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SYNTHESIS OF REACTANCE 4-POLES

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SYNTHESIS OF 4-TERMINAL NETWORKS OF REACTANCES PRODUCING PRESCRIBED INSERTION LOSS CHARACTERISTICS WHEN INSERTED BETWEEN PRESCRIBED RESISTANCES, WITH SPECIAL APPLICATIONS TO FILTER DESIGN

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SYNTHESIS OF REACTANCE 4-POLES WHICH PRODUCE PRESCRIBED INSERTION LOSS CHARACTERISTICS

Including Special Applications to Filter Design

By S. Darlington*

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Introduction

Of the various types of electrical networks which are frequently found useful, one of the commonest is the 4-terminal transducer of reactances, more briefly referred to as the reactance 4-pole.[1] In particular, the selective networks or filters which are commonly used for transmitting certain frequencies while blocking others are almost always reactance 4-poles, and these filters form essential parts of most communication systems.

Detailed methods of designing filters and related reactance 4-poles are well known and have been in general use for a considerable period. For the most part these fit into one general filter design scheme which may be referred to as the image parameter theory, since it is based upon

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[1] Throughout this paper, the term 4-pole will be used to indicate a 4-terminal transducer—i.e. a network with two pairs of accessible terminals subject to the restriction that no external connections can be made between terminals of different pairs. The term has been widely used in this sense and also to indicate a network with four terminals to which external apparatus can be connected in any desired manner.

the concepts of the image impedances and image transfer constants of 4-terminal networks. It has been found that methods based upon this theory can be used to design practical filters with electrical characteristics meeting any ordinary engineering requirements. In recent years it has become apparent, however, that these filters are sometimes unnecessarily costly. At the same time the question of network cost has become increasingly important.

This paper describes a theory of reactance 4-pole design which differs from the image parameter theory in such a way that it sometimes leads to more advantageous choices of element values in filters of conventional image parameter configurations. While more complicated than the image parameter theory, it sometimes permits the realization of substantially greater network economy or of superior electrical characteristics with no increase in cost. Instead of starting with the concepts of image impedances and image transfer constants, this theory is based upon the general problem in network synthesis of finding reactance 4-poles yielding prescribed insertion loss versus frequency functions when inserted between prescribed resistance terminations. Because of this it has come to be called the insertion loss theory.

While the insertion loss theory applies particularly to filter design, it is capable of more general applications. As a matter of fact, it is theoretically possible to use the theory to design physical reactance 4-poles which, when terminated in prescribed resistances, produce insertion loss characteristics identical with those of any general finite passive 4-poles with the same terminations. Although the possibility of designing at least simple reactance networks on an insertion loss basis is common knowledge, when other than very simple circuits are considered extensive special theory such as that developed in this paper is necessary if hopelessly complex computations are to be avoided.

Although very general applications of the insertion loss method of design are theoretically possible, even with the theory developed here the numerical complications are such as to limit its practical usefulness. In many design problems, for instance, economies in the cost of actually constructing filters might be obtained by using insertion loss designs rather than image parameter designs but these economies frequently are off-set by the added cost involved in obtaining the insertion loss designs. As a result, the insertion loss theory applies principally to the design of filters which are to be made in such large numbers that construction economies will justify high design costs or which must meet requirements not easily satisfied with image parameter designs.

The basic insertion loss theory applies only to networks of pure reactances. Modifications are included, however, permitting the design of certain types of dissipative reactance networks with prescribed loss characteristics. These have greatly increased the practical advantages obtainable by the use of the theory.

Limitations of the Image Parameter Theory

The great simplicity of the image parameter filter theory is obtained by adopting certain definite restrictions which limit to a considerable extent the choice of the element values. It is the possibility of avoiding these restrictions by using the more complicated insertion loss theory that sometimes renders this theory advantageous as a basis of filter design. While the use of the insertion loss theory in filter design normally involves the introduction of other restrictions, these are such as to lead to the optimum choice of element values for meeting certain types of filter specifications. In order to clarify the status of the insertion loss theory, it will be best to introduce at this point a brief description of the fundamentals of the image parameter theory and of the restrictions which it involves.

The image parameter theory is, of course, an outgrowth of the artificial line theories of Pupin (1) and Campbell (2). The image impedances and image transfer constants in terms of which it is developed correspond to the characteristic impedances and propagation constants of these artificial lines. In its most familiar form the image parameter theory deals directly with the so-called composite filters introduced by Zobel (3), which are made up of chains of sections with matched image impedances but different transfer constants. In the more general form developed by Bode (4), however, it deals with the equivalent restrictions upon the image impedances and transfer constants of complete networks, without actually requiring chains of tandem sections.

The image attenuation of a non-dissipative filter is identically zero over finite ranges of frequencies. If the filter is terminated in its image impedances, the corresponding transducer loss will also be zero.[2] Although the image impedances must vary with frequency, they can

[2] By the transducer loss of a filter is meant the difference in level between the received power and the maximum power obtainable from the generator with any passive network. The image attenuation of a filter is equal to its transducer loss when terminated in its image impedances at all frequencies where these impedances are real. The transducer loss of any network terminated in resistances can be obtained from the insertion loss by adding the reflection loss corresponding to the ratio of the terminations.

approximate constant resistances over the greater parts of the ranges of zero attenuation, or theoretical pass bands. Consequently, if these resistances are used as the actual terminations the transducer loss will be small over the ranges of good approximation.

The situation described above leads to the introduction of the following requirement, which is responsible for the fundamental limitations of the image parameter theory: The image attenuation of a non-dissipative filter is required to be identically zero in continuous frequency ranges including those to be freely transmitted between the actual terminations and is required to be other than zero at all other frequencies. For the types of filters commonly encountered these restrictions reduce by almost half the constants which could otherwise be chosen arbitrarily.

Without further investigation, it might appear that good filter characteristics could be obtained only by satisfying these restrictions at least so closely that permissible departures from them would be of no practical interest. Actually, these requirements are not necessary but instead are artificial or arbitrary restrictions leading to a simple design procedure.

As an illustration, suppose that the elements of a filter designed on the image basis are changed from their design values by various amounts of the order of perhaps 10 or 20 percent. In general this will split the theoretical pass band into a number of theoretical pass bands separated by narrow theoretical attenuation bands.[3] The image attenuation corresponding to these theoretical attenuation bands, however, will ordinarily be very moderate. In addition, the actual transducer loss may be only a fraction of the image attenuation. A transducer loss of less than 0.4 decibels may be obtained, for instance, even though the corresponding image attenuation is as high as 2.5 decibels. As a result, modifying the elements may leave unchanged the frequencies included in the *effective* pass band and may result in an actual decrease in the corresponding transducer loss.

When the dissipation required in actual filters has a marked effect the arbitrariness of the restrictions of the image parameter theory becomes much more striking. Under these conditions the image attenuation is generally far from uniform over the frequency ranges in which it is zero on a non-dissipative basis. The special restrictions of the image parameter theory then amount to the requirement of transmission range

[3] In general the theoretical attenuation band will also be split up by narrow theoretical pass bands, but it will be sufficient for purposes of illustration to consider only the splitting of the pass band.

insertion losses approximating characteristics which are anything but ideal.

The above discussion indicates the arbitrariness of the restrictions forming the basis of the image parameter method of filter *design* or *synthesis*. It is well known that the *circuit analysis* problem of determining the operation of a known network can be solved in terms of the image impedances and image transfer constant of the network even though it does not satisfy the special restrictions of the design theory. When these restrictions are abandoned, however, the image parameters are no longer convenient in *design* problems involving the determination of resistance-terminated reactance networks with prescribed properties.

Other Previous Theories

Cauer (5), Gewertz (6) and others have studied the synthesis of perfectly general reactance 4-poles having prescribed open- and short-circuit impedances. These investigations, however, have not yielded useful methods of designing reactance 4-poles with resistance terminations except under the restrictions of the image parameter theory. On the other hand, they have produced such useful information as the necessary and sufficient conditions satisfied by sets of open- and short-circuit impedances corresponding to physical reactance 4-poles. In addition, they have shown that physical networks of certain so-called canonical configurations can readily be designed to have any specified set of impedances satisfying these conditions. The canonical configurations, however, were chosen purely for their general realizability and ease of design and are rarely of practical interest.

The development of the present insertion loss theory started with the previous theory developed by Norton (7) which permits the design of two or more filters producing a constant resistance at their common terminals when connected in series or in parallel at one end. Norton's theory involves the problem of designing a reactance 4-pole terminated in an open- or short-circuit at one end. This amounts to the special case of the insertion loss theory in which one of the specified terminations is zero or infinite. Norton's detailed design procedure, however, assumes a rather restricted type of prescribed insertion loss characteristic which is of little interest except in the design of filters to operate in constant resistance pairs.

Principal Operations Involved in the Design of Filters on an Insertion Loss Basis

Returning now to the subject of this paper, filters can readily be designed on an insertion loss basis provided certain fundamental operations can be carried out. There is no problem in fitting these operations together into straightforward design procedures but the operations themselves cannot readily be carried out without special mathematical machinery. The principal object of the insertion loss theory is to supply this mathematical machinery.

The logical order to follow in developing the mathematical machinery of the insertion loss theory turns out to be rather different from the order in which the various parts are used in carrying out actual designs. It will therefore be best to motivate the more detailed analysis by introducing at once a brief description of the various fundamental design operations, rather than introducing each operation at the time the corresponding mathematical machinery is taken up.

