in both cases. Such considerations show that unless the tubes be properly designed and the plate voltages correctly chosen to satisfy equation (26), there is a practical limit to the number of stages of amplification that can be used. If the limitations imposed by equation (26) are not taken regard of, the process of amplification can result in a considerable amount of distortion, which is a serious matter when using the device for amplifying telephonic currents. When using it for telegraph purposes, such as the amplification of radio telegraph signals at the receiving station, the distortion produced results in a waste of energy in harmonics.

VIII. Experimental Verification of Amplification Equations

The circuit shown in Figure 11 is not of the type customarily used in practice, but was designed to test the equations developed in the previous paragraph. This type of circuit was

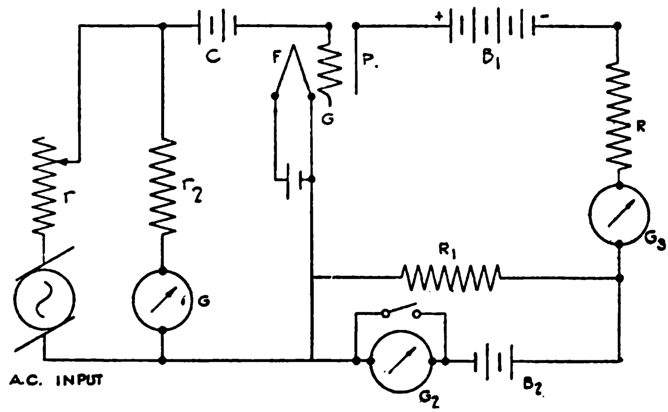

Figure 11

necessitated by the following reasons. Referring to equation (24), it is seen that if the input voltage $e \sin pt$ is zero, the current thru the tube is given by the first term of the equation, which is larger than the alternating current term. For finite values of $e \sin pt$, the resulting alternating current established in the output circuit, which is to be measured, can not be separated in the usual way from this direct current with the help of appropriate inductances and capacities, since then the measured amplification would be largely determined by the constants of the circuit. On the other hand, it is not possible to make the amplifier work simply into a straight non-inductive resistance alone, since the direct current that would flow thru the galvanometer is in most cases large compared with the output alternating current, so that a galvanometer which would be capable of carrying the direct current would not be sensitive enough to measure the output alternating current with any degree of accuracy. This was overcome by using a balancing circuit shown in Figure 11. R and R_1' are two non-inductive resistances stretched upon a board. Parallel to R_1 was shunted a sensitive alternating current galvanometer, G_2, and a balancing battery B_2. This battery was so adjusted that when no alternating current input was applied to the tube, the current thru the galvanometer was zero, that is, the direct current in the output circuit went thru the resistance R_1. This resistance was large compared with that of the galvanometer; hence practically all the alternating current established in the output when the input voltage was impressed, went thru the galvanometer G_2. The effective resistance into which the tube worked was, of course, given by R. The input was varied with the help of the resistances r_1 and r_2. The whole system was carefully shielded and care was taken to avoid any effects due to shunt and mutual capacity of the leads and resistances. The input voltage was varied from a few hundredths of a volt to several volts and the frequency from 200 to 350,000 cycles per second.

Some of the results are shown in the following figures. Figure 12 shows the output voltage (that is, the voltage across the external resistance R) as a function of the input voltage for a frequency of 1,000 cycles per second. The linear relation indi-

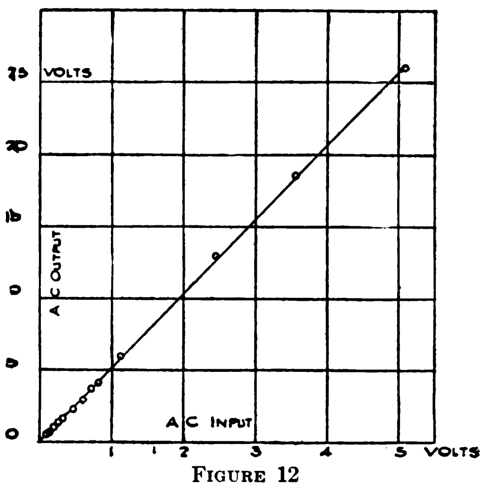

FIGURE 12

cates that the voltage amplification is independent of the input voltage; hence also the power amplification is independent of the input power.

Figure 13 shows the results obtained when the voltage amplification was measured as a function of the external resistance R. The circles show the observed values, while the curves were calculated from equation (28). It is seen that the agreement is quite good. In this case the input voltage was 0.45 volt and the value of μ_o for this tube, as measured by the direct current method explained in paragraph V. was 10.2.

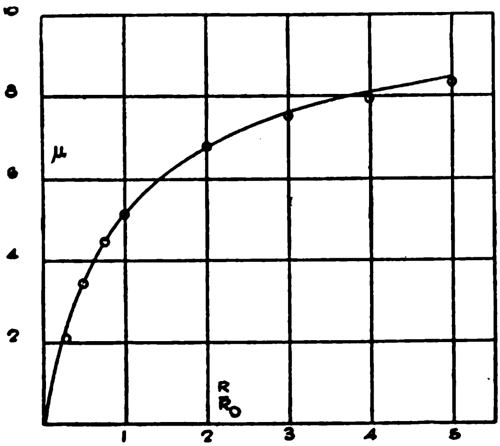

FIGURE 13

$\mu_o = 10.2$, input $= 0.45$ volt

In another experiment, the output power was determined as a function of the external resistance. According to equation (30) this should be a maximum when the external resistance R is equal to the impedance R_o of the tube. In this case the input voltage e_1 was 3.55 volts, and $\mu_o = 10.2$. The impedance of the tube was kept constant at 14,800 ohms. This was done by always adjusting the plate voltage so that when the external resistance was changed the current thru the tube was kept constant. From the results given in Figure 14 it is seen that the maximum occurs at $R = 15{,}000$, which is very nearly equal to the impedance of the tube. This result is in accordance with equation (30). Furthermore, the maximum power computed from equation (30) is 22.2 10^{-3} watt, which is sufficiently close to the observed value, 23 10^{-3} watt, to verify equation (30). This equation does not give the maximum power that can be handled by the tube but merely the power developed in the external resistance R for a given value of the input voltage e_i. The maximum power is obtained when the input voltage has the value given by equation (26).

The circuit shown in Figure 11 was used merely to test the equation derived in this paper. It is not suitable for practical purposes where it is necessary to test a large number of tubes with the speed and facility called for in practice. If the tube is to be used as a voltage amplifier all that is necessary is a determination of the amplification constant μ_o. The actual voltage amplification

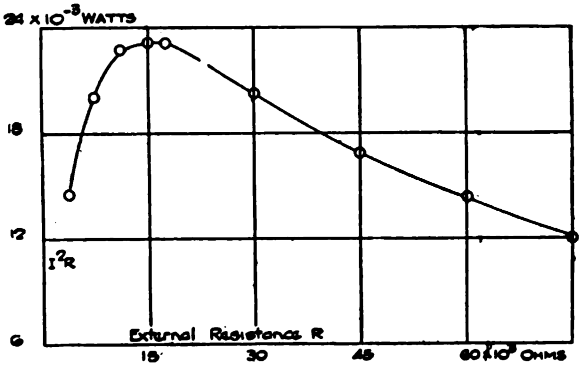

FIGURE 14

obtainable in any particular circuit can then be deduced from equation (28). When the tube is to be used as a power amplifier, a transmission test is used which is common in telephone practice. If it is desired to operate the tube as power amplifier in a certain circuit, its amplification is tested in an equivalent circuit which is arranged so that a note of, say, 800 cycles can be transmitted either straight to a telephone receiver or, by throwing a switch, to the receiver thru the tube and an artificial telephone line, the attenuation of which can be adjusted until the note heard in the receiver is of the same intensity for both positions of the switch. When this is the case the amplification given by the tube is equal to the attenuation produced by the line. The attenuation can be computed from the constants of the line and is usually expressed in terms of miles length of cable of specified constants. This method of measuring and expressing the amplification is convenient in practice. But it must be remembered that this notation has very little meaning and is apt to lead to confusion unless the constants of the cable are definitely specified or previously agreed upon. It simply means that the amplification is equivalent to the attenuation which would be produced by so many miles of a certain sort of cable having a certain definite attenuation constant. Thus, the current amplification produced by the tube can be expressed in terms of length, d, of cable by the following equation:

$$d = K \log_{10} \frac{i_2}{i_1}.$$

where K is determined by the attenuation of that cable. When dealing with power amplification K must be divided by 2. If we adopt as standard the so-called "standard number 19 gauge cable" used by the Western Electric Company, the length, d, is expressed in miles when the constant, K, has the value $K = 21.13$. The fact that this constant is already finding its way into common vacuum practice would suggest its general adoption when speaking of the amplification of a tube. On the other hand, since the unit of measurement is not a cable but a constant, it might have been more desirable to adopt for K the simple value 20, which would have simplified computations. However, the main point is that it is very important to have a common agreement on the value of K.

The foregoing considerations show that the structural parameters play a very important part in the operation of the tube. On them depend the constants μ_o and R_o which appear in the amplification equations and which are involved explicitly and implicitly in the fundamental equation of the characteristic (equation 6). Proper structural design manifests many latent possibilities of this type of device, and enables us to meet the many conditions that must be complied with in order to obtain satisfactory operation in its ever-increasing number of applications. By proper choice of the structural parameters, tubes have been designed to have voltage and power amplification covering a wide range. A power amplification of 3,000-fold was found possible with a single tube using a plate voltage of only 100 volts. It is not difficult to obtain a voltage amplification of several hundred fold, but in building tubes of such high voltage amplification regard must be taken of the increase in impedance with increase in μ_o, as shown by equations (12) and (13).

Altho the simple theory of operation given in this paper applies specifically to the case in which the tube is used as an amplifier, it has also been of considerable help in designing and developing vacuum tube oscillation generators and detectors. The design of a good detector tube depends very much upon operating conditions. The detecting qualities of a tube can easily be increased by designing it to operate on comparatively high voltages. This is sometimes desirable in radio stations where high voltages are available and it is desired to use the heterodyne method of reception. On the other hand, it is often important to use tubes of such design that they can operate efficiently on low voltages and with small power consumption. In this connection it may be said that detectors have been designed to give satisfactory operation with two volts on the filament and a plate voltage of 6 volts and less. In the absence of any satisfactory way of expressing the efficiency of a detector, it is unfortunately not possible to say just how good such a detector is. The problem of measuring the detecting efficiency will be reserved for a future paper.

The indebtedness of the writer is due to Mr. E. H. Colpitts and Mr. H. D. Arnold for valuable advice and kind interest which greatly facilitated the work; and to Mr. H. W. Everitt for able assistance in carrying out the experiments.

SUMMARY: The theory of operation of the three-electrode thermionic vacuum tube given in this paper is based on the fundamental equation of the family of characteristic curves for various plate and grid voltages. This equation, which may be written

$$I = a\left(\frac{1}{\mu_o}\Sigma E_B + \Sigma E_c + \varepsilon\right)^2$$

where E_b and E_c are the plate and grid voltages respectively, and a, μ_o and ε are the structural parameters of the device, is obtained empirically with the help of the relation previously discovered by the author, which states that the voltage between filament and plate bears a linear relation to the effective voltage produced by it between the filament and a plane coincident with that of the grid. From this it follows that an electromotive force e impressed upon the grid circuit produces an electromotive force $\mu_o e$ in the plate circuit. With the help of these fundamental relations the amplification equations of the tube are derived in terms of the structural parameters of the tube and the constants of the circuit.

Methods are given for experimentally determining the constants of the tube, the two most important of which are the amplification constant μ_o and the internal output impedance.

Experiments are described which were performed to test these equations, and they indicate that the first order approximation made give results sufficiently accurate for amplification purposes.

THE OPERATIONAL CHARACTERISTICS OF THERMIONIC AMPLIFIERS*

By

Stuart Ballantine

(Radio Engineer, Navy Department)

Introduction

Remarkable strides have been made in the past few years in connection with the definition and measurement of the operational constants of three electrode vacuum tubes. This has given great impetus to the transition of the device from the domain of the scientist to that of the engineer, with the result that it is now possible to design intelligently vacuum tubes for quite a variety of purposes. With the definition of the tube constants well in mind, and with the aid of appropriate methods for measuring them, we may proceed to accumulate by careful research, data connecting the variation of physical dimensions with the constants so defined, which will serve adequately as a basis for later design work. For this purpose, a very interesting course of procedure would consist in holding the plate-filament distance constant and varying the grid-filament space in small steps, plotting the constants against the latter distance as an independent variable with the plate-filament distance taken as a parameter. The information accruing from an investigation of this sort is of the greatest engineering value. Its accumulation has been made possible by the definitions and methods of measurement that have been developed by such workers as Hazeltine,[1] Langmuir,[2] Miller,[3]

* Received by the Editor, November 29, 1918. Presented before The Institute of Radio Engineers, New York, December 11, 1918.

[1] Hazeltine, L. A., "Oscillating Audion Circuits," Proceedings of The Institute of Radio Engineers, volume 6, page 63, 1918.

[2] Langmuir, I., "The Pure Electron Discharge and Its Applications in Radio Telegraphy and Telephony," Proceedings of The Institute of Radio Engineers, volume 3, number 3, page 261, 1915.

[3] Miller, J. M., "A Dynamic Method for Determining the Characteristics of Three-Electrode Vacuum Tubes," Proceedings of The Institute of Radio Engineers, volume 6, page 141, 1918.

Vallauri,[5] and van der Bijl.[4] The object of the present paper is to display the general character of the variation of tube constants with the variation of grid or controlling potential, and also to describe additional methods for the measurement of three very important constants.

Constants of Vacuum Tubes

It may be of interest at the outset to enumerate briefly and define the most important tube constants. This data is contained in tabular form in Table I, the symbols and dimensions being given together with the names of the persons to whom their definition is due.

TABLE I

Factor	Defined by	Symbol	Dimensions
Voltage Amplification	van der Bijl	μ	$\frac{dE_p}{dE_g}$
	Miller	k	$\frac{dE_p}{dE_g}$
Internal Impedance	van der Bijl	R_o	$\frac{dE_p}{dI_p}$
Mutual Conductance	Hazeltine	g	$\frac{dI_p}{dE_g}$
	van der Bijl	s	$\frac{dI_p}{dE_g}$
	(This Paper)	ρ	$\frac{dI_p}{dE_g}$
Detection (without grid condenser)	" "	D	$\frac{d^2I_p}{dE_g^2}$
Detection (grid condenser)	" "	σ	$\frac{d^2I_g}{dE_g^2} \cdot \frac{dI_p}{dE_g}$

[4] Van der Bijl, H. J., "The Theory and Operating Characteristics of the Thermionic Amplifier," PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 7, number 2, April, 1919.

[5] Vallauri, Professor G., "Sul Funzionamento dei tubi a vuoto a tre elletrodi audion, usati nella radiotelegrafia," "Estratto dal Giornale L'Elettrotecnica," January 25, number 3, and February 5, number 4, 1917.

Coefficient of Voltage Amplification.—This very important factor is an index of the relative effects on the plate current, of the grid and plate potentials. Specifically it is the ratio, $\frac{E_p}{E_g}$. In Langmuir's equation for the thermionic current in the plate circuit

$$I_p = A\,(E_p + k\,E_g)^{3/2} \tag{1}$$

it occupies a prominent place, the k being used as a coefficient to the term E_g, representing the grid or control potential. Dr. H. J. van der Bijl[5] has represented the plate current by another expression, as follows:

$$I_p = a\,(\gamma E_p + E_g + \varepsilon)^2 \tag{2}$$

In this case, γ is the coefficient of his extremely useful conception of the "stray field" and it is obvious that its reciprocal is of the same nature as the k in Langmuir's equation. The value of the voltage amplification constant depends upon the geometry of the tube, increasing as the ratio of the plate-filament, grid-filament distances and inversely as the spacing between grid wires. It is also dependent, as will be shown later, upon the region of the characteristic surface in which operation takes place. This latter relation would seem to make the utilization of the term *factor* preferable to that of *constant*. The manner in which the definition of this factor may be deduced from Vallauri's equation has already been indicated by Miller.

A method of measuring the amplification factor by means of direct currents is outlined in Dr. van der Bijl's paper; also a very useful method has been described by Dr. Miller.[3]

Internal Impedance.—The value of the amplification constant being known, the hypothetical voltage operating in the plate circuit may be calculated as a function of the controlling potential on the grid. If there are no extraneous constants in the plate circuit, the plate current may be calculated by dividing this hypothetical emf. by a factor called the *internal impedance* of the tube. If in the plate circuit other constants are connected, the current may be calculated by the aid of a simplifying theorem suggested by Miller which reduces the plate circuit to a simple series circuit containing the plate resistance *internal impedance*, the added constants in the output circuit, and an operating emf. of value, $E_p = \mu\,E_g$. This simplification is of great value since it places the design possibilities of the vacuum tube circuit on an equal basis with other branches of electrical and radio engineering. It is at all times desirable to design the circuit to fit the characteristics of the tube to be used.

The internal impedance is of the nature of a pure resistance at low frequencies. At radio frequencies, the plate-to-filament capacity of the electrodes, partially subordinates the effect of the internal impedance by acting across it as a branch circuit. This is of particular moment in the design of resistance-coupled amplifiers for short wave length ranges. Since the methods of measurement generally used in determining the magnitude of this factor are based upon audio frequency supply, it will not be necessary to take this into consideration at this time.

The measurement of the internal impedance has been covered by the methods of van der Bijl amd Miller. The first of these methods is based upon the fact that the internal impedance is the slope, $\frac{E_p}{I_p}$ of the d. c. characteristic. It may, therefore, be measured graphically from a plot of this relation, or by means of direct currents. Miller's method is a dynamic one, and is swifter and more accurate. The equation for the computation of R_o from the adjustments, however, involves the factor μ in a very important place, which means that the precision obtainable depends upon the measurement of μ. This is undesirable, not only for this reason, but also because a direct measurement of R_o only cannot be made without first determining the value of μ by a separate balance. These objections are not of a serious nature, but are nevertheless to be considered in the selection of methods for an extended research on vacuum tube problems.

The writer has employed a dynamic null method for the measurement of the internal impedance, which partially overcomes the above objections and gives much better minima in the indicating telephones. The circuital arrangement of apparatus is shown in Figure 1.

This is virtually a Wheatstone bridge connection. The audio frequency alternating current is introduced at the terminals of the slide wire, R_1 R_2. The resistance, R, may be made adjustable if desired and for maximum sensitiveness should be approximately equal to the internal impedance of the tube under measurement. The internal impedance of vacuum tubes may range from 5,000 to 500,000 ohms, and particularly at the higher values it may be advantageous to employ resistances at R made by a spluttered film process. Specially wound Curtis coils may also be used and may be depended upon to be practically free from inductance. This matter becomes of importance only for the higher audio frequencies. The method of indicating the null point is of interest. In the writer's set-up

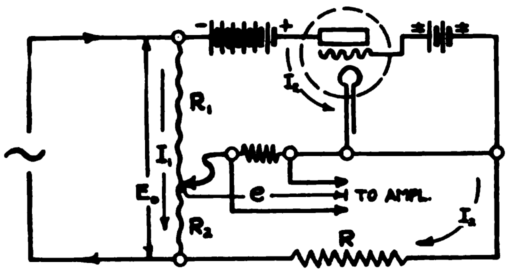

FIGURE 1

a small resistance of known value was inserted in the balance lead, the drop of potential being amplified by a two-stage vacuum tube system. It was found that this arrangement gave audibilities equal to those obtained with telephones inserted directly into the circuit, and possessed the additional advantage of reducing the extraneous inductance and resistance in the circuit to a minimum. The effect of the telephones in the measurements to be described was of the order of several per cent; for this reason the complication seems justified.

The theory of the arrangement shown is very simple. The alternating voltage, E_o operating across the terminals of the slide wire, causes two branch currents to flow, I_1 and I_2 as marked in the figure. For the condition of silence in the telephones, e and the current in the balance circuit containing the indicating apparatus must be zero. This means that:

$$R_2 I_1 = R I_2 \tag{3}$$

but

$$I_1 = \frac{E_o}{R_1 + R_2} \quad \text{and} \quad I_2 = \frac{E_o}{R_o + R} \tag{4}$$

Substituting into (3) gives:

$$\frac{R_2}{R_1 + R_2} = \frac{R}{R_o + R} \tag{5}$$

from which

$$R_o = \frac{R_1}{R_2} R \tag{4}$$

The computation of the internal impedance, R_o, from this relation is very simple and involves nothing but the ratio $\frac{R_1}{R_2}$ and the resistance, R. The main advantage, as mentioned above, of this method is the ability to obtain perfect null points. With ordinary care in the distribution of conductors it is readily possible to obtain balance points of a degree of ambiguity not exceeding 0.2 of one per cent. The measurement under these conditions becomes a matter of some precision.

MUTUAL CONDUCTANCE.—The ratio between the plate current and the corresponding grid or control potential is of great importance in determining the figure of merit of the device as an amplifier and oscillation generator. As pointed out by Hazeltine, it is of the dimensions of a conductance, and was therefore termed the *mutual conductance*, and represented by the symbol g. It seems to the present writer that the use of a symbol of this sort, having general engineering utility is undesirable and leads to confusion, and suggests the use of the Greek letter "rho" (ρ) for this purpose. The mutual conductance is evidently related to the other tube constants just defined by the expression:

$$\rho = \frac{\mu}{R_o} \tag{5}$$

and may, therefore, be computed from a knowledge of the factors involved. It is the slope of the "static" characteristic of the tube, plotted on the basis of I_p versus E_g (plate current versus control potential). It may be evaluated graphically by taking the slope of the tangent at any point on this curve. Since this method is laborious and may easily lead to considerable error, it may be better to measure the amplification constant and internal impedance separately and compute the value of ρ from the relation (5) above. While this is the better of the two methods available, it is still an indirect one, and it is regrettable that the measurement of this most important tube constant should be accompanied by so much labor.

The writer has developed a dynamic method for measuring ρ which retains the recognized advantages of null methods and possesses the additional one of giving the value of the mutual conductance directly. The method becomes particularly simple if the slide wire, R, is calibrated directly in terms of the mutual conductance represented.

Dynamic Method for the Determination of the Mutual Conductance

The circuital arrangement of apparatus is shown in Figure 2.

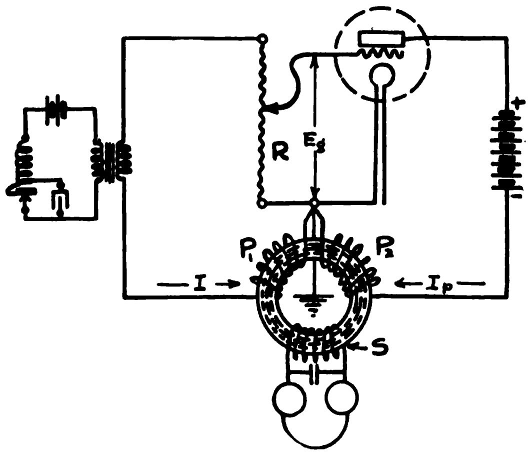

Figure 2

The alternating current for the measurement is supplied to the slide wire, R thru the primary, P_1, of the transformer. This may be of any frequency and may be supplied from an alternator, vacuum tube or buzzer source of the type shown in the figure. If the buzzer arrangement is employed, it is usually necessary to shunt the contacts with a large condenser in order to suppress undesirable harmonics tending to distort the secondary current from the true sine form. The leads to the apparatus from the generator should also be covered with metal sheathing and thoroly grounded, and also arranged so that the induction into the grid circuit will be at a minimum. The core of the transformer and the filament circuit of the tube are inter-connected and grounded in order to minimize residual sound at the balance point.

In the circuit shown, the input and output circuits of the tube are coupled by means of the toroidal core transformer, P_1 P_2 S, to a tertiary circuit containing the indicating apparatus. The theory of the arrangement is very simple.

If the leakage in the transformer is negligible, as will be the case if the core is a torus of the form shown in Figure 2, the induction into the tertiary circuit from the grid circuit primary will be:

$$E' = k\, t_1\, I \tag{6}$$

where k is a constant of little interest. Similarly from the plate circuit we have the induction:

$$E'' = k\, t_2\, I_p \tag{7}$$

If now, the windings are so connected that the two emfs. are vectorially opposed, the resulting emf. operating in the circuit will be:

$$E' - E'' = k\, (I\, t_1 - I_p\, t_2) \tag{8}$$

The indication being taken as the condition of silence in the telephones (or amplifier) connected in this circuit, this state of affairs leads to:

$$I\, t_1 = I_p\, t_2 \tag{9}$$

The value of the plate current depends upon the grid voltage as indicated by the relation, $I_p = \rho E_g$, and

$$I_p = \rho E_g = \rho\, R\, I \tag{10}$$

so that

$$t_1 = \rho\, R\, t_2 \tag{11}$$

and finally

$$\rho = \frac{t_1}{t_2\, R} \tag{12}$$

which defines the value of the mutual conductance, ρ, in terms of the resistance, R, across the grid circuit. For practical reasons, it is well to keep the value of R required for normal measurements, at a low value of the order of 100-1,000 ohms. This matter may be adjusted nicely by selecting a suitable ratio t_1/t_2, such as, for instance, 10:500. If the ratio is kept at a constant value, the relation becomes very simple, permitting the calibration of the slide wire, R, directly in terms of ρ. The method then becomes both swift and accurate and should be particularly useful in connection with acceptance tests of vacuum tubes intended for use as amplifiers and oscillators where a high value of ρ is desirable. The latitude of tolerance may be marked directly upon the slide wire thus reducing the mental labor of such tests considerably. The whole matter then becomes a mechanical proposition, and may be safely intrusted to inexperienced operators. Measurements on certain tubes by this method will be described in a later paragraph.

DETECTOR CONSTANTS.—In addition to its use as an amplifier and oscillator, the vacuum tube has a distinct field of utility as a detector of radio frequency oscillations. When used for this purpose, two methods of operation are possible: (a) without grid condenser but with biasing battery in the grid-filament circuit, and (b) with grid condenser. The phenomena attending the latter mode of functioning have been well described in the classic paper by Armstrong[6] as well as in numerous other publications.

In general, when the grid condenser is omitted, operation takes place upon a region on the characteristic surface

$$I_p = \phi\ (E_g, E_p)$$

which is curved or bent, the indication in the telephones being the result of a variation in the mean value of the plate current due to the superposition of the radio frequency oscillations. Two regions of operation are possible, as shown by the static characteristic curves of Figure 5. It will be observed that the second derivative of I_p with respect to E_g, which determines the curvature, becomes appreciable at two points. One of these represents saturation and involves considerable space current, the other occurs when the curve is leaving the axis, $I_p = 0$. The latter region is usually the best operating point on account of greater curvature.

The curvature of the static characteristic curve at either of these points is a direct index of the detecting merit of the tube. The truth of this proposition may be readily established as follows:

Assume that the curve shown in Figure 3 represents the plate current as a function of the grid or controlling potential, this being the graph of some function:

$$I_p = \phi\ (E_g) \tag{13}$$

The operation is to be performed around some point on this curve, such as E_g marked in the figure. An increment equal to $\Delta\, e_g$ is given the control potential, followed by an equal one of opposite sign. We have then:

$$\begin{aligned} I_p + \Delta_1 I_p &= \phi\ (E_g + \Delta\, e_g) \\ I_p - \Delta_2 I_p &= \phi\ (E_g - \Delta\, e_g) \end{aligned} \tag{14}$$

[6] PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 3, number 3, September, 1915.

The response in the telephones is proportional to the variation of the mean value of the change in plate current, or

$$\frac{\Delta_1 I_p - \Delta_2 I_p}{2} \tag{15}$$

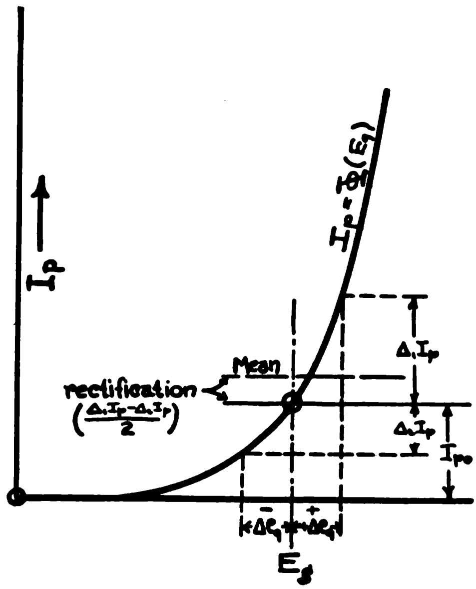

FIGURE 3

from the steady state value, I_p. For a given stimulus, Δe_g, the rectification index is:

$$\frac{\Delta_1 I_p - \Delta_2 I_p}{2 \Delta E_g} \tag{16}$$

or is proportional to the difference in the slopes of the curve at the points corresponding to the constraints of operation. The function being continuous, assuming the increments to be infinitesimal and going to the limit this ratio becomes:

$$D = \frac{d^2 I_p}{d E_g^2} \tag{17}$$

or for small variations of E_g the rectification depends upon the second derivative of the plate current with respect to the grid potential. This, it will be remembered, is also the derivative of the mutual conductance with respect to E_g. The value of this differential is fundamental in determining the merit of the device as a detector without grid condenser, and may be called the *detector constant without grid condenser,* and denoted by the symbol, *D*. It may be readily determined graphically from the curve of mutual conductance as a function of the controlling potential, or computed from measurements of the internal impedance and amplification constant at two points as close together as possible. These methods are open to the objection of being indirect, but a suitable direct method does not seem to be available.

(b) OPERATION WITH GRID CONDENSER.—As is well known, operation of the tube with a series condenser in the grid circuit involves the rectification of the impressed oscillating emf. on the grid to produce a charging current thru the condenser which is uni-directional and hence charges the grid condenser continuously to produce an increasing negative potential on the grid. The radio frequent voltage is super-imposed upon the changing mean potential of the grid, and the point of operation instead of remaining fixed upon the static characteristic curve, slides down along the curve and produces a variation of the mean plate current upon which is superposed the radio frequent changes. On this basic theory we are obviously interested in two changes; first, the rectification in the grid circuit, and second, upon the slope of the $\frac{I_p}{E_g}$ curve. A definition of the merit of the tube used in this manner must contain both of these factors. The first effect, in the light of the previous discussion, we may expect to be dependent upon the second derivative of the grid current-grid potential curve. The second is obviously determined by the value of ρ. Superimposing these effects, an appropriate definition of the detecting action would seem to be:

$$\sigma = \frac{dI_p}{dE_g} \cdot \frac{d^2 I_g}{dE_g^2} \tag{18}$$

In some tubes, the region of maximum curvature in the grid current-voltage curve may correspond to working points involving finite positive values of grid potential, in which case it is desirable to use a "grid-leak" resistance in order to place the starting point at the proper place on the characteristic curve. This does not effect the definition, however, and is of very little interest here. It is also to be noted that as the oscillation persists and the charging of the grid condenser continues, the region of operation shifts, producing simultaneously a change in the curvature of the $\frac{I_g}{E_g}$ curve so that the above definition cannot be regarded as complete except when the stimulus is weak and of short duration or a grid leak is used and the equation of the grid current as a function of the impressed grid emf. may be placed in constant exponential form. Otherwise, the rectification is no longer a function of the curvature of the characteristic alone, but also depends upon the magnitude of the initial oscillatory pulse. In short, the precise nature of the phenomena attending the operation with the grid condenser being complex and imperfectly understood it may be well to bear in mind the possibility of discounting the value of the above definition. Its main value lies, however, in its usefulness in indicating the general possibilities of the tube as a detector used in this connection.

The value of the detecting constant with grid condenser, denoted by the symbol (σ) may be determined by the aid of a dynamic method similar to that used for the measurement of the internal impedance. The circuital arrangement is shown in Figure 4.

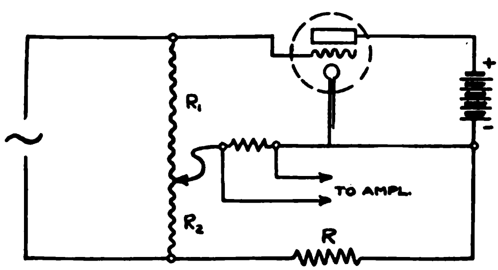

FIGURE 4

For the condition of silence, as in the previous case, we have for the effective resistance of the grid circuit of the tube:

$$R_g = \frac{R_1}{R_2} R. \tag{19}$$

The reciprocal of this quantity, represents a conductance and determines the slope of the grid current-voltage curve. If two measurements are made at points not widely separated, of the reciprocal (g), substitution in the expression:

$$\frac{g' - g''}{e_g' - e_g''} = \frac{R_g'' - R_g'}{R_g' R_g'' (e_g' - e_g'')} \tag{20}$$

yields an approximate value of the second derivative, which when multiplied by the value of ρ corresponding to the mean value of the grid potential taken in the preceding measurements, will give the approximate experimental value of the detecting constant. The value obtained in this manner, while not theoretically accurate, is nevertheless of more value from a practical viewpoint since operation never takes place over a region which is theoretically infinitesimal. The method is perfectly simple and straightforward, but for the reasons cited above, the value of the result as a final index of the operation is questionable.

Experimental Investigation of the General Relation Between Tube Parameters

The summarization of definitions and methods of measurement contained in the preceding paragraphs may be advantageously applied to an investigation of the general character of the relations between the constants of vacuum tubes. In the following paragraphs will be reproduced some curves which have been experimentally obtained with the methods described above and which have been made upon a typical, widely used form of thermionic amplifier. Dr. Miller in illustrating the use of his methods, has already published some curves connecting the amplification coefficient and internal impedance with the plate voltage. The use of this independent variable gives a result, however, which is only a very small part of the whole story. Information of much greater value may be obtained by plotting the variation of the tube parameters in parallel with the static characteristic, or against E_g taken as the independent variable with E_p as parameter. A complete investigation for a given tube would involve the repetition of this process for various values of filament temperature. This introduces a total of three independent variables and involves considerable experimental labor, but this would be compensated for by the completeness of the study and fertility in deductions made possible. A complete and logical research might be planned upon this course of procedure with each tube, by varying the geometry of the electrode system, that is to say, by varying the grid-filament spacing with plate-filament spacing being regarded as a parameter. It is unnecessary to observe that a complete investigation undertaken along these lines, while involving considerable time and experimental labor, would permit of a correlation of structural methods and results which would be of the greatest value in the future design of vacuum tubes.

The general character of the variation of the tube constants defined above with the grid or controlling potential (E_p being taken as a parameter), is of primary interest in the study of a certain type of tube. In order to obviate the necessity for repeating this work with different values of filament temperature, one temperature should be selected for the work which is *optimum,* and represents the maximum value of the product of life and desirability, the latter being defined by the increase in the important constants. In the curves to be displayed, an investigation having previously been made on this point, a filament current was used which was considered to be an optimum value.

The family of *static* characteristic curves in Figure 5 are representative of the tube used.

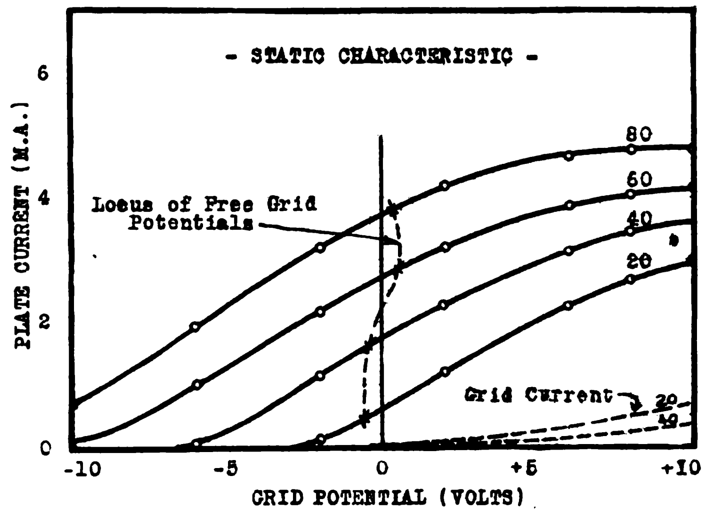

FIGURE 5

In this figure, two of the grid currents have been represented by the dotted curves. The free grid potentials or the potential which must be applied to the grid in order that the plate current may have a value identical with that which is obtained when the grid is disconnected, are represented by the dashed curve marked "locus of free grid potentials." This is a matter of secondary interest, but is sometimes helpful in making deductions concerning the operation of the tube.

The amplification coefficients were measured with the same values of plate potential and with the same independent variable, E_g. Miller's method proved very useful for this purpose. Great care is necessary in the arrangement of apparatus to reduce electrostatic induction effects which tend to obscure the true balance point. The value of such initial care in the set-up of the apparatus is well shown by the agreement between the final smoothed curve and the observation points. The results are depicted in Figure 6.