The distinguishing feature of the insertion loss theory is the use of an exactly prescribed insertion loss versus frequency function to fix the element values of the final network. Since exactly prescribed loss functions are rarely included in design specifications, the first operation in the design of a network on the insertion loss basis is usually the choice of the specific insertion loss function to be obtained. This choice is guided by the following considerations. First, the loss function must be consistent with the design specifications, e.g., it must represent sufficient suppression of the unwanted frequencies. Second, the function must be of a form leading to an economical network as regards the number of elements involved. Third, it must usually correspond to a particular type of network configuration, such as a ladder or a lattice.

After a specific insertion loss function has been chosen, the determination of a corresponding network involves two principal operations. The first of these is the determination of some or all of the open- and short-circuit impedances of the network from the loss function, the second being then the determination of the actual element values from these impedances.[4] It turns out that there are only a finite number of sets of open- and short-circuit impedances corresponding to reactance networks of minimum complexity producing a specific insertion loss function when terminated in a specific pair of resistances. After one of these sets of impedances has been selected, the method of determining the element values of a corresponding network depends upon the type of

[4] This rule, however, is not without its exceptions. In particular, symmetrical lattices with equal terminations are most easily determined from their insertion loss functions by procedures making no direct references to their open- and short-circuit impedances.

configuration chosen, equivalent networks of different configurations frequently requiring quite different methods.[5]

The parasitic dissipation which must be present in actual reactance networks sometimes influences the loss characteristics to a small enough extent to justify neglecting it in the design of the networks. On the other hand, the influence of the necessary dissipation is sometimes so important that it is highly desirable to compensate for it in some manner. The logical way to obtain this compensation is to design dissipative reactance networks which themselves produce more or less exactly prescribed loss functions. Under certain conditions this can be accomplished by modifying the first part of the non-dissipative design procedure in such a way as to obtain the open- and short-circuit impedances corresponding to the removal of the dissipation from the final network.[6] This permits the element values to be computed from the impedances on a strictly non-dissipative basis.

In order to clarify further the general design procedure outlined above, the procedure used in a simple illustrative case will now be described in more detail. It should be borne in mind, however, that the detailed design procedures used in other cases may differ considerably from this illustrative case even though they involve the same general types of operations.

Illustrative Special Case—Choice of Insertion Loss Function

Assuming for the time being that dissipation can be neglected entirely, consider the design of a low-pass filter of ideal reactance elements consistent with the following specifications. First, the insertion loss shall not be greater than a prescribed maximum α_p^0 in an effective pass band extending from zero frequency to a prescribed cut-off frequency f_1^0. Second, the insertion loss shall not be less than a prescribed minimum α_a^0 in an effective attenuation band extending from a second prescribed frequency f_2^0 to infinity. Third, these loss specifications are to be met

[5] As would be expected, when a particular configuration is desired care must usually be exercised to make sure that an appropriate set of impedances is chosen. The various sets corresponding to a single loss function, for instance, usually occur in inverse pairs leading to inverse configurations.

[6] The possibility of doing this depends upon the following situation. When the dissipation required in a network satisfies certain restrictions as regards its variation from element to element the effect of the dissipation on complex impedances and complex voltage ratios corresponds to simple transformations of the functions of frequency representing the impedances and voltage ratios, these transformations being independent of the specific network configuration and element values.

with a network which is electrically symmetrical and is terminated in equal resistances and which has the same configuration as a mid-series low-pass ladder filter of the image parameter theory even though it may have more general element values.[7]

The above specifications on the insertion loss are indicated graphically in Fig. 1A, it being required that the curve of actual insertion loss shall fall within the shaded areas at frequencies less than f_1^0 or greater than f_2^0. The required configuration is indicated in Fig. 2.

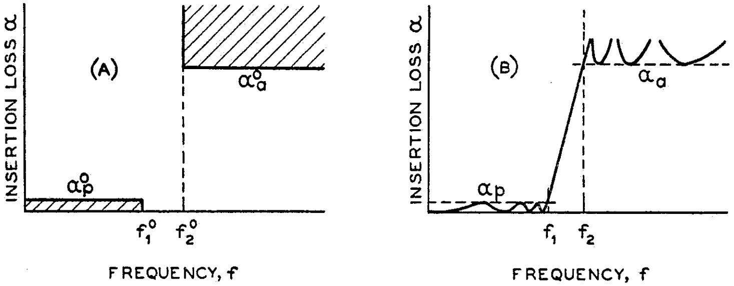


Fig. 1. A) Design specifications—If $f < f_1^0$ or $> f_2^0$, α must lie in a shaded area. B) Form of α characteristic meeting the design specifications.

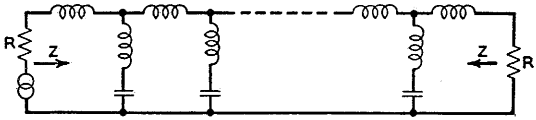


Fig. 2. Configuration to meet specifications of Fig. 1A

The theory to be developed shows that the above specifications on the form of the final network require the insertion loss α to be described by an equation of the form

$$e^{2\alpha} = 1 + \left[S_0 \frac{\omega(\omega_1^2 - \omega^2) \cdots (\omega_\eta^2 - \omega^2)}{(1 - \nu_1^2\omega^2) \cdots (1 - \nu_\eta^2\omega^2)} \right]^2 \quad (1)$$

In this equation, η represents the number of "sections," while S_0 and the ω_σ's and ν_σ's are arbitrary constants. The specific problem is to

[7] It turns out that the requirement that the network shall be either symmetrical or of a type producing inverse image impedances at the two ends tends to lead to efficient filter characteristics. The inverse impedance case is excluded here in order to simplify the illustration. Similarly, the special loss specifications described above are chosen for their simplicity rather than for their generality. Somewhat similar methods can still be used, for instance, when the minimum permissible attenuation band loss varies with frequency.

choose the arbitrary constants in such a way as to satisfy the specifications indicated in Fig. 1A with the smallest possible value of η, i.e., the smallest number of sections.[8]

The best choice of the ω_σ's and ν_σ's turns out to be that leading to the form of loss characteristic indicated in Fig. 1B for the special case of $\eta = 3$. A lengthy analysis will later be outlined which shows that the particular rational "power ratio" function $e^{2\alpha}$ which has the form of (1) and which corresponds to the "equal ripple" type of loss characteristic indicated in Fig. 1B can be obtained by requiring $e^{2\alpha}$ to be determined by the following set of equations relating both $e^{2\alpha}$ and ω to a new variable u.

$$e^{2\alpha} = 1 + (e^{2\alpha_p} - 1)\, sn^2\left[(2\eta + 1)u\frac{K_1}{K}, k_1\right] \qquad (2)^9$$

$$\frac{\omega}{2\pi\sqrt{f_1 f_2}} = \sqrt{k}\, sn(u, k)$$

$$k = \frac{f_1}{f_2}$$

$$q_1 = q^{2\eta+1}$$

In these equations, α_p, f_1, f_2 represent the maximum pass band loss and the limits of the effective pass and attenuation bands, as in Fig. 1B. The symbols K, K_1 and q, q_1 represent constants appearing in the general theory of elliptic functions, K, K_1 being the complete elliptic integrals of the first kind of moduli k, k_1, respectively, and q, q_1 the corresponding elliptic modular constants. These constants are uniquely related to k, k_1 in a manner represented in most elliptic function tables by tabulations of K and $\log_{10} q$ vs $\sin^{-1}k$.

[8] It turns out that choosing the constants in this way will normally lead to a physical network of the form of Fig. 2, although it is possible for difficulties to be encountered if α_p^0 is exceptionally small.

[9] The appearance of elliptic functions in a problem involving an algebraic function producing equal maxima and equal minima is not surprising to those familiar with the elliptic functions appearing in the "Tchebycheff parameter" version of the image parameter theory introduced by Cauer (16). The rational character of $e^{2\alpha}$ considered as a function of ω depends upon the equivalence of $\mathrm{sn}\left[(2\eta + 1)u\frac{K_1}{K}, k_1\right]$ to an odd rational function of sn (u, k), which can be compared with the equivalence of sin $[(2\eta + 1)u]$ to an odd polynomial in sin (u). The correspondence of (2) to the particular special case of (1) illustrated by Fig. 1B depends upon the periodic properties of α and ω considered as functions of u.

The insertion loss corresponding to (2) is completely determined by the choice of α_p, f_1, f_2 and the number of sections η, for f_1, f_2 determine k, which determines k_1 through the relation between q and q_1, while k, k_1 determine K, K_1. The corresponding value of the minimum attenuation band loss α_a indicated in Fig. 1B is thus also determined by the choice of α_p, f_1, f_2. If minor approximations are made, this relationship can be expressed in the following form:

$$\alpha_a = [10 \log_{10}(e^{2\alpha_p} - 1) - 10(2\eta + 1) \log_{10}(q) - 12.04] \text{ db} \quad (3)$$

With the help of (3) it is a simple matter to determine the minimum value of η for which the specifications of Fig. 1A can be satisfied, i.e., for which α_p, α_a, f_1 and f_2 can be chosen in such a way that

$$\begin{array}{ll} \alpha_p \leqq \alpha_p^0 & f_1 \geqq f_1^0 \\ \alpha_a \geqq \alpha_a^0 & f_2 \leqq f_2^0 \end{array} \quad (4)$$

In this way, a definite set of constants can be chosen which determine, with the help of (2), a loss characteristic leading to a final network of the required type and of the minimum number of sections permitted by the loss specifications.[10]

Illustrative Special Case—Determination of a Network Producing the Insertion Loss Chosen

The open- and short-circuit impedances of reactance networks producing the particular insertion loss function chosen cannot be expressed directly in terms of the loss function itself. They can readily be evaluated, however, in terms of the roots of the corresponding power ratio (1).[11] In the special case under consideration these roots can be computed by means of straightforward formulae derived from the elliptic function relations (2).