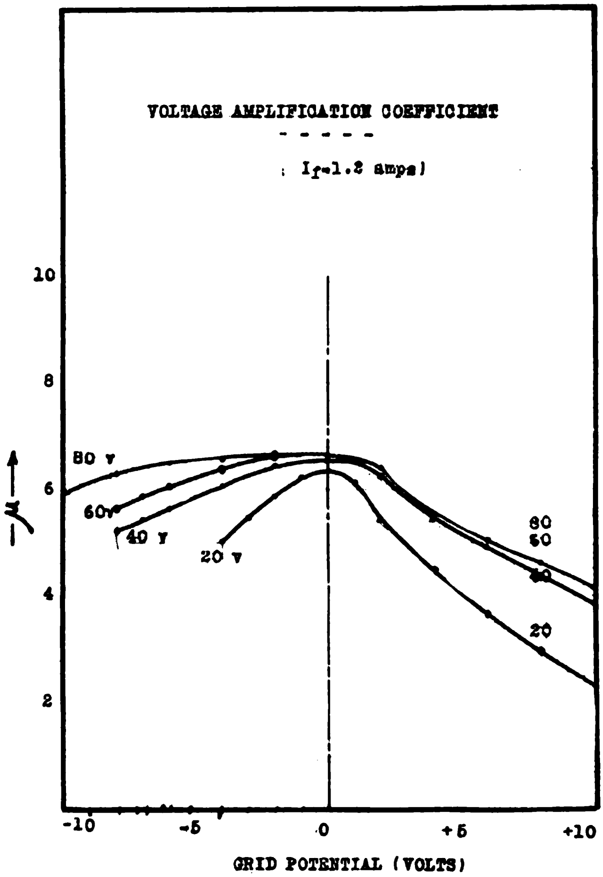

FIGURE 6

The salient feature of these curves lies in the fact that the amplification factor is not constant, as has been intimated from measurements with direct currents by Dr. van der Bijl, but depends greatly upon the grid potential in the region in which operation takes place. The falling off for positive values of grid potential and smaller plate voltages is particularly noticeable. The reason for this is not apparent, *a priori*, from the theory, since the relation between the effects of stray field and the grid potential would seem to be constant and independent of any value that the grid potential might assume. However, the equation upon which the theory rests represents conditions only within limited constraints on the characteristic surface, particularly in the region where the plate current is small and the grid potential is negative or near zero. After the point of inflection on the curve is passed, the expressions of Langmuir and van der Bijl are not even correct in form, so that a wide discrepancy between theory and the results is not seriously disturbing. The following explanation of the effect in question has suggested itself to the writer.

If the exact exponential variation of the plate current with respect to the inter-electrodic forces is assumed to be arbitrary, and the concept of the relative effects of the plate and grid potentials only is retained, we may represent the plate current as some function of these forces as follows:

$$I_p = f\,(E_p + \mu\, E_g) \tag{21}$$

The only matter of interest in this connection is the relative effects upon the plate current, of the grid and plate forces, since this determines the amplification of voltage in the tube. Assume that the distribution of forces and scalar potentials may be approximately represented by the diagram of Figure 7.

In this figure the inter-electrodic distances are represented by the abscissas, the ordinates representing the value of the scalar potentials, V. F is the emitting surface, and the dashed line represents the grid electrode. The positive potential is applied to the plate (P in the figure) and when the tube is cold the scalar potential may be represented by the straight line AB. In this case the grid is not connected. When the grid is connected (the practical case) the only field existing at G is that due to the plate potential acting thru the grid wires as pointed out by Dr. van der Bijl. This he has very appropriately termed the *stray field*. It is represented in the figure by the dotted line Ab. When the tube is hot and the surface F is emitting electrons, AB and Ab become roughly, as shown in the figure, the curves marked AB' and Ab', respectively. The curvature is due to the effect of the electronic atmosphere, which consti-

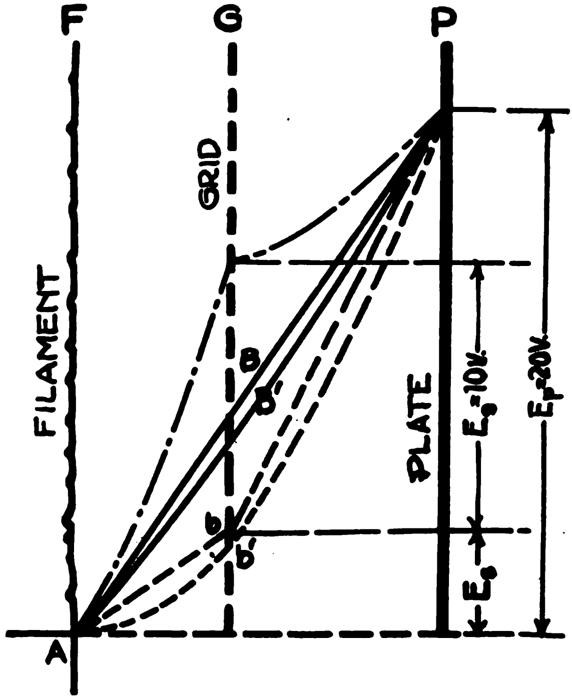

FIGURE 7

tutes a negative cloud. At the operating filament temperature the slope of the curve at the filament is finite and the saturation effect of the *space charge* is not reached when the grid is disconnected. When the grid is connected, however, space charge saturation between grid and filament is undoubtedly attained and the slope of this curve at the emitting surface is zero. The emission velocities at the surface of the filament are ignored as being inappreciable at a temperature of 1,300 degrees Kelvin, the point of operation. The reason for the slope of the potential function attaining a zero value when the emission is copious has been well brought out by Richardson,[7] with the aid of Poisson's equation. This is the initial state of affairs when the grid potential is zero. When a positive potential is placed upon the grid, the potential curves probably assume the general form shown in the figure by the dot and dash curves. In this case the vector potential, $\nabla \cdot V$, the force acting upon the

[7] Richardson, O. W., "The Emission of Electricity from Hot Bodies," Longmans, Green and Co., London, 1916.

electron is greater in the region between the grid and filament than in the plate-grid space, this increasing as the plate potential is decreased. There being now, more electrons in the plate grid space than before, the effect of space charge is beginning to be felt and the curve in this region sags slightly as shown. The force tending to overcome this condition is that due to the plate and for small plate voltages it seems readily possible for the slope of the potential curve at the grid to be zero. This means, of course, that there is no force operating upon the electron. The plate contributes nothing. This effect disappears at higher values of plate potential and the curve straightens out.

From the above statement of the condition of affairs in the inter-electrodic space, it would appear that when the potential on the grid is in the neighborhood of zero, and the grid-filament is enjoying the effects of space charge, the field intensity between plate and filament being high enough to attract strongly any electrons that manage to escape the congestion, we may expect the grid potential to have a greater effect upon the electronic flow than that of the plate. This carries with it the idea of a large value of μ from the relation (21) above. Also when the grid potential becomes positive with respect to the filament, the space charge effect mentioned is broken up and the conduction is increased. This is accompanied by a shift of the space charge from the filament-grid region to the plate-grid region, and, as noted above, the curve between the plate and grid electrodes sags to correspond. In this case there are plenty of electrons available at the grid plane, but the reactive force is the space charge. A slight increment in the plate force results in a partial neutralization of this effect by taking some of the electrons out of the space, so that now the plate potential becomes the predominating force since any increase in grid potential will obviously only increase the tendency to space charge. This results, of course, in a lower value of μ, which effect increases as the ratio $\frac{E_g}{E_p}$ decreases. The gist of the whole matter is that the space charge, with its accompanying saturation, shifts from one region to the other and changes to correspond, the relative effects of the grid and plate potentials and thus the value of μ.

The variation in the internal impedance is well shown by the curves of Figure 8.

The measurements represented were made with the aid of the bridge arrangement described in this paper, the minima obtained with Miller's method being ill defined. The results are homogenous and very satisfactory. The salient feature of the curves is that the resistance of the tube undergoes extreme variation,

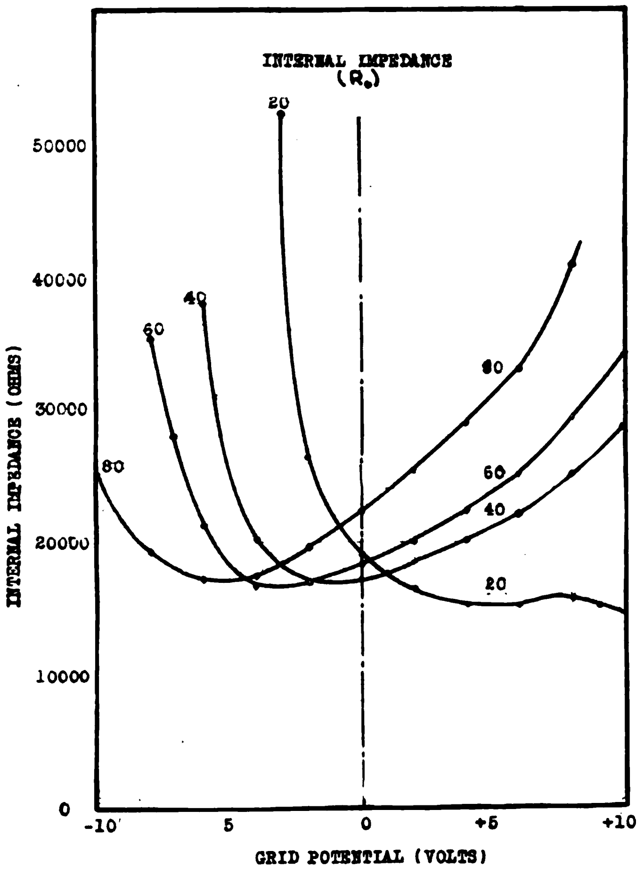

FIGURE 8

the minima representing roughly the points of inflection on the static characteristic curves.

The factor of prime importance in connection with the use of the tube as an amplifier or oscillator is the mutual conductance. This may be computed from the values of μ and R_o measured above from the relation (5) or it may be directly determined by the writer's method previously described. In Figure 9, the solid curves represent the results of computation from the amplification constant and the internal resistance, while the observed points, using the method in question, are represented by the small circles. The agreement between the two methods seems to be satisfactory and the results are very interesting.

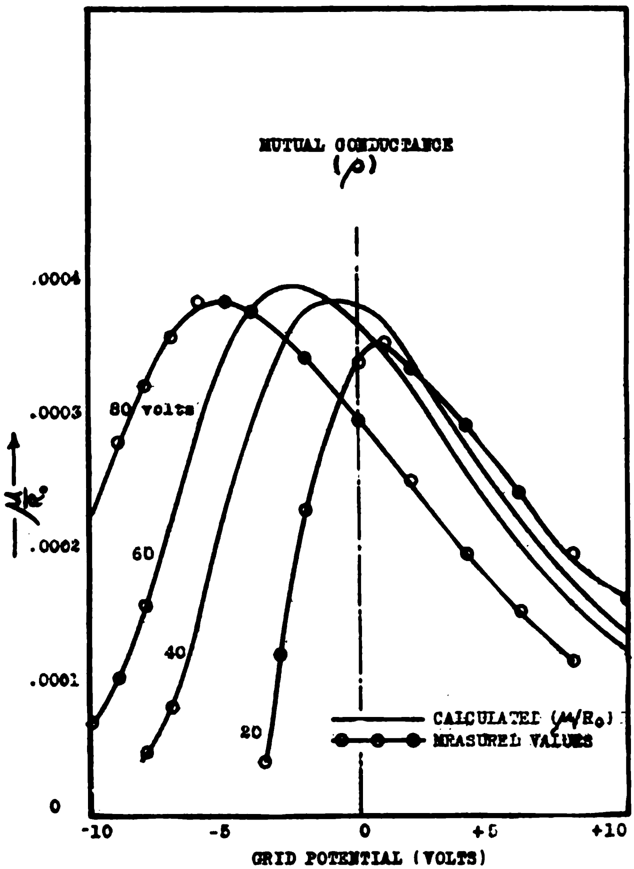

FIGURE 9

It will be noted that none of the maxima are very broad which shows that amplification at these points will be accompanied by distortion if an extended operating region is involved. Also when using the tube for amplification purposes in connection with radio receiving, the point of operation should be carefully selected by means of a biasing potential of suitable value applied to the grid. In selecting the operating region for this purpose, other considerations are pertinent. For instance, the maxima are not equally desirable since their magnitudes are different, hence the optimum value of plate potential should also be selected. This should also be preferably situated on the left hand side of the zero center line, so that the grid or input circuit offers no conductance when connected across the terminals of the condenser or the output circuit of the preceding tube. Having determined by intelligent methods upon the proper point for the operation of the tube, we may now proceed to design the connecting link between the tubes which will introduce the optimum impedance into the output circuit and give the maximum voltage output across the input circuit of the next tube. In this very practical matter, we are aided considerably by a knowledge of the internal resistance of the tube. The design of amplifiers is a matter of some engineering importance and will be considered at length in another paper.

The detecting constant, D, for operation without grid condenser may be evaluated by taking the slope of the tangent to the ρ curve at any point. This is not an entirely accurate procedure but is the best method at present available. The results of this method applied to the tube used in this work are represented in Figure 10.

The truth of the belief that the better operating point corresponds to low values of plate current is well brought out by these curves. Also it will be noted that low plate voltages are favorable to such operation. From the point of view of plate battery economy this is an important point. In general tubes operate better as detectors with grid condensers, and in special cases, with grid-leak resistances in addition.

The statement just made concerning the operation of the tube with grid condenser emphasizes the importance of studying detector constants under these conditions. This is somewhat of a tedious process but the results are interesting. The Wheatstone bridge arrangement described may be used to measure the effective resistance of the grid filament circuit at various grid potentials. The reciprocal of this quantity is the grid conductance, $\frac{I_g}{E_g}$. In Figure 11 this conductance has been plotted against positive values of grid potential.

There is no conduction for negative values of E_g if the emission is pure (that is, there is no positive emission) and the tube has been well evacuated, so that negative values of E_g have no interest. The slope of the conductance curve is evidently the second derivative of the grid current function with respect to the applied potential. This is one of the quantities entering

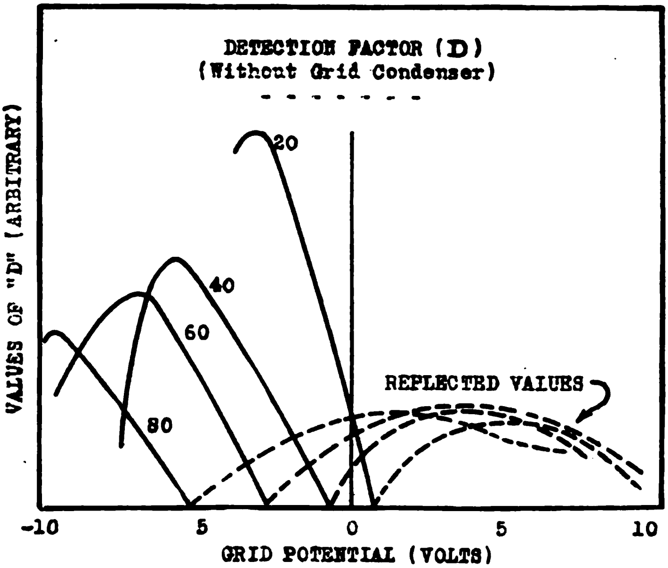

FIGURE 10

into the definition of the detecting constant with grid condenser. The dashed curves in Figure 11, represent the magnitude of the second derivative and have been derived from the conductance curves by taking the tangent at various points. It is interesting to note that the larger values correspond to the lower plate potentials. The complete definition of the detecting factor involves in addition to the grid rectification index, the slope of the $\frac{I_p}{E_g}$ curve, or ρ. The curves in Figure 12 represent the addition of this factor and exhibit completely the detecting action of the tube.

Several practical deductions may be immediately made from these results. In the first place, when the tube is to be employed as a detector (with grid condenser), a grid leak is desirable, especially when strong signals are to be impressed upon the system. This leak resistance should be placed between the grid and the positive leg of the filament and should have a value necessary to place the starting point of the grid potential at a point on or above the maxima on the curves in Figure 12. A

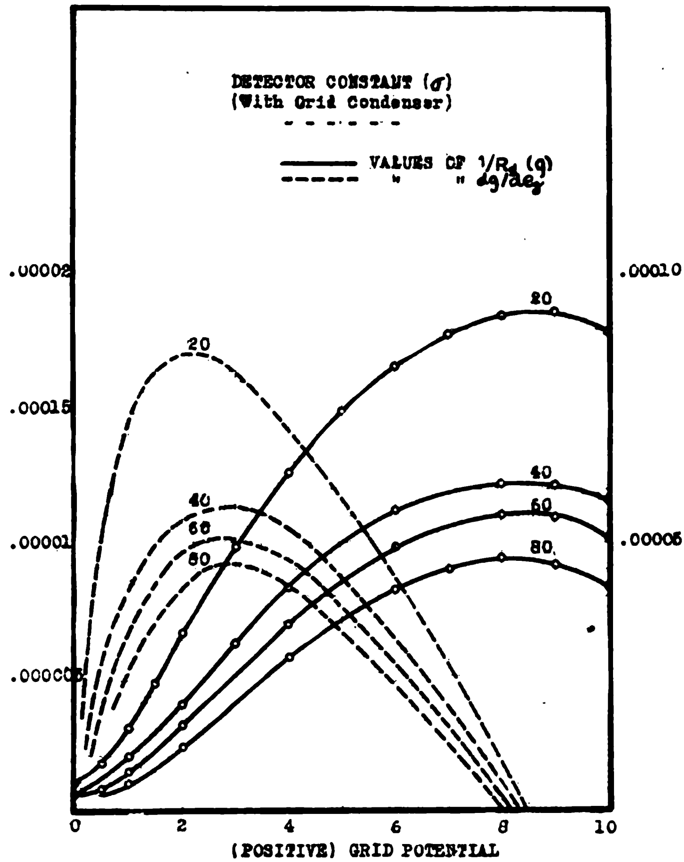

FIGURE 11

plate voltage of 20 or possibly less (this point not having been investigated) should be used, since this gives the greatest value of σ. In the case of weak signals, whether or not a grid leak should be employed is problematic, since theoretically when the grid current-voltage curve intersects the axis, $I_g = 0$ the curvature is infinite, and a singular point condition is obtained. The detecting factor is then infinite also, and the rectification is perfect. This will only be the case for very weak signals and for general purposes, it may be well to sacrifice if necessary, the response for weak signals to gain efficiency on the stronger signals by employing a grid leak resistance. In the region between the zero grid voltage and the voltage corresponding to maximum values of σ, the change in σ is great. Similarly after the maxima have

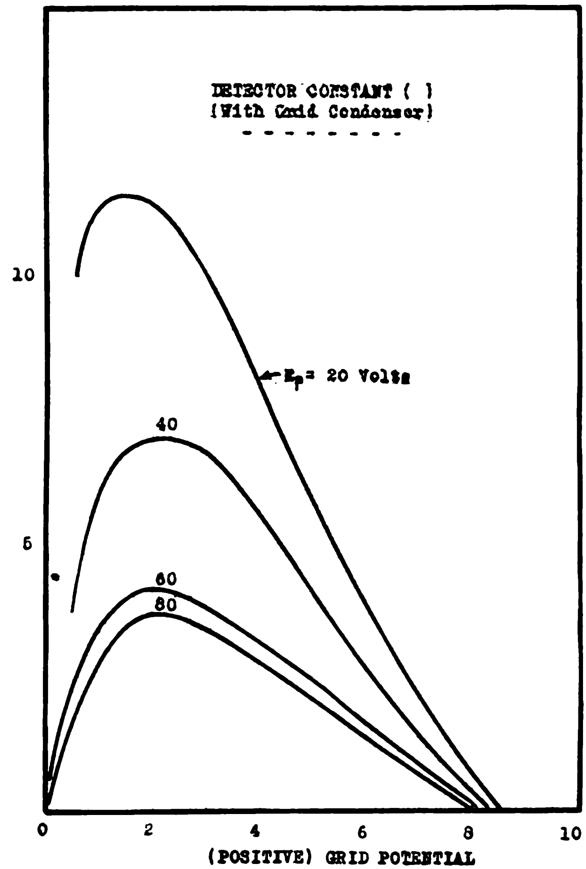

FIGURE 12

been passed, but in this case, the irregularity is not serious since the grid potential will shift to the negative when the charging starts, passing thru the maximum value and being augmented in the process. As a matter of fact, this consideration leads to the question as to whether or not it may be advantageous, in order to cover all possibilities in the matter of signal strength, deliberately to place the initial working point upon positive values of E_g beyond the maxima. Just how far this may be carried out without sacrificing to too great an extent, the response on small stimuli, will be largely a matter of experiment.