If reactance networks of minimum complexity are assumed, and also a definite pair of (equal) terminating resistances, it turns out that there are only four sets of open- and short-circuit impedances consistent with

[10] Any three of the four relations of (4) can be made exact equalities but the fourth will then be an equality only in very special cases. Keeping three of the parameters α_p, α_a, f_1, f_2 fixed and changing the number of sections η makes discrete changes in the fourth parameter. Since some margin is thus usually available even though η has the smallest permissible value, this margin is usually best distributed among the various relations of (4).

[11] If the requirement of a loss function appropriate for a symmetrical network is abandoned it is usually necessary to include also the roots of a linear function of the power ratio $e^{2\alpha}$.

(1). Of these only one set is appropriate for the mid-series low pass ladder configuration of Fig. 2. Formulae for computing the element values of this configuration from the impedances will be included in the more detailed development of the insertion loss theory.[12]

Illustrative Special Case—Compensation for Effects of Dissipation

Turning now to the question of compensating for effects of dissipation, consider first the problem of attempting to design a network of reactance elements of prescribed dissipativeness producing exactly the same insertion loss and phase as the non-dissipative filter considered previously. Assuming such a network to be possible, consider the insertion loss and phase that would be produced by the pure reactance network formed by simply removing the dissipation from all the elements. Provided the elements are all to be equally dissipative, it turns out that this new loss and phase can be evaluated directly from the original loss and phase and the extent of the dissipation, with no further information as to the actual element values or specific configuration.[13] Thus any pure reactance network which could be designed to produce this "predistorted" loss and phase would produce the original loss and phase upon the addition of the required dissipation.

As described above, the "predistorted" insertion loss and phase cannot actually be produced by a pure reactance network. On the other hand, if the required dissipation varies properly with frequency and is not too great, the predistortion can be modified in such a way as to lead to a physical network of the configuration of Fig. 2 which will produce a close approximation to the sum of the original insertion loss and an added constant loss when the required dissipation is added. This modified predistortion amounts to nothing more than the addition of a

[12] When the insertion loss function is not necessarily appropriate for a symmetrical network there are usually more than four sets of open- and short-circuit impedances corresponding to networks of minimum complexity. There may be a considerable variety, for instance, realizable with various networks having a single configuration, such as the configuration of Fig. 2.

In the special case of symmetry under consideration the element values of the final ladder may be computed directly from the equivalent lattice rather than from open or short-circuit impedances. The formulae for doing this, however, are derived by expressing the open-circuit impedances in terms of the impedances of the lattice arms.

[13] Actually the corresponding "complex insertion voltage ratio" is more convenient to use than the insertion loss and phase. The original voltage ratio can be readily evaluated in terms of the roots of (1) used in computing the open- and short-circuit impedances on a non-dissipative basis.

constant to each root of the complex voltage ratio function corresponding to the original loss and phase plus the addition of a constant factor representing an added constant loss.[14] Although the general configuration of Fig. 2 is obtained, however, the network yielded by the predistortion method will not satisfy the previous requirement of electrical symmetry nor will the terminations be equal.[15]

Major Divisions of the Theoretical Development

As was indicated previously, the logical order to follow in developing the insertion loss theory is rather different from the order in which the various parts are used in carrying out actual designs. Specifically, the logical procedure is to divide the development of the theory into the sequence of four parts described briefly below, which represent rather different theoretical problems even though they may all be employed in designing a single filter.

Part I deals with the determination of the open- and short-circuit impedances of reactance 4-poles corresponding to prescribed insertion loss characteristics and pairs of terminations. The necessary and sufficient conditions upon insertion loss characteristics yielding physical sets of impedances are established as well as the procedure for determining the impedances.

Part II deals with the determination of reactance 4-poles of special configurations from prescribed open- and short-circuit impedances. These include a new canonical reactance 4-pole made up of tandem sections and also networks which have the same configurations as the familiar lattices and ladders of the image parameter theory but which are not required to satisfy the restrictions on actual element values imposed by the image parameter theory.

Part III deals with the problem of choosing specific insertion loss functions for filtering purposes. In particular, it deals with certain special types of loss functions which satisfy the conditions for physical realizability established in Part I, which represent the optimum choice

[14] The roots of the voltage ratio are determined in evaluating it from the roots of (1).

[15] When the dissipation varies from element to element or varies improperly with frequency, partial compensation can be obtained by assuming dissipation of the proper type representing an average of the actual dissipation. It is also possible to reduce the restrictions on the dissipation assumed if the non-dissipative design procedure is further modified. It is possible for instance, to assume all coils to be equally dissipative and all condensers to be equally dissipative to a different extent.

of arbitrary constants for meeting certain types of filter requirements, and which lead to simplifications in the more general design procedure.

Part IV deals with modifications of the previous theory permitting the design of certain types of dissipative networks with prescribed insertion loss characteristics. It is shown that within certain severe limits it is possible to design dissipative filters with loss characteristics differing appreciably from those of non-dissipative filters of the same configuration only by added constant losses.

Because of the extent of the material to be covered, detailed derivations and proofs will be only outlined in developing the four parts of the theory described above. In addition, a variety of useful but unessential modifications of the theory will be omitted entirely. These modifications have to do with the application of the theory to various special design problems rather than forming a part of the central theory itself.

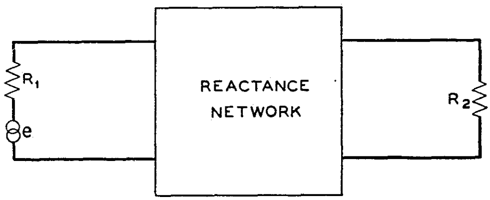


Fig. 3. Reactance 4-pole with resistance terminations

Part I. Open- and Short-Circuit Impedances of Networks with Prescribed Insertion Loss Characteristics

Statement of Problem

As is indicated in the above outline, the first two major divisions of the insertion loss theory are not restricted to the design of networks specifically for filtering purposes. Finite 4-terminal networks are assumed and these are required to be made up of pure reactances and to be terminated in constant resistances as in Fig. 3, but they are not necessarily required to have filter-like characteristics. The terminating resistances R_1, R_2 are assumed to be prescribed in each case and also the exact function of frequency representing the insertion loss α due to the presence of the reactance network. The specific problem considered is the design of the reactance network in accordance with these specifications and subject to the requirement of physical realizability.

The first division of the theory solves the problem of determining sets of open- and short-circuit impedances describing physical reactance networks which would produce the prescribed insertion loss when inserted between the prescribed terminations. When combined with Cauer's method (5) of realizing prescribed impedances with reactance 4-poles of his so-called canonical configurations, it amounts to an academic solution of the general design problem described above. The design of more important equivalent configurations will be described in the second division of the theory.

To insure the possibility of physical corresponding networks the prescribed insertion loss function α must be assumed to satisfy certain necessary and sufficient conditions. By definition, α satisfies the equation

$$e^{2\alpha} = \left| \frac{V_{20}}{V_2} \right|^2 \tag{5}$$

in which V_{20} and V_2 represent the complex voltages received before and after the insertion of the network between the prescribed terminations. Since it represents the square of the magnitude of the "complex insertion voltage ratio" $\frac{V_{20}}{V_2}$, the "insertion power ratio" $e^{2\alpha}$ must be an even rational function of frequency with real coefficients and must be positive at all real frequencies. Because the power delivered by a generator of voltage e and internal resistance R_1 cannot exceed $\frac{e^2}{4R_1}$, while the received power corresponding to V_{20} is $\frac{R_2 e^2}{(R_1 + R_2)^2}$, the power ratio $e^{2\alpha}$ must not only be positive at real frequencies but must not be less than $\frac{4R_1 R_2}{(R_1 + R_2)^2}$. In other words, the relation

$$e^{2\alpha} \geqq \frac{4R_1 R_2}{(R_1 + R_2)^2} \tag{6}$$

must be satisfied at all real frequencies.

The above conditions upon $e^{2\alpha}$ are obviously necessary whether or not the network inserted between the terminations must be made up entirely of reactances. It will be shown, however, that these conditions are sufficient to insure the existence of physical corresponding networks of the pure reactance type.

Form of Solution

It turns out that the open- and short-circuit impedances of a reactance network depend upon the insertion loss in a manner involving a rather complicated combination of the roots of the insertion power ratio and a related function, which are normally of fairly high degree. As a result, explicit equations expressing the impedances directly in terms of the insertion loss cannot well be obtained. On the other hand, a simple sequence of formulae has been derived which indicates the relationship and which permits the impedances to be calculated in numerical cases.

Instead of actually deriving the formulae, it will be sufficient to begin by merely stating them without proof. A derivation is unnecessary in that elementary circuit analysis can be used to show that any networks producing the impedances determined by these formulae would yield the prescribed insertion loss when inserted between the prescribed terminations. After the statement of the formulae, however, it will be necessary to demonstrate that the impedances can actually be realized physically.