Philadelphia, Pennsylvania,
November 24, 1918.

SUMMARY: After giving the definitions of various tube constants for amplification and detection and the methods of measuring them, the author discusses certain special questions.

Among these are grid condensers and grid condenser leaks, and the variation of the various tube parameters with tube construction and operating conditions.

ADDENDUM I*

In the paper a method for measuring the mutual conductance was described which involved the use of a toroidal core balancing transformer. Since this was written an alternative arrangement has been devised which may be used for this purpose with equal success, and which does not require the special transformer of the original arrangement. This method should therefore be more popular with those who do not care to undertake the somewhat tedious task of constructing a transformer with a winding of five hundred turns on an iron torus.

The circuital arrangement is shown in Figure A:

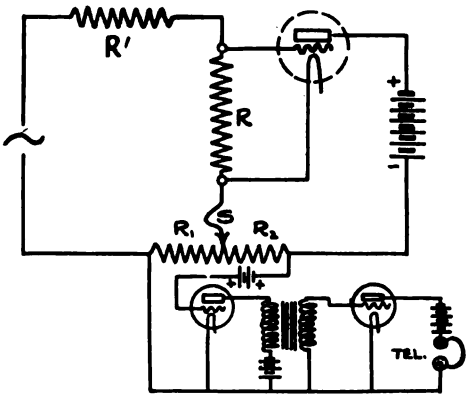

FIGURE A

* Received by the Editor, January 15, 1919.

Here, as will be noted, the transformer primaries, P_1P_2 have been replaced by the resistances, R_1 and R_2. The large resistance R' has been inserted into the supply circuit in order that the main current may not be greatly affected by the variation of R_1 in series. The resistance R_2 is variable and a part of the plate circuit, however, but being of the order of from 70 to 100 ohms with a possible variation of 30 ohms, its effect on the mutual conductance is quite ignorable. The resistance, R, across which the grid is connected is fixed at about 1,000 ohms, this value being chosen for the simplification of numerical computations of ρ from the settings.

The position of the sliding contact, S, is found for which the emf. across the terminals of the slide wire is zero and there is no indication in the telephones connected to the amplifier system. In order that this condition may be attained the emf. across the resistance R_2 must be equal in magnitude and opposite in sign to that across R_1. The proper phase relationship is provided by the mode of connection. The satisfaction of the other condition may be identified with the expression:

$$I\,R_1 = I_p\,R_2 \tag{a}$$

But, by definition,

$$I_p = \rho\,E_g = \rho\,R\,I \tag{b}$$

which, substituted in (a), leads to

$$I\,R_1 = \rho\,R\,I\,R_2 \tag{c}$$

and

$$\rho = \frac{1}{R} \cdot \frac{R_1}{R_2}. \tag{d}$$

Also, since R is purposely set at 1,000 ohms, we have finally

$$\rho = .001\,\frac{R_1}{R_2} \quad \text{amps/volt} \tag{e}$$

which is a very simple relation and therefore one particularly suited to the rapid execution of a large number of measurements.

The terminals of the slide wire, R_1R_2 are connected to the first tube of an audio frequency vacuum tube amplifier in the manner shown. It is rather important that the biasing battery, C, be connected in series with the grid circuit in order that the resistance of this circuit shall be infinite and there will be no tendency for the currents in the slide wire to flow thru the grid circuit, which would happen if the grid were allowed to assume a positive potential at the operating point. Under these conditions, if care has been taken in the arrangement of the wiring, the minima will be well defined. The resistance of the slide wire may well be of the order of 100 ohms so that the potential drops may be greater and a better indication may be had with a minimum of amplification. In the writer's set-up this slide wire consisted of the slide wire portion of the "Student's Potentiometer" manufactured by the Leeds-Northrup Company, which had a total resistance of 116 ohms, and which proved to be very well suited to the work having a scale graduated in 0.5 divisions from 0 to 100.

ADDENDUM II†

As an illustration of the great ease and simplicity with which problems relating to vacuum tube circuits may be handled with the aid of the fundamental plate circuit theorem and the conceptions of the voltage amplification factor and internal impedance, we may profitably consider the oscillation circuit depicted in Figure B, which has previously been treated by Hazeltine.[1]

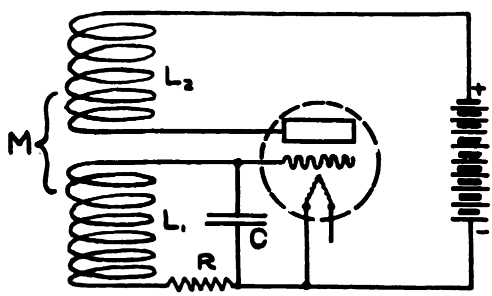

FIGURE B

This circuit is extensively used for receiving purposes in connection with the autodyne method of reception of continuous wave signals. It also enjoys extensive application in transmitting, but when used for this purpose, the entire characteristic curve is utilized, and the grid potential may vary over widely separated limits rendering the rigorous analysis of the operation difficult, if not impossible. In order to construct the differential equations for the potentials, it would be necessary first to formulate the dependence of the amplification factor and internal

† Received by the Editor, January 15, 1919.
[1] Hazeltine, previous citation, page 79.

impedance upon the instantaneous value of the grid potential, e_g. This could be done, possibly, with the aid of empirical expressions covering the variation of μ and R_o as experimentally determined, but it is doubtful if a simple integration of the equations so formed could be effected. For this reason and for the sake also of simplicity, in the present discussion it will be assumed that operation involves only a very small part of the characteristic surface, and that in the region considered the values of μ, and R_o are fixed. This is not so bold an approximation as it would at first seem, since the most favorable part of the static characteristic curve is at the point of inflection and in this region the amplification factor and internal impedance are practically constant. This is shown by the experimental curves displayed in the paper. After the oscillation has started, of course, the operating region expands until the slope of the curve decreases below the minimum possible value of ρ determined by the coefficients of the circuit. The problem consists in formulating the relation between the tube parameters and the circuital coefficients which will render sustained oscillation of the system possible. As mentioned above, this matter has been discussed by Hazeltine in a very valuable paper covering various types of oscillation circuits. In his discussion, the assumption is made that the plate current is a simple function of the grid voltage, the relation being $i_p = \rho\, e_g$. Any reaction existing between the two circuits is thereby not considered, a result being obtained, which, while very simple and useful, cannot be regarded as a complete definition of the criterion for the oscillation in actual systems. The following mode of treatment is more fundamental and the effect of reaction is completely determined.

Referring to Figure B, the circuital constants are denoted by the conventional symbols, the internal impedance of the plate circuit being inserted in series with the inductance, L_2 and an emf. of value $\mu\, e_g$, in accordance with the usual method of considering the plate circuit. The equations for the potentials may be constructed as follows:

$$L_1 \frac{di}{dt} + R\,i + \frac{1}{C}\int i \cdot dt + M\frac{di_p}{dt} = 0 \qquad (1)$$

$$L_2 \frac{di_p}{dt} + R_o\, i_p + M\frac{di}{dt} + \frac{\mu}{C}\int i \cdot dt = 0 \qquad (2)$$

since

$$e_p = \mu\, e_g = \frac{\mu}{C}\int i \cdot dt.$$

We are interested primarily in the variable, i, representing the current in the oscillatory circuit $L_1 C$, which may be isolated by solving (1) for $\frac{d i_p}{dt}$ and substituting the value found in (2). After multiplying thru by $(-M)$, differentiating twice with respect to t, and collecting, we have:

$$(L_1 L_2 - M^2)\frac{d^3 i}{dt^3} + (R L_2 + R_o L_1)\frac{d^2 i}{dt} + \left(\frac{R R_o C + L_2 - \mu M}{C}\right)\frac{di}{dt} + \frac{R_o}{C} i = 0 \qquad (3)$$

which may be placed in the more convenient form:

$$\frac{d^3 i}{dt^3} + \beta \frac{d^2 i}{dt} + \gamma \frac{di}{dt} + \delta = 0 \qquad (4)$$

This is a linear differential equation of a familiar type with constant coefficients in the operating region considered of the form:

$$\beta = \frac{R L_2 + R_o L_1}{L_1 L_2 a}$$
$$\gamma = \frac{R_o R C + L_2 - \mu M}{L_1 L_2 a C} \qquad (5)$$
$$\delta = \frac{R_o}{L_1 L_2 a C}$$

where a is the coefficient of leakage of dimensions, $1 - k^2$, and depends upon the coupling between the grid and plate circuits.

Proceeding as usual with an assumed integral of the form:

$$i = A \varepsilon^{a t} \qquad (6)$$

substitution in (4) yields the cubic auxiliary:

$$a^3 + \beta a^2 + \gamma a + \delta = 0 \qquad (7)$$

to be solved for the a's. The complete solution is:

$$i = A \varepsilon^{a_1 t} + B \varepsilon^{a_2 t} + C \varepsilon^{a_3 t} \qquad (8)$$

where A, B, and C are constants of integration to be evaluated, when amplitudes are of interest, from a knowledge of the initial configuration of the system. The solution of the cubic may be effected by Ferreo's method, but the expanded result being practically unmanageable, this method of attack may well be abandoned. We may obtain some light on the probable nature of the roots by considering the physical phenomena. In order that these may be real and periodic, as expected, it will be necessary for one of the roots to be real and the others complex. On this basis, we may write:

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= \Delta + j\,\omega \\ a_3 &= \Delta - j\,\omega \end{aligned} \tag{9}$$

A fundamental theorem in the theory of equations permits us to write also:

$$\begin{aligned} a_1 + a_2 + a_3 &= -\beta \\ a_1\,a_2 + a_1\,a_3 + a_2\,a_3 &= \gamma \\ a_1\,a_2\,a_3 &= -\delta \end{aligned} \tag{10}$$

which applied to the present case establishes the relations:

$$-a_1 - 2\,\Delta = \frac{R L_2 + R_o L_1}{L_1 L_2\,a} \tag{11}$$

$$2\,\Delta\,a_1 + \Delta^2 + \omega^2 = \frac{R\,R_o C + L_2 - \mu\,M}{L_1\,L_2\,a\,C} \tag{12}$$

$$-a_1\,(\Delta^2 + \omega^2) = \frac{R_o}{L_1\,L_2\,a\,C} \tag{13}$$

To approximate the roots of (7) we will resort to a very useful artifice employed by Dr. Fulton Cutting[2] in connection with his excellent treatment of the transient phenomena in radio transformer circuits, which consists essentially in taking the ratio of the equations (12) and (13)

$$-\frac{1}{a_1}\left(1 + \frac{2\,\Delta\,a_1}{\Delta^2 + \omega^2}\right) = \frac{R\,R_o C + L_2 - \mu\,M}{R_o} \tag{14}$$

thus obtaining a factor in the left hand member which is quite ignorable in comparison with unity. This is particularly permissible in the present case since, at the radio frequencies we are dealing with, ω is large compared with the damping. Proceeding with this we obtain:

$$-a_1 = \frac{R_o}{R\,R_o C + L_2 - \mu\,M} \tag{15}$$

and from the relation (11) the more important damping term:

$$\Delta = \frac{R_o}{2\,(R\,R_o C + L_2 - \mu\,M)} - \frac{R L_2 + R_o L_1}{2\,L_1 L_2\,a} \tag{16}$$

which are excellent approximations. The general solution may be written in the form:

$$i = A\,\varepsilon^{-a_1 t} + \varepsilon^{\Delta t}\,(B\,\varepsilon^{j\omega t} + C\,\varepsilon^{-j\omega t}) \tag{17}$$

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[2] Fulton Cutting, PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 4, page 160, 1916.

Physically, the first term in the above solution is of the nature of a transient and disappears when the oscillations have reached their equilibrium state. The second term is of particular interest since it represents the periodic phenomena under investigation. In order that the oscillations may be continuous and self sustained, it is necessary that the effective damping of the circuit, represented by Δ, shall be zero. The satisfaction of this condition leads to:

$$\frac{R_o}{R\,R_o C + L_2 - \mu M} = \frac{R L_2 + R_o L_1}{L_1 L_2\, a} \tag{18}$$

as an expression relating the coefficients of the circuit and the parameters of the tube for the condition of sustained oscillation. This may be solved for ρ giving:

$$\rho = \frac{\mu}{R_o} = \frac{RC}{M} + \frac{L_2}{M\,R_o} - \frac{L_1 L_2\, a}{M\,(R L_2 + R_o L_1)} \tag{19}$$

It is interesting to note that when the plate circuit inductance L_2 is ignored, and the induction effect into the grid circuit being retained thru the concept of a fictitious mutual inductance, this degenerates into:

$$\rho = \frac{C\,R}{M} \tag{20}$$

which is Hazeltine's result.

In most practical circuits, the term RL_2 may be ignored in comparison to $R_o L_1$ so that equation (19) may be simplified to:

$$\rho = \frac{C\,R}{M} + \frac{M}{L_1 R_o} \tag{21}$$

A graphical representation of this expression is shown in Figure C. In constructing this curve the following circuital constants were assumed:

$L_1 = 60\ \mu$h.

$L_2 = 180\ \mu$h.

$C = 0.0002\ \mu$f.

$R = 20\ \Omega$

$R_o = 15{,}000\ \Omega$

The component curves are shown by dashed lines, their dimensions being designated. The hyperbola is identical with the result obtained by Hazeltine, while the straight part increasing with M represents the effect of the reaction between the circuits. It is evident, from physical considerations, that an optimum value of M exists which is most favorable to oscilla-

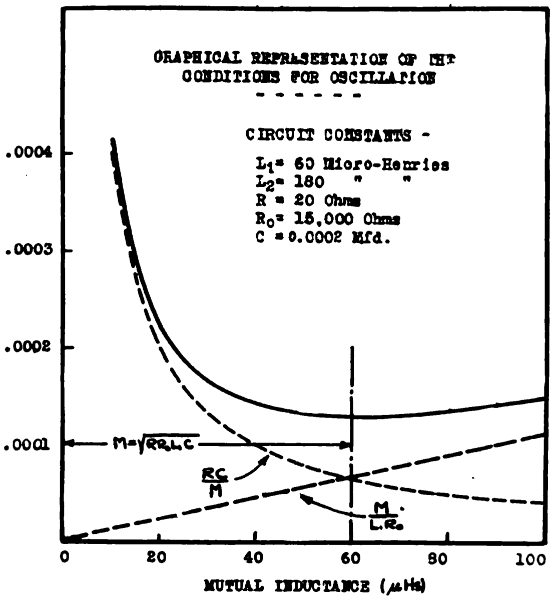

FIGURE C

tion. Differentiating (21) with respect to M and equating to zero we find this to be:

$$M_{opt} = \sqrt{R\,R_o L_1 C} \qquad (22)$$

If it is assumed that this value of M is selected, and substituted in equation (21) we find that the minimum ρ is:

$$\rho = 2\sqrt{\frac{C\,R}{L_1\,R_o}} \qquad (23)$$

or is just double its value in the absence of circuital reaction. Equation (22) may be re-written in terms of the wave length as:

$$M_{opt} = \frac{\lambda}{2\pi v}\sqrt{R\,R_o} \qquad (24)$$

In practical systems, a few trial calculations will show that the point of optimum mutual inductance is seldom found, the circuit used in the above curves being somewhat of an ultimate limit. If the inductance in the plate circuit is increased so as to increase the coupling, an optimum coupling could be obtained with higher wave lengths and higher resistances in the oscillatory circuit, were it not for the fact that this brings us to the consideration of the inherent capacity of the plate circuit acting across the plate inductance. Under these conditions the plate and grid circuits may be more or less in resonance and the above equations are not applicable. This does not decrease in any way the practical value of (21) since the reaction, no matter how small, is always effective in reducing the tendency toward oscillation.

The above equations may be further utilized in the study of the regenerative operation of the vacuum tube, by impressing emf.'s of suitable form upon the input circuit of the tube. The above roots are applicable in the formulation of the free solution, the forced solution being found in the usual way. This investigation will be undertaken in another paper.

SUMMARY OF ADDENDUMS: In a first addendum, the author shows a later method of measuring directly the mutual conductance of tubes, which method is free from certain possible objections of the earlier procedure.

In a second addendum, an autodyne oscillator with inductive coupling between grid and plate circuits is analytically considered. The condition for sustained oscillation is derived, a numerical illustration given, and the value of the optimum mutual inductance between grid and plate circuits for oscillation is obtained.

DISCUSSION

H. J. van der Bijl (by letter): Without entering into a full discussion of Mr. Ballantine's paper, I wish to call attention to a few outstanding points.