Determination of Polynomials N and P from $e^{2\alpha}$

To determine the open- and short-circut impedances corresponding to the prescribed insertion loss α, the power ratio $e^{2\alpha}$ is first expressed in the form

$$e^{2\alpha} = \frac{N}{P^2} \tag{7}$$

in which N and P denote even polynomials in the familiar variable p representing $2\pi if$ and are required to have real coefficients. An expression of this type can be readily obtained, since $e^{2\alpha}$ must be an even rational function of the frequency f, while f^2 is proportional to $-p^2$. If the simplest rational fraction expression for $e^{2\alpha}$ does not have the form $\frac{N}{P^2}$ it is only necessary to multiply numerator and denominator by identical factors. As a matter of fact, N and P can always be chosen in a variety of ways, because of the possibility of associating the squares of arbitrary identical factors with N and P^2.

Determination of Polynomials A and B from N

After N and P have been determined, a second pair of polynomials A and B must be evaluated. These are even polynomials with real coefficients defined by the pair of equations

$$\begin{aligned} N &= J^2(p_1^2 - p^2)(p_2^2 - p^2) \cdots (p_n^2 - p^2) \\ A + pB &= J\,(p_1 - p\,)(p_2 - p\,) \cdots (p_n - p) \end{aligned} \tag{8}$$

in which J must be a real constant, all the p_σ's must have zero or negative real parts, and any complex p_σ's must occur in conjugate pairs. The fact that N must be positive at real frequencies in order for the power ratio $\frac{N}{P^2}$ to be positive is sufficient to insure the possibility of computing J and the p_σ's from their squares, as determined by N, in such a way as to meet the special restrictions imposed upon them.[16] As a matter of fact, there are always two solutions, although only two, because of the arbitrariness of the sign of J. The individual polynomials A and B can obviously be determined by expanding $(A + pB)$ and associating even powers of p with A and odd powers with pB.

The following expression for the complex insertion voltage ratio offers an alternative definition of A and B and indicates their physical significance:[17]

$$\frac{V_{20}}{V_2} = \frac{A + pB}{P} \tag{9}$$

That this voltage ratio corresponds to the original power ratio $e^{2\alpha}$ can be demonstrated by noting that the even polynomials A, B, P are real at real frequencies, while p is imaginary. This requires the power ratio to be

$$e^{2\alpha} = \frac{A^2 - p^2 B^2}{P^2} \tag{10}$$

[16] Since the p_σ's are determined by extracting the square roots of the p_σ^2's determined by N, the signs can be so chosen as to meet the real part requirement. Complex p_σ's with finite real parts will then automatically occur in conjugate pairs corresponding to conjugate p_σ^2's. Pure imaginary p_σ's, which have zero real parts, can also be chosen in conjugate pairs provided the corresponding negative real p_σ^2's occur in identical pairs. Single negative real p_σ^2's cannot be encountered since they represent real frequencies at which N and therefore $\frac{N}{P^2}$ change sign.

[17] In special cases, it may be possible to cancel identical factors out of the numerator and denominator of this expression. Unless the factors are constants or have roots only at real frequencies, however, their cancellation destroys the evenness of the denominator. The even and odd parts of the numerator then lose their utility in regard to the computation of the open- and short-circuit impedances.

The determination of a voltage ratio from a power ratio by (8) and (9) was introduced by Norton (7) in connection with his theory of filters terminated at one end in open- or short-circuits, which forms a part of his theory of constant-resistance groups of filters.

Since reversing the sign of p does not change the even polynomials A and B, the second equation of (8) requires

$$A - pB = J(p_1 + p)(p_2 + p) \cdots (p_n + p) \tag{11}$$

Hence forming the product of $(A + pB)$ and $(A - pB)$ indicates

$$A^2 - p^2B^2 = N \tag{12}$$

and reduces (10) to (7).

The voltage ratio (9) would correspond to the original power ratio even though roots of $(A + pB)$ had positive real parts, provided complex roots occurred in conjugate pairs. The exclusion of roots with positive real parts is merely a condition necessary for physical realizability. This condition is necessary in that the roots represent modes of free oscillation of the complete network consisting of the reactance 4-pole and its terminations.

Determination of Polynomials A' and B' from N and P

In addition to A and B, a very similar pair of polynomials A' and B' are needed. They are even polynomials with real coefficients defined by the following pair of equations, comparable to (8):

$$N - \frac{4R_1R_2}{(R_1 + R_2)^2} P^2 = J'^2(p_1'^2 - p^2)(p_2'^2 - p^2) \cdots (p_n'^2 - p^2)$$

$$A' + pB' = J'(p_1' - p)(p_2' - p) \cdots (p_n' - p) \tag{13}$$

In this case, J' must be real and any complex p_σ''s must occur in conjugate pairs but any of the p_σ''s are permitted to have positive real parts. These conditions can always be satisfied because of the fact that (6) and (7) require $\left(N - \frac{4R_1R_2}{(R_1 + R_2)^2} P^2\right)$ to be non-negative at real frequencies. In general, there are actually a number of solutions for A' and B' because of the arbitrariness of the signs of the p_σ''s as well as that of J'.

In the case of A' and B' there is again an alternative definition indicating physical significance, corresponding to equation (9) defining A and B. Suppose Z_1 and Z_2 represent the input impedances of the terminated network measured at the ends terminated in R_1 and R_2, respectively, as indicated in Fig. 4. Then A' and B' can be defined by the equations[18]

$$\frac{R_1 - Z_1}{R_1 + Z_1} = \frac{A' + pB'}{A + pB} \qquad \frac{R_2 - Z_2}{R_2 + Z_2} = \frac{-A' + pB'}{A + pB} \tag{14}$$

[18] In special cases, it may be possible to cancel out common factors from

The square of the magnitude of $(\pm A' + pB')$ is $(A'^2 - p^2B'^2)$, which in turn satisfies the equation

$$A'^2 - p^2B'^2 = N - \frac{4R_1R_2}{(R_1 + R_2)^2} P^2 \tag{15}$$

corresponding to (12). Hence (14) requires

$$\left|\frac{R_1 - Z_1}{R_1 + Z_1}\right|^2 = \left|\frac{R_2 - Z_2}{R_2 + Z_2}\right|^2 = 1 - \frac{4R_1R_2}{(R_1 + R_2)^2} e^{-2\alpha} \tag{16}$$

This expression can be checked by examining the relation between the input power and input impedance and recalling that input and output powers are identical in the case of a reactance network.

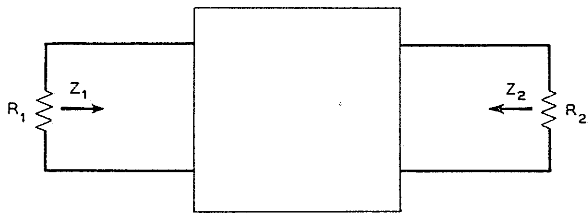


Fig. 4. Input impedances Z_1 and Z_2

The possibility of roots of $(A' + pB')$ with positive real parts is due to the fact that they are the roots of the non-physical impedance $(R_1 - Z_1)$. The simple relation between the two reflection coefficients indicated in (14) can be checked by simple circuit analysis. The following important relation between the A's and B's represents a combination of (12) and (15):

$$A'^2 - p^2B'^2 = A^2 - p^2B^2 - \frac{4R_1R_2}{(R_1 + R_2)^2} P^2 \tag{17}$$

Computation of the Open- and Short-Circuit Impedances from the A's, B's and P

In terms of the A's, B's and P the desired open- and short-circuit impedances are determined by the following formulae:

---

one or both of the equations for the reflection coefficients as in the case of the voltage ratio (9). If the denominators are not $(A + pB)$, however, the numerators lose their direct application to the computation of the open- and short-circuit impedances.

$$Z_{01} = R_1 \frac{A - A'}{p(B + B')} \qquad Z_{S1} = R_1 \frac{p(B - B')}{A + A'}$$

$$Z_{02} = R_2 \frac{A + A'}{p(B + B')} \qquad Z_{S2} = R_2 \frac{p(B - B')}{A - A'} \tag{18}$$

$$Z_{012} = \frac{-2R_1R_2}{(R_1 + R_2)} \frac{P}{p(B + B')} \qquad Z_{S12} = \frac{(R_1 + R_2)}{2} \frac{p(B - B')}{P}$$

In these formulae Z_{01}, Z_{S1}, Z_{02}, Z_{S2} are the familiar open- and short-circuit driving-point impedances of the reactance network as measured at the terminals next to terminations R_1 and R_2, respectively. Z_{012} and Z_{S12} are the less familiar open- and short-circuit transfer impedances defined in Fig. 5.[19] That the impedances determined by (18) correspond to the original power ratio and to the related voltage ratio and reflection coefficients of (9) and (14) can be checked by means of simple circuit analysis.[20]

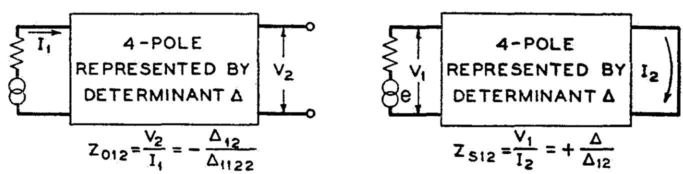


Fig. 5. Open- and short-circuit transfer impedances Z_{O12} and Z_{S12}

Physical Realizability of the Impedances Determined by (18)

The necessary and sufficient conditions for the realizability of a set of open- and short-circuit impedances with a physical reactance 4-pole can be stated as follows: First, the four driving-point impedances Z_{01}, Z_{02}, Z_{S1}, Z_{S2} must be separately realizable as the impedances of reactance 2-poles. Second, the two transfer impedances Z_{012}, Z_{S12} must be odd rational functions of p with real coefficients even though they need not be realizable as 2-terminal impedances. Finally, the various

[19] In terms of more conventional notation, these impedances would be referred to as the transfer impedance and the reciprocal of the transfer admittance. The more specific description is called for in the design of tandem section networks such as those to be described in Part II, which sometimes involves the use of short-circuit impedances or open-circuit admittances.

[20] The demonstration may also require the use of equation (17) relating the A's and B's if certain possible expressions for the voltage ratio and reflection coefficients in terms of the impedances are chosen as a starting point.

impedances must be inter-related by the following identities:

$$Z_{01}Z_{02} - Z_{012}^2 = Z_{01}Z_{S2} = Z_{02}Z_{S1} = -Z_{012}Z_{S12} \tag{19}$$

The above conditions differ from the more familiar necessary and sufficient conditions introduced by Cauer (5) in that the requirement of 2-pole realizability is imposed explicitly on all four driving-point impedances, instead of on only two, while Cauer's requirement on the residues of a transfer impedance is abandoned. It turns out that the requirement of four physical 2-pole impedances, together with (19), is sufficient to insure the satisfaction of Cauer's residue requirement. This can be demonstrated by analyzing the behavior of (19) in the neighborhood of a pole of Z_{012}.

An inspection of (18) shows that the expressions for the driving-point and transfer impedances are all odd rational functions of p with real coefficients. The required identities (19) are readily shown to be satisfied by merely replacing the impedances by the corresponding expressions of (18) and then making use of (17). To prove the complete realizability of the impedances it therefore remains only to show that the four driving-point impedances are not only odd rational functions of p with real coefficients but are actually realizable as the impedances of separate reactance 2-poles.

The independent realizability of the four driving-point impedances can be demonstrated with the help of the following theorem. If A_x and B_x are even polynomials in p with real coefficients such that $(A_x + pB_x)$ has no roots with positive real parts, then $\frac{pB_x}{A_x}$ is realizable as the impedance of a physical reactance 2-pole. The truth of this theorem is indicated by the argument outlined below in terms of the theory of positive real functions, or impedances of general 2-poles, as developed by Brune (8).

In the first place, the combination $\frac{A_x}{A_x + pB_x}$ must be a positive real function because of the fact that its poles lie in the left half of the p plane while its resistance part, $\frac{A_x^2}{A_x^2 - p^2 B_x^2}$, is non-negative on the real frequency axis.[21] Hence the reciprocal, $\left(1 + \frac{pB_x}{A_x}\right)$, is a positive real

[21] The polynomial $A_x + pB_x$ is permitted to have roots on the real frequency axis as well as in the left half of the p plane but these are always roots of A_x and pB_x and produce no pole of $\frac{A_x}{A_x + pB_x}$.

function. This requires $\frac{pB_x}{A_x}$ to be a positive real function, since subtracting a real constant from a positive real function leaves a positive real function provided the real part is still non-negative on the real frequency axis. Finally, the fact that the positive real function $\frac{pB_x}{A_x}$ is odd requires it to be realizable as the impedance of a 2-pole of the reactance type.

Dividing the four driving-point impedances of (18) by R_1 or R_2 and then replacing the open-circuit functions by their reciprocals yields the four functions corresponding to different choices of the signs in $\frac{p(B \pm B')}{(A \pm A')}$. Thus the four driving-point impedances will be realizable with separate reactance 2-poles provided $[(A \pm A') + p(B \pm B')]$ has no roots with positive real parts. That this condition is satisfied can be demonstrated by showing $\frac{[(A \pm A') + p(B \pm B')]}{A + pB}$ to be a positive real function. The positive realness of this new function is proved by the following conditions: First, the poles all lie in the left half plane, since $(A + pB)$ has no roots with positive real parts; second, the real part is non-negative at real frequencies, as can be shown by recalling that the A's and B's are real at real frequencies, by evaluating the real part of the function on this basis, and by then using equation (17) relating the A's and B's.[22]

Multiplicity of Solutions

As was indicated previously, a multiplicity of solutions for the impedances can be obtained in a variety of ways. In the first place, squares of identical factors can always be associated with the polynomials N and P^2, in the power ratio $\frac{N}{P^2}$. These factors cancel out in the final impedance formulae only if they are constants or if all their roots occur at real frequencies. In addition, the dependence of A, B, A', and B' upon N and P leaves the sign of $(A + pB)$ arbitrary and normally permits a number of solutions for A' and B'.

The choice of N and P determines the complex insertion voltage ratio $\frac{A + pB}{P}$ except as to sign, while the voltage ratio in turn determines the complexity of corresponding networks, which is normally the same

[22] Roots of $(A + pB)$ on the real frequency axis turn out to be roots of A, pB, A', and pB' and produce no poles of $\frac{[(A \pm A') + p(B \pm B')]}{A + pB}$.

for all choices of A' and B' permitted by N and P.[23] Making the degree of N as low as is permitted by the prescribed insertion loss naturally makes the degree of the voltage ratio as low as possible and leads to the simplest networks. When the squares of identical factors with roots at complex frequencies are combined with N and P^2, more complicated networks producing a modified voltage ratio are obtained. Some of these networks correspond to the addition of familiar all-pass phase-shifting sections to the networks of minimum complexity. An additional multiplicity of networks is normally possible, however, corresponding to impedances which cannot be realized in this way.

The analysis outlined above indicates the existence of a number of physical solutions for the impedances, all assuming a particular form. On the other hand, it does not show that there may not be other solutions of quite different forms. A more complicated analysis, however, shows that there can be no such alternative solutions. In other words, it can be demonstrated that the proper choice of N and P and of the corresponding A's and B's permits the determination of any set of impedances realizable with a reactance network corresponding to the prescribed insertion loss function. This will not be demonstrated here, for the object is merely to show that at least some physical sets of impedances can be obtained.

Practical Design Procedure

For practical purposes it is usually desirable to restrict the insertion power ratio $e^{2\alpha}$ somewhat further than is necessary merely to insure physical realizability. Suppose that identical factors other than constants or the simple factor p^2 must be combined with numerator and denominator of the simplest expression for $e^{2\alpha}$ in order to obtain the form $\frac{N}{P^2}$. It turns out that the corresponding network complexity will then be the same as though these added factors were not identical.[24] On the other hand, requiring the factors to be identical usually leads to less desirable loss characteristics than could be obtained by making

[23] In very special cases, special choices of A' and B' may lead to networks of complexities which are less than normal for the degree of the voltage ratio.

[24] The reason for this is as follows: If Δ is the determinant of the complete network formed by the reactance 4-pole together with its terminations, the insertion power ratio is proportional to $\frac{|\Delta|^2}{|\Delta_{12}|^2}$, in which the cofactor Δ_{12} is independent of the terminations and therefore is an odd or even polynomial in p. Thus the denominator of the simplest rational fraction expression for $e^{2\alpha}$ is the square of an even or odd polynomial in the frequency except when the element values are such that Δ and Δ_{12} have roots in common.

them different. Thus it is generally desirable to require $e^{2\alpha}$ to be so chosen that no identical factors other than constants or the simple factor p^2 need be combined with the simplest corresponding rational fraction in forming $\frac{N}{P^2}$. This merely requires all the poles of $e^{2\alpha}$ to occur in identical pairs.

After the power ratio has been chosen the corresponding open- and short-circuit impedances can be computed in a straightforward manner by means of the formulae described above. Corresponding networks of the canonical type devised by Cauer (5) can then readily be designed from the partial fraction expansions of various of these impedances or of the corresponding admittances. More frequently, however, it is preferable to use equivalent tandem section networks, which can be designed by methods to be described in the next section.

In the determination of the impedances considerable labor may be encountered in the extraction of the roots of the polynomials N and $\left(N - \frac{4R_1R_2}{(R_1 + R_2)^2} P^2\right)$, which are needed in the formation of the polynomials $(A + pB)$ and $(A' + pB')$ in accordance with (8) and (13). The required root extraction can now be expedited, however, by the use of machines which have recently been constructed for the determination of the roots of any polynomials of reasonable degree.[25] In the design of networks specifically for filtering purposes, moreover, the special polynomials encountered are usually such that the greater part of the root extraction labor can be avoided by the use of special methods to be described in Part III.

The theory described above assumes non-dissipative reactance networks. The distortion due to the parasitic dissipation which must be present in actual networks can be readily estimated as soon as the complex voltage ratio is determined provided the dissipation is uniformly distributed. This can be accomplished by making use of the theory of uniformly dissipative networks as developed by Mayer (10) and Bode, which shows how the effects of uniform dissipation can be estimated from the derivative of the insertion phase.[26] Ways of avoiding distortion due to dissipation will be described in Part IV, which will be devoted to methods of designing certain types of dissipative networks producing prescribed insertion loss characteristics.

[25] See, for instance, the description of Fry's isograph recently published by Dietzold (9).

[26] Bode developed a theory similar to Mayer's independently but published only a brief reference to it. This reference is in the paper "Ideal Wave Filters" (11) which he published in collaboration with Dietzold.