The method he gives for measuring the mutual conductance is very interesting and obviously allows of a simple and rapid determination of this important quantity. It might, however, be well to point out that the method, as it stands, is only applicable to cases in which the output impedance of the tube is large compared with that of the coil connected in the output circuit. As long as this is so, the alternating current, i_p, in the plate circuit can (for small impressed oscillations) be given by the simple expression used by Mr. Ballantine, namely: $i_p = \rho\, e_g$ where $\rho = \frac{\mu}{R_o}$, the mutual conductance, and e_g is the a. c. input voltage. If, on the other hand, the external impedance, Z, into which the tube works is not negligibly small, the current is given by

$$i_p = \frac{\mu\, e_g}{R_o + Z}$$

which follows directly from the amplification equations I gave in my paper cited. (μ as used here is the same as μ_o in my equations.) This then gives

$$\frac{R_o}{\mu} = \frac{e_g}{i_p} - \frac{Z}{\mu}$$

where $\frac{R_o}{\mu}$ is the inverse slope of the static characteristic and $\frac{e_g}{i_p}$ the inverse slope of the dynamic characteristic, or writing ρ for the true mutual conductance of the tube and ρ' for that of the tube and circuit:

$$\frac{1}{\rho} = \frac{1}{\rho'} - \frac{Z}{\mu}.$$

Mr. Ballantine does not state the impedance of the coil used in his experiments, but it was undoubtedly small in comparison with that of the tube which, at the voltages used, had a rather high impedance. The second term of the above equation can, however, become very disturbing when using low impedance tubes, and tubes having an impedance of a few thousand and even a few hundred ohms are not uncommon. In any case the external impedance Z must be made negligibly small, otherwise the necessity of taking it into consideration, thus compelling the use of the above equation, instead of the simple one given by Mr. Ballantine, destroys the usefulness of his method which aims at being quick and simple.

This consideration shows that the mutual conductance depends on the constants of the circuit in which the tube is operated, having a limiting value, $\frac{\mu}{R_o}$, which obtains when the tube works into a negligibly small impedance, and which can be characterized as the mutual conductance of the tube itself. The reason for this becomes readily apparent when considering that the external impedance tends to straighten out the characteristic. Even if the external impedance were a pure reactance, the dynamic characteristic would tend to straighten out and have a smaller slope than the static characteristic which would not be affected by the reactance. If, for example, the current be increased by increasing the grid potential, the voltage between filament and plate decreases, due to the increased voltage drop in the coil. In other words, both E_g and E_p in the characteristic equation are variables, E_p being always 180° out of phase with E_g. Mr. Ballantine's method could, of course, be used as it stands to determine the particular value the mutual conductance would have in any particular circuit, (say the circuit in which the tube is to be operated in practice) by choosing an impedance in the test circuit equal to that to be used in practice. This would, however, not be so simple as computing the mutual conductance of the tube itself from μ and R_o and then determining its value for any circuit with the help of the above equation. I think that a determination of μ and R_o is more important because a direct determination of the mutual conductance tells nothing about μ or R_o, the values of which must be known in order to determine the operating range of the tube, the impedance to be used in the output, and so on. On the other hand, a separate determination of μ and R_o allows the computation of the mutual conductance for any type of circuit. It is very important to distinguish between the static characteristic of the tube, which is governed by the fundamental characteristic equation, and the dynamic characteristic of the tube *and circuit*, which is governed by the amplification equations I gave in my paper in this issue of the PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS.

The other important point is the limits of operation of the tube. For example, referring to Figure 6 of his paper, which gives the amplification constant μ as a function of the grid voltage, Mr. Ballantine states that "The salient feature of these curves lies in the fact that the amplification factor is not constant, as has been intimated from measurements with direct currents, by Dr. van der Bijl, but depends greatly upon the grid potential in the region in which operation takes place." The amplification constant is a function only of the geometry of the tube, and must therefore be constant. When it is measured by the dynamic null method it will be found to be constant provided the tube is operated within the limits that I specified. These limits apply only to the grid voltage. (The equations are not by any means limited to small plate current.) As soon as the grid becomes sufficiently positive to take current the effective grid voltage is reduced, and unless this reduction is taken into consideration μ will come out too small, as can easily be seen by considering the circuit for the dynamic measurement of μ. It is obvious that this effect increases as the plate voltage is decreased because a larger plate voltage would have a greater influence in pulling the electrons thru the grid.

For negative values of the grid potential the limitation is imposed mainly by $\frac{E_p}{\mu}$, and will be more effective the lower the value of E_p. These considerations, I think, explain the variation in μ with grid potential observed by Mr. Ballantine. The tube with which he obtained his curves has a very small operating range of grid voltage which, of course, depends on the plate voltage used. In a tube of these general constants the value of ε in my characteristic equation becomes an important quantity, as it always is when $\frac{E_p}{\mu}+E_g$ becomes small. In fact ε depends to a great extent on the nature and surface condition of the electrodes. If it is positive the μ, E_g-curve will drop more readily for positive values of the grid potential, and if negative the effect will be on the negative side. I have obtained curves like these with a Western Electric Company tube of the telephone repeater type and found that with 150 volts on the plate, μ which had a value of 5.0 remained constant to within 10 per cent over a range of grid voltage from -20 to $+20$. This is because this type of tube has a greater operating range and is, therefore, not so easily overtaxed.

L. A. Hazeltine*: 1. MUTUAL CONDUCTANCE. The writer wishes to congratulate Mr. Ballantine on the clever arrangement (Figure 2) that he has devised for directly measuring the mutual conductance of an amplifier bulb. This method readily gives the mutual conductance for any combination of direct-current adjustment (grid potential, plate potential and heating current), as shown by Mr. Ballantine's curves. It may be noted, however, that mutual conductance in general depends also on certain conditions of the alternating-current circuits, as follows:

(a) *On the frequency.* It is conceivable that the mutual conductance obtained by a dynamic method might differ from the slope of the static characteristic curve obtained with direct current; and it is also conceivable that values at radio frequency might differ from those at audio frequency. While differences undoubtedly occur between the static and dynamie characteristics in bulbs showing positive ionization, no such frequency effects are to be expected in high-vacuum bulbs, nor have any been found, so far as the writer is aware. If desired, however, Mr. Ballantine's method may readily be extended to radio frequencies by substituting for the simple buzzer a buzzer-excited oscillating source and for the audio-frequency balancing transformer a radio-frequency transformer together with a detector. The method may also be extended to direct-current operation by substituting two resistances and a galvanometer for the balancing transformer and telephone; but some modifications are necessary on account of the normal direct current of the plate.

(b) *On the amplitude of oscillation.* With a low impressed alternating voltage, the variations of the grid voltage and plate current will be over such a short arc of the characteristic curve that this virtually coincides with its tangent. The slope of the latter therefore represents the mutual conductance for small amplitudes of oscillation. For higher impressed voltages the curvature of the characteristic will become appreciable, and the mutual conductance will then be rather crudely represented by the slope of the secant line connecting the extreme points of the oscillation[1]. If Mr. Ballantine's method is to be used for large amplitude of oscillation, the effect of wave distortion must be

*Revised and amplified from the oral discussion at the Institute meeting, New York, December 11, 1918. Received by the Editor, January 9, 1919.

[1]For the exact definition of the mutual conductance with large amplitudes of oscillation, see the writer's paper, PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 6, page 64, foot-note.

eliminated, either by sharply tuning the telephone circuit to the supply frequency or by substituting for it a sharply tuned vibration galvanometer.

(c) *On the presence of an alternating plate voltage.* If the plate potential is varied proportionally to the grid potential but in an opposite sense (as must happen in any use of the bulb in which power is given out from its plate circuit), the changes in plate current will be smaller than if the plate potential remained fixed. The "effective mutual conductance" (see Article 3 of this discussion) is therefore lowered by the presence of an alternating plate voltage. Mr. Ballantine's method can be extended to give this effective mutual conductance by including part of the resistance R (Figure 2) in the plate circuit.

In calling attention to the possible extensions of Mr. Ballantine's method, the writer appreciates that the low-amplitude value of mutual conductance, unmodified by the presence of an alternating plate voltage, is the most fundamental and most useful value, and that the simple method of the paper is far more likely to be of practical value than any of the extensions.

In regard to the symbol for mutual conductance, the writer still prefers g to ρ, considering it very desirable to denote quantities of the same physical dimensions by the same symbol. This has been found particularly true for mutual conductance, as it often occurs in formulas similar to those containing other conductances (for example, compare equations (26) and (27) below). In cases where more than one conductance enters in the same equation, different subscripts or other distinguishing marks may be used. The writer suggests denoting mutual conductance by g_m in such cases, and will use this notation in what follows.

2. Plate Resistance. The term "internal impedance" is open to two objections: (a) the bulb has *two* "internal" circuits, that of the grid as well as that of the plate; and (b) the impedance has the nature of a pure *resistance*, as is assumed in all proposed methods for measuring it, and so for clearness' sake should be called a resistance. The proper term for this constant would seem to be "plate resistance" and the appropriate symbol r_p. The use of the reciprocal, the plate conductance g_p, is sometimes more convenient.

In this connection it might be well to compare the two methods of viewing the plate circuit, considered as a source of alternating-current power. According to the first method, the alternating plate current I_p is made up of two terms, one due to the alternating grid voltage E_g and the other due to the alternating plate voltage E_p:

$$I_p = E_g g_m - E_p g_p, \qquad (1)^{2}$$

where g_m is the mutual conductance and g_p is the plate conductance. Interpreting this equation, we may say that *the plate circuit acts as a generator whose total current generated is $E_g g_m$, of which a part $E_p g_p$ is lost in a shunt path, or "leak," of conductance g_p.* To lead to the second method of viewing the action, we may solve equation (1) for E_p:

$$E_p = \frac{g_m}{g_p} E_g - \frac{I_p}{g_p}; \qquad (2)$$

or

$$E_p = \mu E_g - I_p r_p, \qquad (3)$$

where

$$\mu = \frac{g_m}{g_p} \qquad (4)$$

is the amplification constant, and

$$r_p = \frac{1}{g_p} \qquad (5)$$

is the plate resistance. Interpreting equation (3), we may say that *the plate circuit acts as a generator the generated or internal voltage of which is μE_g and the internal resistance of which is r_p.*[3]

These two methods are equivalent and therefore equally correct. That represented by equation (1) probably corresponds most closely to the physical action, and is most useful in treating oscillating circuits. That represented by equation (3) has the advantage that one of the constants (μ) is nearly independent of the direct-current adjustments of the bulb, and is most useful in treating amplifiers and detectors.

3. USE OF "EFFECTIVE MUTUAL CONDUCTANCE." In many calculations of thermionic oscillators it is found convenient

[2] This equation was given by the writer, PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 6, page 67. It is the alternating-current equivalent of Vallauri's equation, which in corresponding notation is

$$i_p = e_g g_m + e_p g_p + c.$$

Equations of this form seems to have been first employed by Latour, "Electrician," (London), December 1, 1916 (in discussing amplification), and by Bethenod, "La Lumière Electrique," December 16, 1916 (in discussing oscillation).

[3] The equivalent of equation (3), its derivation and its interpretation, were given by Miller, PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 6, page 143.

to deal with a single constant, rather than with two independent constants (g_m and g_p, or μ and r_p). That is, the "effective mutual conductance" g_m' is employed instead of the "normal mutual conductance" g_m, and is defined as the quotient of the (r.m.s.) plate current I_p by the (r.m.s.) grid voltage E_g, taking into account the presence of such alternating plate voltage as may exist under operating conditions:

Effective mutual conductance, $$g_m' = \frac{I_p}{E_g}. \tag{6}$$

Substituting (1),

$$g_m' = g_m \left(1 - \frac{g_p}{g_m} \cdot \frac{E_p}{E_g}\right); \tag{7}$$

or

$$g_m' = g_m \left(1 - \frac{n}{\mu}\right), \tag{8}$$

where

$$n = \frac{E_p}{E_g} \tag{9}$$

is the ratio of plate voltage to grid voltage as determined by the circuit conditions. The expression $\left(1 - \frac{n}{\mu}\right)$ thus enters as a correction factor, which frequently differs from unity by a percentage less than the uncertainties in the values of g_m with various bulbs or various adjustments, and so may then be disregarded. On the whole, the effective mutual conductance g_m' is of more service in the calculation of oscillating-current circuits than the normal value; it is the value given by the equations for g in the writer's Institute of Radio Engineers paper previously referred to, some of which are included in Figure 2 herewith.

It is interesting to observe that the effective mutual conductance g_m' for any chosen value of n may be determined with a single setting of Dr. Miller's apparatus, as follows. In Figure 1 herewith[4] the ratio r_2/r_1 is set at the desired value of n, and R is adjusted for a balance. We then have

$$g_m' = \frac{I_p}{E_g} = \frac{E_p/R}{E_g} = \frac{n}{R}. \tag{10}$$

Of course, if n is very small or very near μ [which it cannot exceed according to (8)], the precision will be low.

[4] Figure 1 of Dr. Miller's paper, PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 6, page 144, 1918.

Certain cases arise where it is necessary to employ the correction factor $\left(1-\frac{n}{\mu}\right)$ of equation (8), or to use a value of g_m' differing considerably from g_m. Three of these will be considered here.

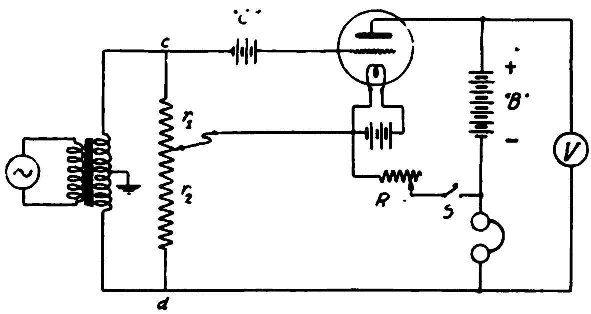

FIGURE 1

(*a*) *Optimum Mutual Inductance.* In Figure 2b herewith (Figure 3b of the writer's paper) the effective mutual conductance (denoted simply as g), is

$$g_m' = \frac{C\,r}{M}; \tag{11}$$

so the normal mutual conductance by (8) is

$$g_m = \frac{C\,r}{M\left(1-\frac{n}{\mu}\right)} = \frac{C\,r}{M\left(1-\frac{1}{\mu}\cdot\frac{M}{L}\right)}. \tag{12}$$

Considering M as variable, it can readily be shown that g_m will be a minimum (or the oscillation will be the strongest) when

$$\frac{M}{L} = \frac{\mu}{2}. \tag{13}$$

Substituting in (12), the value of g_m is found to be $\frac{2\,C\,r}{M}$ or just *twice* the effective mutual conductance in (11). In other words, to maintain a strong oscillation an optimum value of mutual inductance occurs, as given by (13); and for this value the effective mutual conductance has fallen to one half the normal value.[5]

[5]The existence of an optimum mutual inductance was first brought to the writer's attention by Mr. Ballantine, who has derived an expression for its value differing in form from equation (13) but equivalent thereto. (This is found on page 160 of Addendum 2 above.—Editor.)

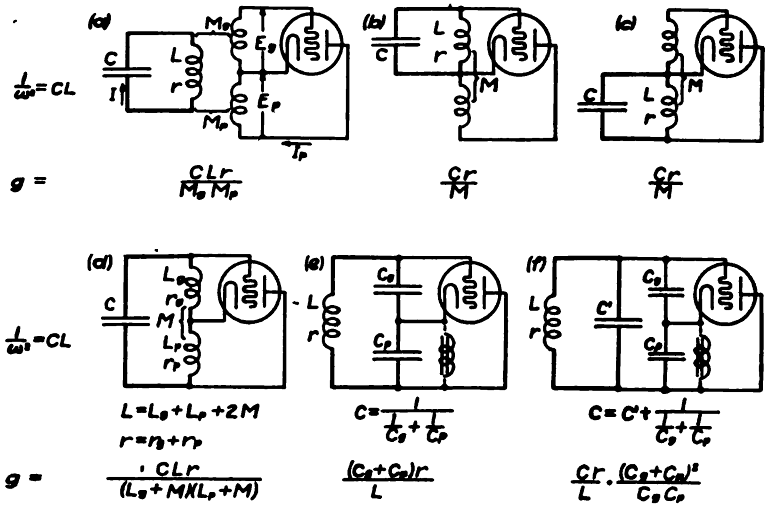

FIGURE 2

(*b*) *Optimum Voltage Division.* In Figure 2d herewith the effective mutual conductance is given as

$$g_m' = \frac{C L r}{(L_g+M)\,(L_p+M)} = \frac{C\,r}{L} \cdot \frac{(E_g+E_p)^2}{E_g E_p} \tag{14}$$

$$= \frac{C\,r}{L} \cdot \frac{(1+n)^2}{n}. \tag{15}$$

This is a minimum when $n=1$; so it has sometimes been thought that in circuits of this form the oscillation is the strongest when the total voltage is divided equally between the plate and the grid. However, introducing the correction factor, we have

$$g_m = \frac{C\,r}{L} \cdot \frac{(1+n)^2}{n\left(1-\frac{n}{\mu}\right)}, \tag{16}$$

which is a minimum for

$$n = \frac{\mu}{\mu+2}. \tag{17}$$

That is, the oscillation will be the strongest when the total voltage is divided between the plate and the grid in the ratio $\mu/(\mu+2)$, or about 0.8 for values of μ usually found in detector bulbs. For this case the correction factor becomes

$$1-\frac{n}{\mu} = \frac{\mu+1}{\mu+2}, \tag{18}$$

which is usually in the neighborhood of 0.9.

The above deduction applies also to Figure 2e with fair accuracy, but to Figure 2f only very roughly, especially when C_g and C_p are small compared with C' and when $C\,r/L$ is high. Exact treatment of the last case requires the complex method, for the reason that on account of the grid and plate conductances the voltages across C_g and C_p may be considerably out of phase with one another, contrary to the assumptions made in the "loss method" used in deriving the formulas of Figure 2. (Both theoretical and experimental results show optimum values of n varying over a considerable range as the circuit constants are varied, and sometimes more than one optimum value occurs in a given circuit.)

(*c*) *Maximum Output.* The circuit of Figure 2a herewith is frequently used in bulb transmitters, for which we wish to choose the circuit constants to secure the greatest output. The effective mutual conductance is given as

$$g_m' = \frac{C\,L\,r}{M_g\,M_p}; \qquad (19)$$

and the voltage ratio is evidently

$$n = \frac{M_p}{M_g}. \qquad (20)$$

Solving for M_g and M_p,

$$M_g = \sqrt{\frac{C\,L\,r}{g_m' n}} \qquad \text{and} \qquad M_p = \sqrt{\frac{C\,L\,r\,n}{g_m'\,n}}. \qquad (21)$$

Now the output of the bulb depends only on g_m' and n (as far as the radio-frequency circuit is concerned); so if their optimum values can be determined M_g and M_p can be at once computed by (21).