One Open- or Short-Circuit Termination

Reactance networks terminated at one end in open- or short-circuits can be considered as limiting cases of the previous networks in which R_1 or R_2 approaches zero or infinity. They are of interest for two principal reasons. For one thing, it is sometimes advantageous to approximate actual open-circuit terminations by means of vacuum tubes. In addition, when groups of filters are to be combined in series or in parallel at one end, it is sometimes best to design the separate filters as though they were to be terminated in open- or short-circuits at their common terminals.[27]

Although the general methods described above can still be used, difficulties with the specific impedance formulae are experienced when one termination approaches zero or infinity. It turns out that all but one of the possible solutions for the polynomials A' and B' permitted by (13) lead to element values which approach zero or infinity with the termination. Even with the one permissible choice most of the open- and short-circuit impedance formulae become indeterminate.[28]

More careful analysis yields the following special relations.[29] If termination R_2 is infinite

$$Z_{01} = R_1 \frac{A}{pB} \qquad Z_{012} = -R_1 \frac{P}{pB} \tag{20}$$

$$R_{Z2} = R_1 e^{-2\alpha}$$

in which R_{Z2} is the resistance measured at the open-circuited end of the terminated network. If termination R_2 is zero

$$Z_{S1} = R_1 \frac{pB}{A} \qquad Z_{S12} = R_1 \frac{pB}{P} \tag{21}$$

$$G_{Y2} = \frac{1}{R_1} e^{-2\alpha}$$

[27] This is the method introduced by Norton (7) for the design of constant-resistance groups of filter. It can also be used to design more efficient filter groups producing approximately constant resistances or conductances at their common terminals at pass band frequencies, the susceptances or reactances being later corrected by the addition of reactance 2-poles.

[28] Some such indeterminacy is necessary, since any arbitrary impedance can be connected in parallel with a short-circuit termination or in series with an open-circuit termination without affecting the operation of the circuit.

[29] The formulae for Z_{O1}, Z_{S1}, R_{Z2}, G_{Y2} are exhibited by Norton (7) as a part of his theory of constant resistance groups of filters.

in which G_{Y2} is the conductance measured at the short-circuited end of the terminated network. Z_{012}, Z_{S12} are the open- and short-circuit transfer impedances while Z_{01}, Z_{S1} are the open- and short-circuit driving-point impedances corresponding to the end of the network terminated in the finite resistance R_1. The polynomials A, B and P are determined from the power ratio exactly as though termination R_2 were finite.

If minimum complexities are assumed, networks can be designed with no additional impedance data. This can be shown by ananalysis of the relations between the open- and short-circuit impedances of physical reactance 4-poles. It will also be clarified by the ladder network theory which will be considered in the next section.

Part II. Realization of Open- and Short-Circuit Impedances with Physical Configurations

Methods have been developed for designing reactance 4-poles of a variety of different configurations producing prescribed open- and short-circuit impedances, such as impedances determined from insertion loss functions by the method described in Part I. Some of these configurations are said to be canonical in that they can be designed to have any set of impedances realizable with physical reactance networks. Others are less general but can frequently be used more advantageously or else amount to special cases of the general configurations for which simpler design methods have been developed.

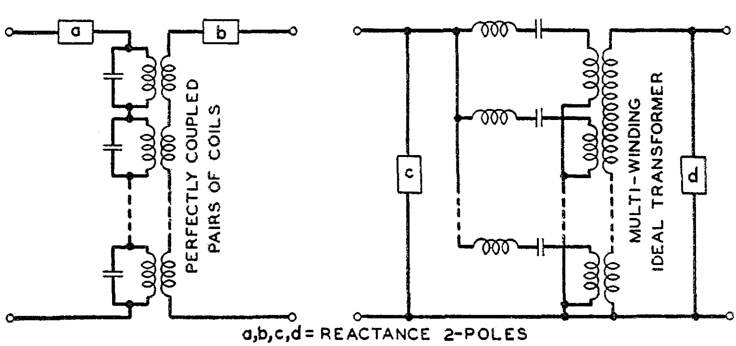


Fig. 6. Cauer's canonical reactance 4-poles

Cauer's Canonical Configurations

The simplest general design theory applies to the design of the two canonical reactance 4-poles introduced by Cauer (5). These have the configurations indicated in Fig. 6. They are designed in terms of partial fraction expansions of the three open-circuit impedances or short-circuit admittances by noting certain relations between partial fractions of the different expansions and the correspondence of these partial fractions to the various network branches. Since Cauer has described these networks very completely, it will not be necessary to describe them in more detail here.

A Canonical Tandem Section Configuration

Cauer's canonical configurations are of particular interest only in theoretical studies of the properties of reactance 4-poles. When it comes to actual construction it is almost always preferable to use equivalent circuits consisting of simple networks or sections connected in the tandem manner indicated in Fig. 7.[30] In the case of selective networks or filters the use of tandem sections is usually a practical necessity. In the first place, the use of tandem sections permits reasonable approximations to theoretical transmission characteristics to be obtained with much less precise adjustment of the elements to their theoretical values.

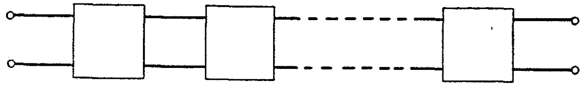


Fig. 7. Tandem 4-poles

In addition, the filters most commonly encountered can be built in tandem section form without the use of the mutual inductances required in equivalent networks of Cauer's canonical configurations.[31]

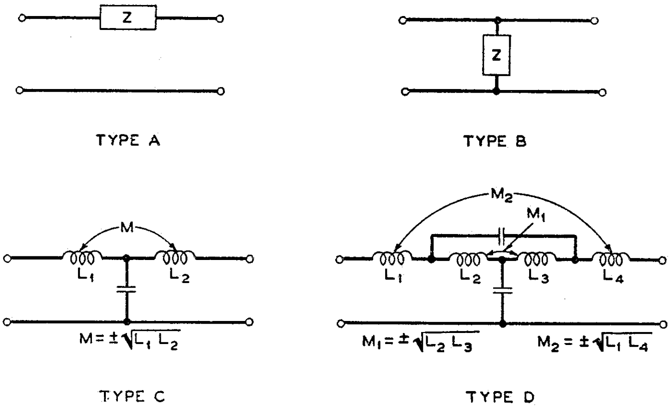
For many purposes the ladder network is the most useful form of tandem section combination. Even under its most general definition as any sequence of alternate series and shunt 2-terminal impedances, however, the ladder configuration cannot be used to realize all open- and short-circuit impedances realizable with more general reactance 4-poles. On the other hand, it can be shown that a slight modification

[30] By section is not necessarily meant a filter section with the properties of those of the image parameter theory but merely the constituent parts of a combination of 4-poles when these are to be connected in tandem.

[31] The requirement of mutual inductances adds substantially to the difficulty of building a network. If perfect coupling between coils is called for, it can be only approximated. If less than perfect coupling is required, it is difficult to obtain simultaneous adjustments of the self and mutual inductances.

or generalization of the ladder configuration can always be realized and hence can be referred to as a canonical reactance 4-pole. This configuration is defined as any tandem combination of sections included in the four types *A*, *B*, *C*, *D* indicated in Fig. 8, plus possibly an ideal transformer in tandem at one end of the network.[32]

In many cases sections of all four types are not required in the canonical network described above and frequently the ideal transformer does not appear. It also turns out that series inductances can often be included in the series branches, or networks of type *A*, in such a way that the inductive coupling in sections of types *C* and *D* can be less than perfect or can even be completely replaced by separate self-inductances in the shunt branches of these sections. In most filters, for instance,



Z = a general reactance 2-pole. Types *C* and *D* may be dissymetrical.

Fig. 8. Types of tandem sections yielding a canonical reactance 4-pole

neither sections of type *D* nor the ideal transformer appear, while the inductive coupling in sections of type *C* can almost always be eliminated.

General Theory of the Tandem Section Configuration

The input impedance of a reactance 4-pole terminated at one end in a constant resistance can readily be determined from the open- and short-circuit impedances. Conversely, it can be shown that the open- and short-circuit impedances can normally be determined from the impedance function corresponding to a prescribed terminating resistance,

[32] Other canonical networks of similar tandem sections can be obtained by modifying somewhat sections of types C and D in accordance with the principles of inverse networks and frequency transformations. It will be sufficient to consider here, however, only the particular type of tandem combination described above.

except for an obvious ambiguity as to the signs of the transfer impedances corresponding to the possibility of interchanging the input or output terminals. The only exceptions to the rule correspond to values of the termination leading to impedances of reduced degree, and there can be only a finite number of such values for any one network.[33]

In accordance with the above principle, one way to obtain prescribed open- and short-circuit impedances of the reactance type is to design a resistance-terminated reactance 4-pole as a 2-pole producing an input impedance computed from the open- and short-circuit impedances. The addition of the proper ideal transformer to the terminated end of the 4-pole will yield the prescribed open- and short-circuited impedances provided the design is so carried out that the degree of the input impedance is normal for the prescribed impedances and for the final configuration. The added ideal transformer is generally necessary in order to obtain the terminating resistance assumed in computing the input impedance from the required open- and short-circuit impedances.

Brune (8) has shown how any prescribed positive real function can be realized as the input impedance of a tandem combination of sections of types A, B and C provided it is permissible to include series resistances between sections. When the roots of the resistance part of the positive real function occur only in identical pairs and are all real or imaginary, Brune's design procedure can be carried out in such a way as to eliminate all the resistances except that forming the termination.[34] The requirement of real or imaginary roots turns out to be unnecessary if Brune's procedure is replaced by a recent modification permitting sections of type D to be used. The configuration obtained can be shown to be such that impedance functions of the same degree as the prescribed function would be obtained with general values of the terminating resistance, i.e., the prescribed function does not represent a special case

[33] In terms of the determinant Δ of the reactance 4-pole and various of its cofactors, the input impedance is $\dfrac{\Delta + R_2\Delta_{22}}{\Delta_{11} + R_2\Delta_{1122}}$. Except in special cases in which the cancellation of identical factors leads to reductions in the degree of the impedance function, this requires $(\Delta + R_2\Delta_{22})$ and $(\Delta_{11} + R_2\Delta_{1122})$ to be proportional to $Kp^n(A_1 + pB_1)$ and $Kp^n(A_2 + pB_2)$, respectively, in which $(A_1 + pB_1)$ and $(A_2 + pB_2)$ are the numerator and denominator of the simplest rational fraction expression for the impedance function. The necessary evenness and oddness of the determinants permits of only two corresponding solutions for the open- and short-circuit impedances. Since it turns out that passing from one solution to the other reverses the sign of Δ_{12}^2, only one solution can be physical.

[34] The resistance part of the function is defined as the even part and represents the real part only at real frequencies.

of reduced degree.[35] Thus a tandem section equivalent can readily be found corresponding to any reactance 4-pole which can produce an input impedance such that the degree is normal for the configuration and such that all the roots of the resistance part occur in identical pairs. This turns out to be a property of all physical reactance 4-poles.[36]

The analysis outlined above proves the canonical nature of the reactance 4-pole formed by adding an ideal transformer to one end of a tandem combination of sections of types A, B, C, D. It can also be shown that the two-terminal impedance formed by closing one pair of terminals through a resistance termination constitutes a canonical general 2-pole. It was indicated above that this type of 2-pole can correspond to any positive real function provided the roots of the resistance part occur only in identical pairs. The restriction on the positive real function turns out to be unnecessary, however, when it is permissible to use special values of the terminating resistance leading to impedance functions of reduced degree.[37] Because of the canonical nature of the reactance 4-pole, this can be demonstrated by merely showing how it is possible to find a set of physical open- and short-circuit impedances of the reactance type leading to an input impedance represented by any prescribed positive real function. How this can be accomplished is indicated below.

Suppose that a reactance network is to be so designed as to produce a prescribed input impedance at one end when terminated at the far end in a prescribed resistance. Because of the identity of the input and received powers, the insertion loss that would be obtained upon terminating the input end in a second resistance can readily be computed from the prescribed impedance.[38] The general theory of Part I can then be used to determine the corresponding sets of physically realizable open- and short-circuit impedances. It is easily shown that one of these

[35] This will be clarified by the subsequent description of the more detailed design procedure.

[36] In terms of the determinant Δ of a reactance network and various of its cofactors, the resistance part of the input impedance is represented by $\frac{R_2\Delta_{12}^2}{\Delta_{11}^2 - R_2^2\Delta_{1122}^2}$. It is obvious that single roots of the denominator can coincide with any of the double roots of the numerator only when R_2 has special values.

[37] Because the impedance function must generally be of less than normal degree for the configuration, there are usually equivalent 2-poles representing substantially more efficient use of the elements. In spite of this practical disadvantage, a knowledge of the canonical nature of this particular 2-pole is useful in general network theory problems.

[38] Specifically, the loss can be computed with the help of (16) of Part I.

will correspond to the prescribed impedance provided it is a positive real function.

Design Procedure

Consider now the actual operations involved in the design of a reactance 4-pole of the tandem section type producing prescribed open- and short-circuit impedances. As indicated above, the first step is to compute the input impedance of either end of the network corresponding to an arbitrary far-end termination subject only to the requirement that the degree of the impedance function must be normal for the open- and short-circuit impedances. A section of type A, B, C, or D is then designed in such a way that it will produce the required input impedance when terminated in a new physical impedance of lower degree. A second section is next designed to produce the required terminating impedance when itself terminated in a new impedance of further reduced degree. This procedure is continued until the required terminating impedance is reduced to a constant resistance.[39] Finally, the required terminating resistance is replaced by the equivalent combination of an ideal transformer terminated in a resistance identical with the termination assumed in computing the original input impedance.

Brune shows how any positive real function with a pole or root on the axis of real frequencies can be realized as the input impedance of a section of type A or B, respectively, terminated in an impedance of reduced degree. Brune also shows how any positive real function with no roots or poles on the real frequency axis but such that the resistance part has a pair of identical real or imaginary roots can be realized as the input impedance of a section of type C terminated in a physical impedance of reduced degree. Finally, if the roots of the resistance part of the input impedance all occur in identical pairs, the terminating impedance required in each case turns out to have this same property.[40] Hence, to complete the explanation of the design procedure it remains only to show how the appearance of identical complex roots of the resistance part permits a section of type D to be made use of in the same

[39] The exact manner in which the degree of the terminating impedance is reduced at each stage in the design is what leads to a final configuration for which the original input impedance is of normal degree.

[40] The computation of the terminating impedance from the input impedance may eliminate a pair of identical roots of the resistance part in reducing the degree of the impedance function and may convert one or more pairs of identical real roots into roots or poles of the impedance function or vice versa. Otherwise the roots of the resistance part remain unchanged.

way that a section of type C can be used when the roots are real or imaginary.

Design of Sections of Type D

Suppose that a prescribed section of type D produces a prescribed input impedance Z when terminated in impedance Z_L. If Δ is used to represent the determinant of the section of type D and if the input and output meshes are numbered 1 and 2, respectively, Z_L is related to Z by

$$Z_L = \frac{\Delta - Z\Delta_{11}}{Z\Delta_{1122} - \Delta_{22}} = \frac{\Delta_{12}^2 Z}{\Delta_{22}(\Delta_{22} - Z\Delta_{1122})} - Z_{s2} \qquad (22)$$

in which Z_{s2} represents the short-circuit impedance of the section of type D at the terminated end. If it is assumed that Z is a positive real function such that its resistance part has a pair of identical complex roots, it can be shown that the following condition always determines a physical section of type D which leads to a physical Z_L of lower degree than Z: The functions Δ_{12}^2 and $(\Delta_{22} - Z\Delta_{1122})$ are required to have coincident pairs of identical complex roots which are also coincident with a pair of identical complex roots of the resistance part of Z.

It is easily shown that the above coincidence of roots of Δ_{12}^2 and $(\Delta_{22} - Z\Delta_{1122})$ leads to the elimination of all roots of the impedance $(Z_L + Z_{s2})$ which are not also roots of Z, which excludes the possibility of roots in the right half of the p plane. If it is also assumed that the section of type D is physical, simple additional analysis shows that $(Z_L + Z_{s2})$ must actually be a positive real function. This proves the positive realness of Z_L itself except that it does not exclude real-frequency poles with negative residues covered up in $(Z_L + Z_{s2})$ by the positive residues of coincident poles of Z_{s2}. Finally, it can be shown that any coincident poles of the two impedances can be separated, without changing the corresponding residues of Z_L, by adding Lp to the original input impedance Z. The reduced degree of Z_L can be demonstrated by showing that the above conditions lead to eight coincidences of roots and poles of the expression $\dfrac{\Delta - Z\Delta_{11}}{Z\Delta_{1122} - \Delta_{22}}$ of (22).

To show that the condition on the roots of Δ_{12}^2 and $(\Delta_{22} - Z\Delta_{1122})$ actually leads to a physical section of type D, definite formulae for the element values are first derived from this condition without considering the question of physical realizability. The input impedance Z appears in these formulae only through the appearance of the values assumed by Z and its derivative at double roots of its resistance part. This permits Z to be replaced in the formulae by the open-circuit impedance of a physical reactance 4-pole which produces the input impedance Z when terminated in the proper constant resistance.[41] It can then be shown that the corresponding capacities and self inductances of the section of type D will be finite real and positive.

Conditions Necessary for Ladder Networks

When all the frequencies of infinite loss are real or imaginary, sections of type D do not have to be included in the canonical tandem section configuration.[42] This is because the frequencies of infinite loss of the complete network are the corresponding critical frequencies of the separate sections, while sections of type D are required only to realize complex frequencies of infinite loss. When sections of type D are absent the network can be considered as a general ladder of reactances, which may be defined as any combination of alternate series and shunt branches consisting of reactance 2-poles. While the coupling between the series inductances in sections of type C renders them somewhat more complicated than alternate series and shunt 2-poles the coupling can be thought of as merely a device for realizing negative inductances appearing in the equivalent T networks.[43]

The Mid-Series Low-Pass Ladder Configuration

Of the large variety of actual configurations possible in ladders of the type described above, only a few are commonly made use of. An extensive special design theory has been developed for these particular configurations in order to permit the element values to be determined

[41] As was indicated previously, the theory of Part I can be used to demonstrate that any positive real function can be realized as the input impedance of a resistance-terminated reactance 4-pole.

[42] By the frequencies of infinite loss of a network is meant the frequencies of infinite loss obtained with general finite resistance terminations excluding any specific terminations which bring roots and poles of the general expression for the insertion voltage ratio into coincidence. Each frequency of infinite loss is included in one or more of the following groups of critical frequencies: roots of the open-circuit transfer impedance, poles of the short-circuit transfer impedance, coincident roots or poles of open- and short-circuit driving-point impedances of the same end of the network, and zeros of the resistance part of an input impedance obtained with a resistance termination.