The writer has previously indicated[6] how the values of n and g_m' for greatest output could be determined by trial from the characteristic curves of the bulb. A more convenient method would probably be by direct experiment, using any of the circuits of Figure 2. Thus if Figure 2c is employed, we may insert a radio-frequency ammeter in the circuit (C, L) and vary M and C (or M and r) until a maximum output is obtained, varying also the direct-current quantities within permissible limits to determine their most desirable adjustments. When the best arrangement has been found, the desired values are

$$g_m' = \frac{C\,r}{M} \qquad (22)$$

[6] PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 6, pages 94 and 95.

and

$$n = \frac{L}{M}. \tag{23}$$

4. DETECTOR CONSTANTS. The constants which Mr. Ballantine denotes by D and σ and calls variously "detection," "detecting constant," "detector constant," "detecting action" and "detection factor," are open to criticism. Neither is defined directly in terms of the quantities in which the user of a detector is interested; and—more unfortunately still—these two constants, "detection (without grid condenser)" and "detection (grid condenser)," are not of the same physical dimensions and so are not numerically comparable with one another, as one might have expected. Several months ago the writer used the term "detector constant" in a private report to represent *the change in plate current per volt square (r.m.s.) impressed on the grid.* This definition leads to the following formulas:

$$\text{Detector constant without grid condenser} = \frac{1}{2} \cdot \frac{d^2 i_p}{d e_g^2}, \text{ and} \tag{24}$$

$$\text{Detector constant with grid condenser} = \frac{1}{2} \cdot \frac{d i_p}{d e_g} \cdot \frac{\dfrac{d^2 i_g}{d e^2_g}}{\dfrac{d i_g}{d e_g}} \tag{25}$$

(where small letters are used, instead of the corresponding capitals of Mr. Ballantine, to represent the instantaneous currents and voltages, as is customary in alternating-current circuit equations). The former [equation (24)] agrees with Mr. Ballantine's D except for the factor $\frac{1}{2}$; but the latter [equation (25)] differs from his σ not only by the factor $\frac{1}{2}$, but also by the factor $\frac{d i_g}{d e_g}$ in the denominator. Mr. Ballantine's omission of this factor is distinctly in error; for if two bulbs are alike in all other respects, that having the lower value of $\frac{d i_g}{d e_g}$ will be a correspondingly better detector—quite irrespective of any of the incidental features of operation which Mr. Ballantine mentions in connection with the definition of this constant.

Very recently the writer has used the term *rectification constant* to represent *the change in the grid generated (d.c.) voltage per volt square (r.m.s.) impressed* and considers this a more fundamental and more useful constant than the above "detector constant." The formulas for this constant without and with a grid or stopping condenser are respectively,

Mutual rectification constant, $$\nu_m = \frac{1}{2} \cdot \frac{\dfrac{d^2 i_p}{d e_g^2}}{\dfrac{d i_p}{d e_g}} = \frac{1}{2 g_m} \cdot \frac{d g_m}{d e_g}, \text{ and} \tag{26}$$

Grid rectification constant, $$\nu_g = \frac{1}{2} \cdot \frac{\dfrac{d^2 i_g}{d e_g^2}}{\dfrac{d i_g}{d e_g}} = \frac{1}{2 g_g} \cdot \frac{d g_g}{d e_g}. \tag{27}$$

The theory of rectification in various forms of detectors and methods of directly measuring the rectification constants are included in a paper now in preparation.

The connections of Figure 4 for determining σ are also open to criticism, for they fix the grid potential arbitrarily. Actually the potential of the grid is fixed by the grid leak resistance (added or inherent), whose value has a very marked influence on the value of σ. This is because the entire direct current of the grid circuit must return thru the leak; so the operating point on the grid characteristic curve will be that for which the quotient of grid potential by grid current is equal to the resistance of the leak. In the report referred to above, the writer used connections similar in principle to Figure 4, but elaborated so as to determine the actual operating potential of the grid and to measure the grid conductance at and near this potential. This method has been used by the writer and by others with satisfactory results.

In regard to the practical value of such "detector constants" or "rectification constants" as measures of a bulb's effectiveness in radio reception, the following may be stated. They do not tell the whole story; for the bulb affects the circuit in more than one way. These constants do, however, give definite measures of certain important features of the bulb; and, together with the other necessary bulb data, they permit the exact calculations of the over-all effectiveness of a bulb used in connection with any given radio receiver. The other independent bulb constants which affect its detecting action are the amplification constant, the grid conductance, and the plate conductance (or resistance). If, for example, we compare the detector action of a bulb without and with a stopping condenser, we find the rectification constant several times higher *with* the stopping condenser. On the other hand, when we calculate the effect of the high grid conductance inherent to operation with a stopping condenser, we find that it greatly reduces the radio-frequency voltage received by the grid and so may more than compensate for the better rectification. This agrees with practical experience, especially with grid circuits having a high quotient of self-inductance by capacity; for in such cases the grid loss may be several times all other radio-frequency losses put together, causing a marked diminution in impressed grid voltage.

V. Bush: Mr. Ballantine's very interesting paper is of particular service in gathering together and carefully defining the various tube constants which have been used by many writers. It is also a step toward a consistent accepted nomenclature of the subject, which will prove very necessary if we are to avoid confusion.

It would be an additional help if Mr. Ballantine could throw some light upon the following point: It seems to me that there should be certain standard conditions under which the constants of a tube are defined. We give the regulation of a generator, for instance, as the rise in terminal voltage from full load to no load under normal speed and field excitation. Could we not similarly more definitely define the internal impedance of a tube as the slope of the E_p, I_p curve under analogous standard conditions as regards plate voltage, filament current, and grid potential? Is it not true that the so-called constants of the tube reviewed in the paper are as yet not constants at all; but that we must give their values by means of a large number of curves for various conditions in order to describe completely the electrical operation of a given tube? Much of this is undoubtedly due to the versatility of the three element tube, so that many statements are needed to describe its action completely. Thus, for instance, we have no definite constant as yet for specifying output capacity. It is to be hoped however that as the art develops some definite set of constants and curves may be settled upon, as few in number as possible, which will describe briefly and completely the electrical behavior of a given tube under any set of conditions usually met with in practice, without adding a great deal of information which will not be needed.

With the ordinary generator we can of course go much further. There are various mechanical constants such as peripheral speed the constants of the iron, numbers of turns, and so on; and when these are known we can by means of well known formulas express the electrical constants in terms of them. That is, we can design consistently in order to obtain a generator for any desired purpose. It is of course too early to hope for such design formulas for thermionic tubes, for they are at present largely empirical.

Mr. Ballantine's paper will assist, however, toward the time when we will order a tube for a given service in exactly the same way that we now specify the behavior of a generator.

Edward Bennett (by letter)*: The use of uniform names and symbols for the constants of tri-electrode amplifiers is a matter of such importance that I venture to supplement the compilation of terms and symbols which the paper contains by a statement of the nomenclature which has been found very useful in an extended investigation of amplifiers under conditions in which the grid current cannot be neglected.

This nomenclature is as follows:

The ratio of the grid alternating current to the grid alternating potential (the plate potential being kept constant) is called the *grid conductance*, G_g. If ΔI_g and ΔE_g represent corresponding increments in the grid current and grid potential as read off from the continuous potential characteristics, then

$$\frac{\Delta I_g}{\Delta E_g} \text{ is represented by } G_g \tag{1}$$

The ratio of the plate alternating current to the plate alternating potential (the grid potential being kept constant) is called the *plate conductance*, G_p.

$$\frac{\Delta I_p}{\Delta E_p} \text{ is represented by } G_p \tag{2}$$

The ratio of the plate alternating current to the grid alternating potential (the plate potential being kept constant) is called the *controlled conductance of the plate by the grid*, or briefly the *controlled plate conductance*, G_{cp}.

$$\frac{\Delta I_p}{\Delta E_g} \text{ is represented by } G_{cp} \tag{3}$$

The ratio of the grid alternating current to the plate alternating potential (the grid potential being kept constant) is called the *controlled conductance of the grid by the plate*, or briefly the *controlled grid conductance*, G_{cg}.

$$\frac{\Delta I_g}{\Delta E_p} \text{ is represented by } G_{cg} \tag{4}$$

* Received by the Editor, January 16, 1919.

For the use of the term "conductance" in connection with the constants of amplifiers we are indebted to Professor Hazeltine. The quantity which in the above notation is termed the *controlled plate conductance* is by Professor Hazeltine termed the *mutual conductance*. It seems to me that the term mutual is open to objection in that the effect is not a mutual effect in the same sense as in mutual inductance or mutual elastance.

As an illustration of the utility of the constants defined above, consider their application to the simple case of non-regenerative power amplification in the circuit of Figure 1. In this figure A represents the source delivering the power which is to be amplified. The resistance, R, in the plate circuit represents the element to which the amplified power is to be delivered.

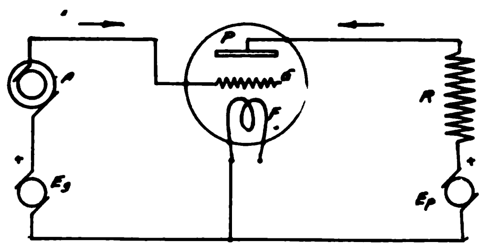

FIGURE 1

Let E represent the r.m.s. value of the alternating voltage of the source A supplying the power which is to be amplified.

Assuming for the moment that no variation occurs in the voltage between the plate P and filament F, the voltage E of the source impressed in the grid circuit will cause the following currents to flow:

$$I_{p1}=G_{cp}E$$

$$I_{g1}=G_gE$$

The passage of the alternating current of the value I_p thru the resistance R will, however, cause the plate voltage to vary by the amount,

$$E_1=-I_p R$$

This variation of the plate voltage will give rise to the following plate and grid currents,

$$I_{p2}=(-I_p R)G_p$$

$$I_{g2}=(-I_p R)G_{cg}$$

Whence the resultant plate and grid currents are as follows,

$$I_p \,(=I_{p1}+I_{p2}) = G_{cp}E - I_p RG_p$$

or

$$I_p = \frac{G_{cp}E}{1+G_p R} \tag{5}$$

$$I_g \,(=I_{g1}+I_{g2}) = G_g E - I_p RG_{cg}$$

or

$$I_g = G_g E - \frac{RG_{cp}G_{cg}E}{1+G_p R} \tag{6}$$

The power expended in the resistance is $I_p^2 R = \left(\frac{EG_{cp}}{1+RG_p}\right)^2 R$

The power delivered by the source A is

$$E\,I_g = E^2 G_g - \frac{RG_{cp}G_{cg}}{1+RG_p} E^2$$

The power amplification =

$$\frac{I_p^2 R}{E I_g} = \frac{G_{cp}^2 R}{(1+G_p R)\,(G_g + G_g G_p R - G_{cp} G_{cg} R)} \tag{7}$$

The value which the resistance R in the plate circuit must have in order to lead to the maximum possible amplification of the power may be determined by taking the derivative of the amplification with respect to R, equating the derivative to zero, and solving the resulting equation for the value of R.

The value of R for maximum power amplification is found to be,

$$R_m = \frac{1}{G_p\sqrt{1 - \frac{G_{cp}G_{cg}}{G_p\;G_g}}} \tag{8}$$

Substituting this value of R in equation (3), the expression for the maximum amplification is found to be,

Maximum power amplification =

$$\frac{G_{cp}^2}{G_g G_p\left[1+\sqrt{1-\frac{G_{cp}G_{cg}}{G_p\;G_g}}\right]^2} \tag{9}$$

In most tri-electrode devices, the fraction $\left(\frac{G_{cp}}{G_p}\right)\left(\frac{G_{cg}}{G_g}\right)$ is small in comparison with unity because of the small value of G_{cg}, or of the second fraction in comparison with the first.

Under these conditions the following approximate expressions may be written for maximum amplification.

$$R_m \text{ should equal } \frac{1}{G_p} \tag{8a}$$

$$\text{Maximum power amplification} = \frac{G_{cp}^2}{4G_g G_p} \tag{9a}$$

$$\text{The corresponding voltage amplification} = \frac{G_{cp}}{2G_p} \tag{6a}$$

The maximum voltage amplification, namely, $\frac{G_{cp}}{G_p}$, occurs only when the resistance R is made infinitely great; in this case the power amplification is 0.

Stuart Ballantine (by letter):* Dr. van der Bijl's remarks relative to the presence of inductance in the plate circuit of the vacuum tube under measurement with its concomitant effect on the value of the mutual conductance obtained, reflect a popular view concerning the accuracy of the method which is quite erroneous. It is true that the output circuit does contain the winding P_2 of the balancing transformer, which, in the absence of the supply voltage, e, will cause the current in this circuit to assume another value and lag behind the plate voltage by a small angle which depends upon the internal resistance of the tube. However, it is to be noted, that in the presence of a current in the winding P_1, and in particular when a condition of balance is attained (which is the state of affairs of primary interest), the flux in the core is reduced to zero and there is no inductive effect in the plate circuit, no magnetic circuit loss and no serious reaction between the primary and secondary circuits involved. The circuital reactions that do exist conspire to preserve the accuracy of the method by their mutual destruction of inductive effects. Few ignorations are therefore necessary and the values of ρ obtained are representative of the tube and not the circuit.

Looking at the thing from another point of view, it is fairly evident that the main current I, flowing thru the winding P_1 induces an emf. in the plate circuit which is of the proper phase to react upon the emf. drop due to the inductance of the winding, P_2. When a perfect balance is obtained, as indicated by silence in the telephones, the neutralization is complete and the plate circuit acts as tho it was devoid of inductance. This may be demonstrated mathematically from the circuital solution, but since space is valuable, reference to the vector diagram of Figure 1 may serve to illustrate the truth of these remarks.

* Received by the Editor, February 18, 1919.

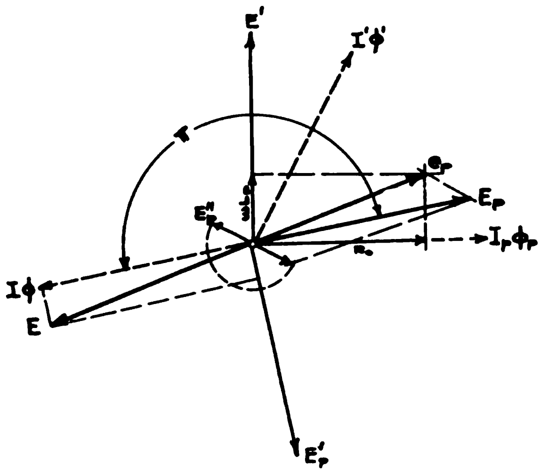

FIGURE 1

In the construction of this diagram no attention has been paid to the proper representation of magnitudes, the phase relations being of paramount interest.

The impressed forces E_p and E are the plate and supply voltages respectively. In addition e_p is the impressed force in the plate circuit on the assumption that there is no reaction from the other primary circuit. When the supply circuit is connected without the emf., e, the flux induces therein the force E' which produces the current, I' and the flux ϕ'. This in turn results in the reactive emf. in the plate circuit, E_p''. The component due to this reaction when combined with e_p gives the final resultant impressed force, E_p. The circuit, as is well known, then acts as if its inductance were reduced and its resistance increased. Due to the connection of the grid across the resistance R in the main circuit, the relation between the main current I and the plate voltage is fixed as shown. The emf. in the main circuit is E; the current produced is I which induces the force E_p' in the plate circuit. Of particular interest here is the fact that this induction is opposed to the normal inductance drop in the plate circuit. This is a crude method of illustrating why the objections mentioned by Dr. van der Bijl tend to disappear when the condition of balance and zero flux is attained. With certain connections, and circuit constants, slight errors due to other causes do exist, as indicated mathematically, but their magnitude is of no practical importance. Furthermore, the method described makes no pretense at much precision, and is sufficiently accurate for most purposes.

In connection with the other point brought out in Dr. van der Bijl's discussion, I cannot agree with him that the "amplification constant is a function only of the geometry of the tube." It seems to me that the important consideration is the relation between the electronic flow and the forces produced by the electrodes, which is not a simple function of their spatial relation, but depends as has been clearly demonstrated by Richardson, also upon the distribution and congestion of the particles themselves. In constructing the explanation given in the paper to account for the falling off of μ for positive grid potentials I was greatly influenced by this view. Furthermore, I think that a few trial computations will indicate that Dr. van der Bijl's explanation based entirely upon the conductivity of the grid circuit, is quite fallacious. As a matter of fact, this was also the first explanation that occurred to me, suggested probably by the increase in the effect for decreasing plate potentials. The "effective" value of the grid potential is, of course, $\frac{R_1 R_g I}{R_1 + R_g}$ instead of $R_1 I$ as was assumed in my measurements. The slide wire used had a total resistance of about 116 ohms; the resistance of the grid circuit was in no instance lower than 10,000 ohms and in most cases was considerably above this (the exact values are deducible from Figure 11). After correcting for the grid conductance I was surprised to find that the resulting curve was identical with the original one almost within the normal width of the curve itself. I am convinced, therefore, that the effect is negligible and that any explanation based solely upon its consideration is inadequate to account for the pronounced falling off of μ observed. I have but limited faith in the explanation given in the paper to account for this effect and do not wish to contend that this is the correct and final answer to the question. It is frankly crude and explains only the falling off for small plate voltages; the variation for constant plate voltage and variable grid potential still remains unexplained. Any theory for this latter effect must indeed be possessed of remarkable flexibility in order to account for some of the vagaries that have been observed.

In order to give Dr. van der Bijl's theory a fair test I made some very careful measurements of μ with tubes having cylindrical electrodes. This type of constructions gives a relation between the plate current (I_p) and the grid potential (E_g) which follows most faithfully the relation formulated in Dr. van der Bijl's theory. On account of this agreement with the theory, we should expect the amplification factor to remain constant as the theory predicts. The results are displayed in Figure 2.

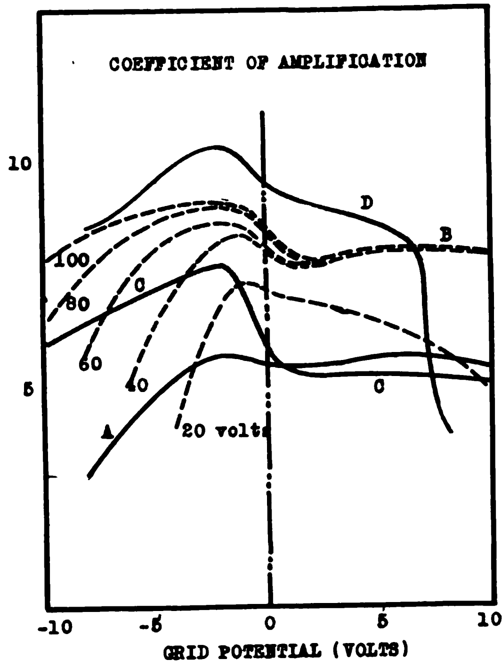

FIGURE 2

The curves "A" and "B" were obtained on tubes with similar plate dimensions and with similar plate-filament spaces. In the case of "A," however, the grid-filament space is greater (by 0.5 millimeter or 0.02 inch) than in the case of tube "B." The falling off of μ for low plate voltages is still noticeable altho not so pronounced as the curves shown in Figure 6, which were obtained with a tube having plane electrodes. The full family of dotted curves is given for tube "B" so that the effect of plate voltage may be indicated. The remaining curves, "C" and "D" were taken with two other tubes of foreign design, tube "D" being of cylindrical form and tube "C" having electrodes of circular form in unilateral arrangement with respect to the filament. All of these curves were taken with the tubes operating on a plate potential of 80 volts. The very pronounced falling off shown by curve "D" is attributable to an incorrect filament temperature. Raising the filament temperature caused this part of the curve to become horizontal. The filaments of each of the four tubes were made of tungsten and an effort was made to adjust them to the same intrinsic brilliancy. The four curves show a mutual tendency to rise in the region of -2 volts on the grid. After this point is reached, as the grid potential is increased, a sudden fall is noticeable, which is apparently inexplicable and certainly not due to grid conductance since a sensitive galvanometer inserted in the grid circuit shows no evidence of grid current.

I am greatly interested in this matter but feel that too much space has already been consumed in the discussion of an abstract question which is properly a matter of pure science; from an engineering point of view, the variation of μ is not important, particularly in the positive plane. The amplification factor is generally of minor importance in the specification of tube merit, the mutual conductance being a dominant consideration.

Professor Hazeltine's interest in the arrangement shown in Figure 2 is very gratifying. Also his discussion on the influence of the frequency and amplitude of the measuring current are interesting but I have found that in all but very extraordinary cases, these effects are of secondary importance. With gas tubes, in which the phenomenon of impact ionization is not completely negligible, the case is quite different. As first intimated by Vallauri,[1] the hysteresis and viscosity effects then become prominent and the results are very erratic. Fortunately, present day tendencies are directed away from the gas tube, so that this does not become a matter of great concern.

The matter of treating analytically problems relating to vacuum tube circuits seems to resolve itself into a question of individual preference. To one accustomed to the use of straightforward methods of solution by means of differential equations or complex-imaginary algebra (in the case of forced solutions), the conceptions of the coefficient of amplification and internal impedance are of great value. The method which seems to be preferred by Professor Hazeltine involving an "effective" mutual conductance possesses the very important advantage of mathematical simplicity, yet the physical reason-

[1] Vallauri, "L'Elettrotecnica," volume 4, number 18, page 335, 1917.

ing which should parallel any mathematical investigation is much more involved in its nature. In this method, the effect of the reactive emfs. and other extraneous emfs. in the plate circuit require formulation as a correction factor to the normal mutual conductance of the tube. I have always preferred the other method since it seems to be more fundamental and involves the parameters of the tube as fundamental quantities. An example of this mode of dealing with such problems is presented in the treatment of the inductively coupled oscillation circuit given in Addendum 2 to the paper.

Professor Hazeltine has remarked that the two methods of treatment give results which are identical. His execution of the investigation is formally correct, and yet the result he obtains is equivalent to that given in Addendum 2, which is only approximate. The necessity for approximation in my investigation was generated in the solution of the cubic. The extent of the error made is not serious, but it is known to be present. In Professor Hazeltine's treatment no approximations are made so that the lack of rigor becomes apparent. The value of the normal mutual conductance given by Professor Hazeltine is:

$$g_m = \frac{C\,r}{M\left(1 - \frac{1}{\mu} \cdot \frac{M}{L}\right)} \tag{a}$$

which is equivalent to that given in the Addendum:

$$g_m = \frac{C\,r}{M} + \frac{M}{L\,R_p} \tag{b}$$

Proper substitution for μ in (a) transforms it to (b). In other respects Professor Hazeltine's deductions are identical to those which I have made.

The standardization of vacuum tube nomenclature is a matter which is destined to become of increasing importance as this remarkable device is developed. I am inclined to agree with the arguments advanced in Professor Hazeltine's discussion, covering the mutual conductance and internal impedance. The symbol μ for the amplification factor as given originally by Dr. van der Bijl seems to remain satisfactory. The term "internal impedance" is perhaps a misnomer, and should be properly changed to *plate resistance.* The impedance of the tube at radio frequencies is rather complex in form and cannot be represented by a pure resistance as is done at audio frequencies, so that, as Professor Hazeltine remarks, the resistance symbol is quite inappropriate. The symbol R_p is undoubtedly the best suggestion.

I do not feel competent to join in Professor Hazeltine's erudite discussion of the functioning of the tube as a detector in the grid condenser connection. I feel that a derivation of the definition for the merit of the tube is not possible in the absence of an explicit specification of the constants of the associated radio frequency circuit. Even if this information was supplied, the resulting expression involving these circuital coefficients would not be representative of the tube itself, but would represent the operation of the circuit as a whole. The criticism of the definition given in the paper for the detector constant *without* grid condenser is well founded. The result which I gave is erroneous, altho the reasoning based upon Figure 3 is perfectly accurate and leads quite naturally to:

$$D = \frac{1}{2} \frac{d^2 i_p}{d e_g^2} \qquad \text{(c)}$$

as a definition of the detector constant. The physical structure of the published definition is correct, the error being simply one of quantity. This definition may be checked by assuming the plate current (i_p) to be any function of the grid potential (e_g) as follows:

$$i_p = f(e_g) \qquad (d)$$

which is expressible in a power series of the form:

$$i_p = A_o + A_1 e_g + A_2 e_g^2 \quad \cdot \quad \cdot \quad \cdot \quad A_n e_g^n \qquad \text{(e)}$$

if the coefficients are experimentally evaluated, this may be written in a Maclaurin expansion as follows:

$$i_p = I_o + \frac{d i_p}{d e_g} e_g + \frac{1}{2} \frac{d^2 i_p}{d e_g^2} e_g^2 + \frac{1}{6} \frac{d^3 i_p}{d e_g^3} e_g^3 + \quad \cdot \quad \cdot \quad \cdot \qquad \text{(f)}$$

The first few terms are of interest only, the indications being that the convergence is complete and rapid in the region in which we are particularly interested. The first term represents the normal plate current at the point of operation; the second term, which is proportional to ρ may be regarded as the amplification. The third term, proportional to the second derivative of the I_p, E_g characteristic apparently represents the rectification and is seen to be identical with (c) above. Succeeding terms may be combined in similar order with the amplification and rectification terms.

In connection with the definition of the detector constant with grid condenser, I do not believe that the phenomenon of detection is quite so simple as indicated by Professor Hazeltine's definitions. In constructing the definition which I used in plotting the variation of σ with the grid voltage, my primary object was not to derive something which was quantitatively comparable to the detector constant without grid condenser, but to express something which was of some physical significance and which could be plotted to show the variation of the factor from one part of the characteristic surface to the other. As stated above. an explicit consideration of the quantitative nature of this factor is not possible without analyzing the entire circuit and any expression to possess the desired utility must involve such factors as L, C, R, R_g, and C_g. The magnitude of the detected output is undoubtedly proportional to the extreme excursion of the grid potential in the negative direction regarded as an argument, with the concomitant change in the plate current as the result, this latter effect being proportional to the mutual conductance. The grid potential charge is proportional to the rectification in the grid circuit so that the superposition of these characteristics seems to be of definite physical significance. Professor Hazeltine's remarks evidence a firmer belief in the value of this detecting factor than I have expressed. I have already elaborated my views concerning this so that further discussion is not necessary. If he believes that the grid potential decrease is proportional to the strength of the oscillation and its decrement and is not influenced by the change in detector factor with changing grid potential I do not think that he has given the matter adequate consideration.

Professor Bush's remarks relative to the desirability of being able to specify the performance of vacuum tubes briefly, and without the use of a number of experimental curves, touch upon a very practical matter. It seems to me that the best method of informing the user concerning the capabilities of the device in its various connections as detector, amplifier, modulator, and oscillation generator, would consist in providing each tube with a name plate giving the following information:

	Oscillator	Detector	Modulator	Amplifier
Figure of Merit.......	ρ, I_p (*max.*)	D, σ	*	ρ
Plate Resistance......	R_p	..	..	..
Amplification.........	μ	..	..	..
Grid Voltage.........	± volts	..	..	..
Plate Voltage.........	+ volts	..	..	..
Filament Current....	Amps.	Amps.	Amps.	Amps.

The factors enumerated are to be measured at the plate and grid potentials corresponding to their maximum values since these are of main interest. This system would not only furnish the figures of merit of the device but would also advise the user of the tube at what part of the characteristic surface to operate. This is obviously desirable and should be helpful in reducing the amount of experimenting necessary to arrange for the proper operating conditions.

* See paper by Dr. J. R. Carson, PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS, volume 7, number 2, 1919.

A THEORETICAL STUDY OF THE THREE-ELEMENT VACUUM TUBE*

By
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The device with the theory of which the present paper is concerned is termed, in default of a generally accepted name, the three-element vacuum tube. Structurally, as is well known, it consists of an evacuated vessel which contains a cathode in the form of an incandescent filament; an anode or plate; and an auxiliary or control electrode which is usually in the form of a grid. The extensive literature which exists concerning this device, and which reflects its scientific and technical importance renders superfluous a description of its structural details or a discussion of the physics of its operation, for a very complete account of which the reader is referred to a recent paper by van der Bijl on the "Thermionic Amplifier."[1]

The purpose of the present paper is two-fold; first, to develop simple formulas which serve as a satisfactory basis on which to construct an elementary theory of the operation of the device in its triple role of amplifier. modulator, and detector, and which indicate the characteristics and factors on which its functioning depends; and secondly, to develop a rigorous mode of dealing with the device by aid of which exact formulas are deducible.

It will be understood that the device dealt with in the following is assumed to be so highly evacuated that the current is transported entirely by electrons emitted from the incandescent cathode, and that ionization of the residual gas molecules plays a quite ignorable part in the mechanism of current conduction.

Secondly it will be assumed that the auxiliary electrode or grid is at all times maintained negative with respect to the

* Received by the Editor, November 12, 1918.

[1] Proceedings of The Institute of Radio Engineers, volume 7, number 2, 1919. This paper also contains a very complete discussion of the theory and operation of the tube as an amplifier.

cathode or filament by means, for example, of suitable "C" battery in the input or grid-filament circuit. When this condition obtains, no current flows in the input circuit and the control is entirely electrostatic.

In the original circuit connections of the audion detector, a condenser was inserted in series with the grid and this arrangement is rather extensively employed to-day. To avoid misunderstanding it should be expressly understood that this arrangement is *not* considered in the present paper which deals with a different mode of detection which does not depend on current rectification in the input circuit.

The physical structure and the fundamental problem involved may be briefly described as follows: The input or grid-filament circuit includes a "C" battery connected with such polarity and of such value as to maintain the grid negative with respect to the filament at all times, and in series therewith a variable emf. corresponding to the impressed effect which it is the function of the device to translate into an amplified, modulated, or detected output as the case may be. The output or plate-filament circuit includes a direct current source of energy termed the "B" battery and in series therewith an external or load impedance z in which is made available the translated output. Corresponding to a variation in the emf. impressed on the input circuit the output circuit current is varied in a manner to be investigated. Our problem, in fact, a solution of which completely predetermines the performance of the device, is to formulate the variation of the output circuit current as a function of the variable input voltage.

With this preliminary understanding we proceed to an analysis of the problem. The basis for this analysis is furnished by the characteristic equation of the tube:

$$I=\phi\left(E_c+\frac{1}{\mu}E_b\right) \tag{1}$$

where I is the output circuit current; E_c is the instantaneous potential difference existing between the grid and filament; E_B is the instantaneous potential difference between plate and filament; and μ is van der Bijl's "amplification constant" of the device, which depends for its value on the design of the tube and, in particular, on the structure and location of the grid. This characteristic equation, while empiric, holds with great accuracy over the operating range of the device; for a full discussion, see van der Bijl's paper.

Since we are concerned not with the total output current but with its variation in response to a change in the input emf. we write:

$$I = I_o + J$$
$$E_c = E_{co} + e \qquad (2)$$
$$E_b = E_{bo} + v$$

Here obviously I_o may be identified with the steady or normal current in the output circuit corresponding to the steady emfs. E_{co} and E_{Bo}; e is the variation in the input emf. which is to be identified with the variation in the grid-filament potential difference corresponding to the effect to be amplified, modulated or detected; and J is the consequent variation in the output current with which we are concerned. E_{co} and E_{bo} are to be identified with the effective "C" and "B" battery voltages. v is the variation in the plate filament potential difference consequent upon the variation J of the plate-filament or output current. It obviously depends on the value of the load impedance z, and a little consideration will make it clear that v is simply the potential drop, with sign reversed, of the current J thru the impedance z. A clear grasp of this fact is essential to the following treatment of the problem.

Substitution of (2) in (1) and expansion in a power series gives:

$$J = P_1(\mu e + v) + P_2(\mu e + v)^2 + P_3(\mu e + v)^3 + P_4(\mu e + v)^4 + \cdots \qquad (3)$$

where $P_1, P_2, \cdots P_n$ are the differential parameters:

$$P_1 = \frac{1}{1!}\left(\frac{\partial I}{\partial E_b}\right)_o$$
$$\cdots\cdots \qquad (4)$$
$$P_k = \frac{1}{k!}\left(\frac{\partial^k I}{\partial E_b^k}\right)_o$$

The subscript "o" denotes that the derivatives are to be evaluated at the point $E_c = E_{co}$, $E_b = E_{bo}$ of the characteristic.

The necessary conditions that the expansion (3) shall be convergent are

$$E_c < o$$
$$E_c + \frac{1}{\mu}E_b > o \qquad (5)$$
$$I < I_s$$

where I_s is the saturation value of the output current. It will be remarked that conditions (5) simply define the operating range of the tube. The sufficient conditions to insure convergence of the expansion can be determined only when the functional form of ϕ is specified; it appears, however, that in actual tubes ϕ is closely represented by a power of the argument and for this case conditions (5) are sufficient as well as necessary. It may safely be assumed, therefore, as is done in the following discussion, that the expansion (5) is valid over the operating range of the tube.

An investigation of series (3) looking to an explicit formulation of J in terms of e requires a knowledge of v which in turn requires that the load impedance z be specified. The general case where z is unrestricted in form is examined later; for the present we take the simple but illuminating case where z is a pure resistance R. This at once leads to the relation:

$$v = -RJ$$

the substitution of which in (3) gives:

$$J = P_1(\mu e - RJ) + P_2(\mu e - RJ)^2 + \cdots \tag{6}$$

Equation (6) defines J as an implicit function of e, whereas we require its formulation as an explicit function. This is accomplished by inversion of the series (6); the simplest method is to assume an explicit expansion of the form:

$$J = a_1 e + a_2 e^2 + a_3 e^3 + \cdots \tag{7}$$

substitute this series for J in (6), and identify the unknown coefficient by direct equation of like powers of e.

Proceeding in this manner we get the following values for the first three coefficients:

$$\begin{aligned} a_1 &= \frac{\mu P_1}{1+P_1 R} \\ a_2 &= \frac{\mu^2 P_2}{(1+P_1 R)^3} \\ a_3 &= \frac{\mu^3 P_3}{(1+P_1 R)^4} - 2R\frac{\mu^3 P_2^2}{(1+P_1 R)^5} \end{aligned} \tag{8}$$

These coefficients have a clearer physical significance if we write:

$$\begin{aligned} P_1 &= \left(\frac{\partial I}{\partial E_b}\right)_o = \frac{1}{R_o} \\ P_2 &= \frac{1}{2!}\left(\frac{\partial}{\partial E_b}\frac{1}{R_o}\right)_o = -\frac{1}{2!}\frac{R_o'}{R_o^2} \end{aligned} \tag{9}$$

In accordance with common practice R_o is to be regarded and defined as the internal resistance of the tube or more precisely of the plate-filament circuit.

With this notation:

$$a_1 = \frac{\mu}{R_o + R}$$
$$a_2 = -\frac{1}{2!}\frac{\mu^2 R_o' R_o}{(R_o+R)^3} \quad (10)$$

and

$$J = \frac{\mu e}{R_o+R} - \frac{1}{2!}\frac{\mu^2 R_o' R_o}{(R_o+R)^3} e^2 + a_3 e^3 + \quad . \; . \; . \quad (11)$$

The higher order coefficients $a_3, a_4,$ . . . can be evaluated without any trouble; for the present, however, we are concerned primarily with the first two coefficients a knowledge of which is sufficient to construct an elementary theory of the operation of the device. In fact, the higher terms of series (11) represent as may be readily shown, departures from the ideal device in any of its three functions under consideration, and satisfactory operation requires that these higher terms shall be small. As a first approximation we are justified, therefore, in ignoring all terms of the series (11) beyond the second; subsequently, if necessary, the error introduced by their ignoration can be examined. We therefore proceed to a discussion of the operation of the device on the basis of the approximate formula:

$$J = \frac{\mu}{R_o+R} e - \frac{1}{2}\frac{\mu^2 R_o' R_o}{(R_o+R)^3} e^2 \quad (12)$$

In dealing with the problem it is convenient to take the impressed emf. e as

$$A \cos p\,t + B \cos q\,t \quad (13)$$

If we are concerned with amplification the two components of e may be regarded as of the same order of magnitude and comparable frequencies; if modulation is under consideration $B \cos q\,t$ may be regarded as a carrier wave of radio frequency and $A \cos p\,t$ as a signal wave of audio frequency; while as regards detection the form of e given by (13) is appropriate for a study of heterodyne receiving, in which case the two components are to be regarded as both of radio frequency with a frequency difference within the audible range.

Substitution of (13) in (12) and simplification gives:

$$J = \frac{1}{R_o+R}\left\{\mu A \cos p t+\mu B \cos q t\right\} - \frac{1}{2}\frac{\mu^2 R_o' R_o}{(R_o+R)^3}\left\{\begin{array}{l} 2 A B \cos pt \cdot \cos q t \\ +\frac{1}{2}A^2 \cos 2 p t+\frac{1}{2}B^2 \cos 2 q t \\ +\frac{1}{2}A^2+\frac{1}{2}B^2 \end{array}\right. \qquad (14)$$

which may be also written as

$$J = \frac{1}{R_o+R}\left\{\mu A \cos p t+\mu B \cos q t\right\} - \frac{1}{2}\frac{\mu^2 R_o' R_o}{(R_o+R)^3}\left\{\begin{array}{l} A B \cos (q-p)\, t \\ +A B \cos (q+p)\, t \\ +\frac{1}{2}A^2 \cos 2pt+\frac{1}{2}B^2 \cos 2 q t \\ +\frac{1}{2}A^2+\frac{1}{2}B^2 \end{array}\right. \qquad (15)$$

Formulas (14) and (15) are fundamental to the following elementary discussion:

Amplification

In amplification the fundamental requirement is that the output shall be a faithful copy of the voltage applied to the input circuit. Consequently, corresponding to an applied voltage as given by (13) the amplified output is to be identified with the term:

$$\frac{1}{R_o+R}\left\{\mu A \cos p t+\mu B \cos q t\right\} \qquad (16)$$

of formula (14) while the remainder represents first order distortion.

The outstanding deductions from formulas (16) may be stated as follows:

The effect of impressing a voltage e on the input circuit of a three-element vacuum tube of internal resistance R_o, amplification constant μ and load impedance R, is equivalent, to a first order approximation, to inserting a voltage (μe) in a circuit of resistance R_o+R.

The available amplified voltage across the load resistance is $\frac{\mu R}{R_o+R}e$ and the available output energy is $\frac{R}{(R_o+R)^2}\mu^2 e^2$. The latter is a maximum when $R=R_o$, a result stated by van der Bijl.

As regards the first order distortion, it is proportional to the curvature of the resistance characteristic of the tube. It diminishes with increasing load resistance as stated by van der Bijl; it diminishes also with a decrease of the amplitude of the impressed emf. The curvature of the characteristic, while responsible for the departure of the device from ideal requirements as an amplifier, at the same time makes possible its employment as a modulator and detector.

Modulation

In discussing the phenomena of modulation formulas (14) and (15) are applicable as they stand, but in this case $B \cos q\, t$ is to be regarded as a carrier wave of radio frequency and $A \cos p\, t$ as a signal wave of audio frequency. Examination of formula (14) shows that the only terms of the same order of frequency as the carrier wave are:

$$\frac{1}{R_o+R}\mu B \cos q\, t - \frac{\mu^2 R_o' R_o}{(R_o+R)^3} A B \cos p\, t \cdot \cos q\, t \qquad (17)$$

The other terms may be disregarded since, owing to their frequencies, they are suppressed or filtered out by the usual tuning adjustment.

Inspection of (17) shows that the first term is an unmodulated carrier wave of constant amplitude while the second is the modulated output proper; that is, a carrier wave the amplitude of which varies in accordance with the signal wave. As regards the modulated output the obvious deductions from (17) are as follows:

Its amplitude is proportional to the product of three factors

$$(\mu^2 R_o' R_o) \cdot (A B) \cdot \left(\frac{1}{R_o+R}\right)^3$$

The modulated output is therefore proportional to the curvature of the characteristic, R_o', and decreases rapidly as the load resistance is increased. The presence of the factor AB leads to the important practical deduction that the modulated output is independent of the relative amplitudes of the carrier and signal waves provided their product is constant.

The available modulated voltage is proportional to $\frac{R}{(R_o+R)^3}$; this is a maximum when the load resistance is adjusted to make $R = \frac{1}{2} R_o$.

The available modulated energy is proportional to $\frac{R}{(R_o+R)^6}$; this is a maximum when the load resistance is adjusted to make $R=\frac{1}{5}R_o$.

These last two deductions which are valid also for detection, are of considerable practical importance in designing the load impedance for efficient operation. While approximate in that they are based on the neglect of higher terms, it is believed that they are in substantial agreement with the facts. In any case it is to be observed that the presence of higher terms must operate to cause a departure from the ideal modulated wave, so that we are justified in regarding (17) as representing the true modulated wave, the conditions for the maxima of which are substantially as given above.

DETECTION

The phenomena of detection are identical with those of modulation, with the essential distinction that we are concerned with a different order of frequencies in the output; consequently formulas (14) and (15) are adapted as they stand to investigate detection by the heterodyne method. In this case, however, $A \cos p\, t$ and $B \cos q\, t$ are both to be regarded as waves of radio frequency, while the detected output is required to be within the audible range. Consequently discarding the radio frequency terms of (15) the detected wave is made up of:

$$-\frac{1}{2}\frac{\mu^2 R_o' R_o}{(R_o+R)^3}\left\{A\, B \cos(q-p)\, t+\frac{1}{2}A^2+\frac{1}{2}B^2\right\} \qquad (18)$$

To fix our ideas let $A \cos pt$ be the transmitted wave, and $B \cos qt$ a locally generated wave; then the first term of (18) represents the familiar "beat-note" characteristic of heterodyne reception. The term $\frac{1}{2}B^2$ represents merely a steady value which may conveniently be lumped with the steady current I_o and excluded from explicit consideration. The term $\frac{1}{2}A^2$ represents a change in the normal plate current for the duration of the signal and it alone is cognizable in the absence of the locally generated wave.

Formula (18) furnishes an immediate answer to a question which has been discussed at some length in the pages of this Journal; namely, the theoretical amplification obtainable by means of a locally generated wave.[2] To answer this question we observe that in the absence of the locally generated wave, the detected effect is proportional to $\frac{1}{2}A^2$, while when a locally generated wave of amplitude B is present the detected current, assuming A as small compared with B, is proportional to $A\,B$. The ratio $\frac{A\,B}{A^2}=\frac{B}{A}$ may be logically regarded as measuring the "heterodyne" amplification, and shows that theoretically it increases without limit as the amplitude of the local wave is increased. Practically, of course, it is limited by the necessity of keeping within the operating range of the tube. The theoretical law is, however, in agreement with the fact that enormous amplification is obtainable by the heterodyne method.

When a modulated wave is to be detected we may identify e of formula (12) with an expression of the form:

$$A\,B \cos q\,t \cdot \cos p\,t + C \cos q\,t$$

where the last term is an unmodulated carrier wave and the first is a wave modulated in accordance with the signal wave $A \cos p\,t$. Substituting in formula (12) and retaining only terms of audio frequency (those comparable with $\frac{p}{2\pi}$ the detected output is:

$$-\frac{1}{2}\frac{\mu^2 R_o' R_o}{(R_o+R)^3}\begin{cases} B\,C\,A \cos p\,t \\ +\frac{1}{4}A^2B^2 \cos 2\,p\,t \\ +\frac{1}{4}A^2B^2+\frac{1}{2}C^2 \end{cases} \qquad (19)$$

Bearing in mind that B and C represent constant amplitudes of the carrier waves, the first term of (19) is directly proportional to the original modulating signal wave; the second term represents a wave of double signal frequency while the last two terms of zero frequency are relatively unimportant and indeed are usually eliminated, as for example, by a transformer. Observe that in the absence of the unmodulated carrier wave of amplitude C, the detected effect is of double frequency; consequently it is obviously desirable to make C large in order to preserve the wave form of the signal wave.

The foregoing discussion is frankly elementary and makes no pretense to being more than a rough approximation to the

[2] See PROCEEDINGS OF THE INSTITUTE OF RADIO ENGINEERS; volume 1, number 3, pages 89, 100; volume 3, number 2, page 185; volume 5, number 1, page 33; volume 5, number 2, page 145; volume 5, number 4, page 247; volume 6, number 5, page 275.—EDITOR.

complicated phenomena. It is believed, however, that it furnishes a good working theory and is in substantial agreement with the phenomena which it represents.

General Solution

We shall now take up the more involved case where the load impedance z is unrestricted in form and is merely any specified function of the frequency. To formulate the solution of this general case we start afresh with the series expansion (3), our problem, as before, being to express J as an explicit function of the impressed voltage e. To make the treatment general, the applied voltage e will be taken as a series of component sinusoids of the form

$$\sum_{k=1}^{k=n} |E_k| \cdot \cos(p_k t + \theta_k)$$

which will be written in the exponential form:

$$e = \frac{1}{2}\sum_{k=n}^{k=n} E_k \, \varepsilon^{j p_k t} + \frac{1}{2}\sum_{k=1}^{k=n} \bar{E}_k \, \varepsilon^{-j p_k t} \tag{20}$$

Here the bar denotes the conjugate imaginary of the corresponding unbarred symbol and the entire expression is, of course, real and equivalent to the cosine summation above.

If the voltage formulated by (20) is applied to a circuit of symbolic or complex impedance z, the resultant current is

$$\frac{1}{2}\sum \frac{E_k}{Z(j p_k)} \varepsilon^{j p_k t} + \frac{1}{2}\sum \frac{\bar{E}_k}{Z(-j p_k)} \varepsilon^{-j p_k t}$$

which will be written as

$$\frac{1}{2}\sum \frac{E_k}{Z_k} \varepsilon^{j p_k t} + \frac{1}{2}\sum \frac{\bar{E}_k}{\bar{Z}_k} \varepsilon^{-j p_k t}.$$

The convenience of this notation will become apparent in the course of the argument. Z is, of course, supposed to be defined as a complex quantity.

Starting with the expansion (3) let us set

$$\begin{aligned} J &= J_1 + J_2 + J_3 + \cdots + J_n + \cdots \\ v &= v_1 + v_2 + v_3 + \cdots + v_n + \cdots \end{aligned} \tag{21}$$

and substitute in (3). The significance of the series formulation of J and v will become apparent in the course of the argument.

At present we observe that the component terms of the series (21) are unrestricted except by the necessity of satisfying (3) and the relation obtaining between J and v, namely, that v is the potential drop, with sign reversed, of the current J thru the load impedance z.

Now let the components of the J and v series be determined to satisfy the following system of equations:

$$\begin{aligned} J_1 &= P_1 (\mu e + v_1) \\ J_2 &= P_1 v_2 + P_2 (\mu e + v_1)^2 \\ J_3 &= P_1 v_3 + 2 P_2 (\mu e + v_1) v_2 + P_3 (\mu e + v_1)^3 \\ J_4 &= P_1 v_4 + 2 P_2 \left[(\mu e + v_1) v_3 + \frac{1}{2} v_2^2 \right] \\ &\quad + 3 P_3 (\mu e + v_1)^2 v_2 \\ &\quad + P_4 (\mu e + v_1)^4 \\ &\quad \ldots\ldots\ldots\ldots \end{aligned} \tag{22}$$

Direct addition by columns of (22), regard being had to the identities of (21), shows that (22) satisfies (3) and is, therefore, a formal solution. The law of formation of the right hand side of each equation will be clear; thus the m^{th} equation includes all terms of order $e^h v^k{}_{m+1-h-k}$ or all values of k and h from 0 to m. We have further to satisfy the relation obtaining between J and v; this will be accomplished if we identify any v_k with the potential drop corresponding to any component J_k.

Now writing $P_1 = \dfrac{1}{R_o}$ as above, (22) becomes

$$\begin{aligned} R_o J_1 - v_1 &= \mu e \\ R_o J_2 - v_2 &= R_o P_2 (\mu e + v_1)^2 \\ R_o J_3 - v_3 &= 2 R_o P_2 (\mu e + v_1) v_2 + R_o P_3 (\mu e + v_1)^3 \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{23}$$

Bearing in mind the assumed physical significance of the v series, it follows at once that any equation say k^{th} is simply that of the current J_k in a circuit of impedance $R_o + z$ in response to a voltage given by the right hand side of the equation. The recognition of this fact furnishes the key to the complete formal solution for J.

To carry this out substitute e as given by (20), in the first of (23); then it follows at once that

$$J_1 = \frac{1}{2} \mu \sum \frac{E_k}{R_o + z_k} \varepsilon^{j p_k t} + \frac{1}{2} \mu \sum \frac{\bar{z}_k}{R_o + \bar{z}_k} \varepsilon^{-j p_k t}$$

If we write

$$R_o + z_k = Z_k$$

$$\frac{z_k}{R_o + z_k} = \rho_k$$

then: $$J_1 = \frac{1}{2}\sum \frac{\mu E_k}{Z_k} \varepsilon^{j p_k t} + \frac{1}{2}\sum \frac{\mu \bar{E}_k}{\bar{z}_k} \varepsilon^{-j p_k t}$$

and (24)

$$v_1 = -\frac{1}{2}\sum \mu \rho_k E_k \varepsilon^{j p_k} - \frac{1}{2}\sum \mu \bar{\rho}_k \bar{E}_k \varepsilon^{-j p_k}$$

Equations (24) state that to a first-order approximation the effect of impressing a voltage e on the input circuit of a tube of amplification constant μ, internal resistance R_o and load impedance z, is equivalent to inserting a voltage μe in a circuit of impedance $R_o + z = Z$. This is the generalized form of the law already stated when z is a pure resistance R and shows that this restriction is unnecessary. J is, of course, to be identified with the amplified output current of the ideal amplifier.

By virtue of (24) and (20), the right hand side of the second equation of the system (23) is known; it is after easy simplification:

$$\frac{\mu^2}{4} R_o P_2 \sum^h \sum^k \begin{cases} (1-\rho_h)(1-\rho_k) E_h E_k \varepsilon^{j(p_h+p_k)t} \\ +(1-\rho_h)(1-\bar{\rho}_k) E_h \bar{E}_k \varepsilon^{j(p_h-p_k)t} \\ +(1-\bar{\rho}_h)(1-\rho_k) \bar{E}_h \bar{E}_k \varepsilon^{-j(p_h-p_k)t} \\ +(1-\bar{\rho}_h)(1-\rho_k) \bar{E}_h E_k \varepsilon^{-j(p_h-p_k)t} \end{cases} \tag{25}$$

The last two terms of the double summation are the conjugate imaginaries of the first two, respectively. This expression may be regarded as the applied emf. which generates the current J_2; consequently J_2 and v_2 are given by

$$J_2 = \frac{\mu^2 R_o P_2}{4} \sum_{h=1}^{h=n} \sum_{k=1}^{k=n} \begin{cases} \frac{(1-\rho_h)(1-\rho_k)}{Z_{h+k}} E_h E_k \varepsilon^{j(p_h+p_k)} \\ + \frac{(1-\rho_h)(1-\bar{\rho}_k)}{Z_{h-k}} E_h \bar{E}_k \varepsilon^{j(p_h-p_k)t} \\ + \frac{(1-\bar{\rho}_h)(1-\bar{\rho}_k)}{\bar{Z}_{k+k}} \bar{E}_h \bar{E}_k \varepsilon^{-j(p_h+p_k)t} \\ + \frac{(1-\rho_h)(1-\rho_k)}{Z_{h-k}} E_h E_k \varepsilon^{-j(p_h-p_k)t} \end{cases}$$

(26)

$$v_2 = -\frac{\mu^2 R_o P_2}{4} \sum_{h=1}^{h=n} \sum_{k=1}^{k=n} \begin{cases} \varrho_{h+k}(1-\varrho_h)(1-\varrho_k) E_h E_k \varepsilon^{j(p_h+p_k)t} \\ +\varrho_{h-k}(1-\varrho_h)(1-\bar{\varrho}_k) E_h \bar{E}_k \varepsilon^{j(p_h-p_k)t} \\ +\bar{\varrho}_{h+k}(1-\varrho_h)(1-\varrho_k)\bar{E}_h \bar{E}_k \varepsilon^{-j(p_h+p_k)t} \\ +\varrho_{h-k}(1-\varrho_h)(1-\varrho_k)\bar{E}_h E_k \varepsilon^{-j(p_h-p_k)t} \end{cases} \tag{27}$$

In these summations Z_{h+k} denotes the complex impedance of the circuit R_o+z to the frequency $\frac{p_h+p_k}{2\pi}$; Z_{h-k} denotes that to frequency $\frac{p_h-p_k}{2\pi}$; while in accordance with the notation adopted

$$\varrho_{h+k} = \frac{z(j(p_h+p_k))}{R_o+z(j(p_h+p_k))} = \frac{z_{h+k}}{R_o+z_{h+k}}$$

With v_2 determined by (27) the right hand side of the third member of the system (23) is known and consequently the third order components J_3 and v_3 can be written down. Owing, however, to the complexity of the resulting expressions a more compact notation is required, and the following symbolic notation commends itself for this purpose. If we let a negative subscript attached to any symbol denote the conjugate imaginary of the same symbol with positive subscript and extend the summations over negative as well as positive values, we write

$$e = \frac{1}{2} \sum_{k=-n}^{k=n} E_k \varepsilon^{jp_k t}$$

which is equivalent to (20). In this same notation J_1 and v_1 are given by

$$J_1 = \frac{\mu}{2} \sum_{k=-n}^{k=n} \frac{E_k}{Z_k} \varepsilon^{jp_k t}$$

$$v_1 = \frac{-\mu}{2} \sum_{k=-n}^{k=n} \varrho_k E_k \varepsilon^{jp_k t}$$

while J_2 and v_2 become:

$$J_2 = \frac{\mu^2 R_o P_2}{4} \sum_{h=-n}^{h=n} \sum_{k=-n}^{k=n} \frac{(1-\varrho_h)(1-\varrho_k)}{Z_{h+k}} E_h E_k \varepsilon^{j(p_h+p_k)t}$$

$$v_2 = -\frac{\mu^2 R_o P_2}{4} \sum_{h=-n}^{h=n} \sum_{k=-n}^{k=-n} \varrho_{h+k}(1-\varrho_h)$$

$$(1-\varrho_k) E_h E_k \varepsilon^{j(p_h+p_k)t}$$

These last two equations are equivalent to (26) and (27) respectively.

The right hand side of the third member of (23) becomes:

$$\frac{\mu^3 R_o}{8}\sum_{g=-n}^{g=n}\sum_{h=-n}^{h=n}\sum_{k=-n}^{k=n}(P_3-2R_oP_2^2\rho_{h+k})(1-\rho_g)(1-\rho_h)(1-\rho_k)\,E_g\,E_h\,E_k\,\varepsilon^{j(p_g+p_h+p_k)}$$

by aid of which J_3 is given by:

$$\frac{\mu^3 R_o}{8}\sum_{g=-n}^{g=n}\sum_{h=-n}^{h=n}\sum_{k=-n}^{k=n}\frac{P_3-2R_oP_2^2\rho_{h+k}}{Z_{g+h+k}}(1-\rho_g)(1-\rho_h)\cdot(1-\rho_k)\,.\,(E_gE_hE_k\varepsilon^{j(p_g+p_h+p_k)t}$$

It is to be regretted that the complexity of the formulas require such an abbreviated notation, but this is inherent in the nature of problem and I know of no other system of symbolic notation by which the results can be written down in compact form. As a check on this solution we can put the load impedance z equal to a pure resistance R, in which case the general solution degenerates into a term by term identity with the series solution derived in the first part of this paper. The complexity of the general solution is of course due to the presence of harmonic frequencies and the fact that the load impedance is a frequency function instead of being a constant.

The solution, it should be observed, while formally correct, is incomplete without an investigation of the convergence of the J and v series, a rigorous discussion of which is beyond the scope of this paper. It can be shown, however, that the formal solution is a Fourier series, the coefficient of each term of which is an infinite convergent series within the range of convergence of the expansion (3). The range of validity of the formal general solution is, therefore, defined by the region of convergence of the original expansion from which it was derived.

September 9, 1918.

SUMMARY: Starting with a functional relation between the plate current and the grid and plate voltages, the plate current variation is deduced to a sufficient degree of approximation in terms of constants of the tube and the grid voltage variation (for pure resistance in the output circuit).

The requirements of amplification, modulation, and detection are then deduced from the equations obtained. Heterodyne amplification is discussed, and it is held that within the working range of the tube this amplification is the ratio of the amplitudes of the locally and remotely generated waves provided the latter is small as compared with the former.

The general problem of output current variation with an impedance in the output circuit is then considered, and the production of harmonic frequencies noted.