[43] In many cases the negative inductance can be eliminated without the introduction of the coupling but the conditions under which this can be done are not subject to simple statement.

without the labor involved in using the general theory of tandem sections described previously. The special design theory is best developed first for the specific type of ladder indicated in Fig. 9, which may be referred to as the mid-series low-pass configuration even though it is not necessarily a mid-series type low-pass filter of the image parameter theory.[44] The other configurations commonly encountered can be designed by means of simple modifications of the theory of this special case.

The earliest special formulae for computing the element values of ladders of the mid-series low-pass configuration indicated in Fig. 9 were developed by Norton (7) as a part of his theory of constant resistance pairs of filters. Although Norton's formulae represented an important step in the development of the theory of mid-series low-pass ladders, it has been found that the computations which they call for in numerical problems are undesirably complicated and must usually be carried to an abnormally high precision.

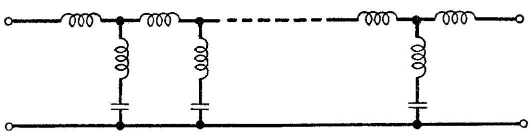


Fig. 9. The mid-series low-pass ladder configuration

As a result an extensive further analysis of the design problem has been carried out leading to the derivation of a new set of formulae. These new formulae are relatively satisfactory for numerical computations and are also useful in a variety of theoretical investigations, such as the determination of what impedances can be realized with mid-series low-pass ladders. In addition, the derivation of the formulae involves the development of an alternative set which are expressed very compactly in terms of determinants and which are useful in certain theoretical investigations even though they have the same disadvantages as Norton's when applied to ordinary numerical problems.

Assumptions and Conditions Leading to Design Formulae

The development of the design formulae for ladders of the mid-series low-pass configuration is simplified if certain simple assumptions are adopted temporarily. The procedure to be followed when the assumptions are not satisfied can best be investigated after the formulae have

[44] This type of ladder is obviously equivalent to sections of type C generally combined with one or more sections of type A consisting of simple series inductances.

been derived. In the first place, it is simplest to start by assuming a set of open- and short-circuit impedances to have been specified which are known in advance to be appropriate for the configuration, leaving until later the question of what impedances have this property. It is also best to assume further that the impedances are such that the multiplicity of solutions for the elements which are normally encountered are all possible, even though special sets of impedances can be found which require certain of the solutions to be excluded. Certain difficulties are also avoided by the temporary assumption that all the frequencies of infinite loss are different and that all the open- and short-circuit impedances are of normal degree for the configuration even though some may be of reduced degree in special cases.

The following relations, which can be shown to apply to any ladder of the mid-series low-pass configuration consistent with the above assumptions, form the basis of all known formulae for the element values.[45] First, the resonances of the shunt branches are identical with the frequencies of infinite loss except for a single infinite loss point at infinity.[46] Second, the value assumed by any of the open- and short-circuit driving-point impedances at a shunt branch resonance is independent of the elements separated from its terminals by the shunt branch, which acts as a short-circuit across the ladder. Finally, under the assumption of impedances of normal degree for the configuration the derivatives of the driving-point impedance functions with respect to the frequency have this same property.

It turns out that the relations stated above are sufficient to determine all element values from one open-circuit or short-circuit driving-point impedance together with the frequencies of infinite loss, except for the far end inductance in the case of an open-circuit impedance.[47] The

[45] These relations were introduced by Norton as the basis of his design equations.

[46] Under the present assumptions, the finite frequencies of infinite loss are the roots of the open-circuit transfer impedance and also the finite poles of the corresponding short-circuit impedance.

[47] This at first seems contrary to the well known fact that it takes three impedances to fix a 4-pole. The additional data are here supplied by the assumption of impedances appropriate for a specific configuration subject to special restrictions. Recall that Cauer's canonical reactance 4-pole of the shunt or admittance type can be designed from one short-circuit driving-point impedance and the short-circuit transfer impedance except for a two terminal shunt branch across the far end of the network. For the particular circuit under consideration the terminal shunt branch would be absent in the equivalent shunt type canonical network while the short-circuit transfer impedance could be found from a short-circuit driving-point impedance and the frequencies of infinite loss.

multiplicity of solutions which are normally obtainable are due only to the fact that the finite frequencies of infinite loss can be distributed arbitrarily among the shunt branches as their individual resonances. In order to obtain a unique solution it is expedient to assume at the outset that a particular distribution has been chosen. The problem then becomes that of realizing a known two-terminal impedance of the reactance type as an open-circuit or short-circuit impedance of a ladder of the midseries low-pass configuration with prescribed shunt branch resonances.

Continued Fraction Expansion Forming Basis of Design Problem

The development of the formulae for the element values calls for the introduction of extensive special notation so chosen as to reduce the design problem to the determination of a particularly simple continued fraction expansion of a known function. In the first place, instead of dealing directly with the element values, it is simpler to consider the

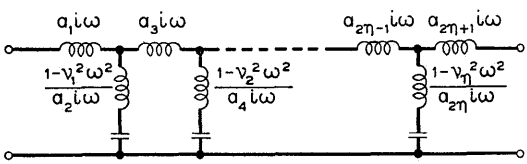


Fig. 10. Designation of the impedance branches of a mid-series low-pass ladder

constants in the designations indicated in Fig. 10 for the impedances of the various network branches, expressed in terms of ω rather than $p = i\omega$.[48] The constants $\nu_1, \cdots, \nu_\eta$ represent the reciprocals of the shunt branch resonances or finite points of infinite loss in terms of ω and are thus assumed to be known.[49] The problem under consideration therefore amounts to the determination of the so-called ladder network coefficients $a_1, \cdots, a_{2\eta+1}$, since there can be no difficulty in determining the element values from these coefficients together with the ν_σ's. In order to avoid ambiguity in the design formulae referred to previously, which are merely specific formulae for the individual a_σ's, it is convenient to include the additional requirement that the numbering of the coefficients a_1, a_2, etc., shall begin at the terminals of the particular

[48] Although the variable p is more convenient in formulating the general theory of Part I, ω turns out to be more convenient in the ladder network theory under consideration here.

[49] The reciprocals of the values of ω at infinite loss points are more convenient than the values themselves when the infinite loss points are later permitted to become infinite.

open-circuit or short-circuit driving-point impedance from which they are to be computed.

It is well known that the problem of realizing a 2-terminal impedance as the input impedance of a ladder network of prescribed configuration amounts to the problem of obtaining a prescribed form of continued fraction expansion of the impedance or of some related function.[50] For the particular network under consideration, the required continued fraction is simplest if the function F to be expanded is derived by dividing the impedance function by p, or by its equivalent $i\omega$. In other words, F is best defined by

$$F = \frac{Z}{i\omega} \tag{23}$$

where Z is the open- or short-circuit impedance from which the coefficients a_1, a_2, etc. are to be computed.

Since Z is an odd rational function of $i\omega$, the quantity F must be a function of ω^2. This suggests replacing ω^2 by a new variable in order to decrease the degree of F. It turns out, however, that a simpler continued fraction is obtained if the reciprocal of this variable is used. Hence the following additional notation is introduced:

$$z = \frac{1}{\omega^2} \tag{24}$$

$$z_\sigma = \nu_\sigma^2 \qquad \sigma = 1, \cdots, \eta$$

in which z represents the new variable, while the z_σ's indicate the values of z corresponding to the frequencies of infinite loss. In terms of this notation, the required continued fraction expansion of F takes the following form:

$$F = a_1 + \cfrac{1}{\cfrac{a_2}{z_1 - z} + \cfrac{1}{a_3 + \cfrac{1}{\cfrac{a_4}{z_2 - z} + \cdots}}} \tag{25}$$

The problem is to solve this identity for the a_σ's assuming the constants z_σ to be known and also the function F of the variable z.

In deriving the more useful formulae for the a_σ's which have been developed by solving the above problem the first part of the analysis

[50] A variety of ladders described by Fry (12), for instance, correspond to Stieltjes' Fractions.

is devoted to the derivation of the alternative formulae referred to previously as being expressed compactly in terms of determinants. The final formulae are then derived by expanding the determinants in terms of the partial fraction representation of the function F. Since the two parts of the derivation both involve long and complicated algebraic manipulation, they will be no more than briefly outlined here. Greater clarity will be obtained if the statement of the preliminary formulae in terms of determinants precedes the outline of their derivation.

Formulae for the a_σ's in Terms of Determinants

The determinants appearing in the preliminary solution for the a_σ's are formed from the quantities H_{qr} defined as follows in terms of the notation introduced above:

$$H_{qr} = \frac{F_q - F_r}{z_q - z_r} \qquad H_{qq} = F'_q \tag{26}$$

where F_q, F'_q, etc., are used to represent the values assumed by F and $\frac{dF}{dz}$ at $z = z_q$, etc. The use of the notation H_{qr} and H_{qq} is consistent since H_{qq} is the limit approached by H_{qr} as z_r approaches z_q.

The determinants themselves are of three different types. The determinant U_k is defined as $|H_{qr}|$ in which q and r take the values 1 to k. In other words,

$$U_k = \begin{vmatrix} H_{11} & H_{12} & \cdots & H_{1(k-1)} & H_{1k} \\ H_{21} & H_{22} & \cdots & H_{2(k-1)} & H_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{k1} & H_{k2} & \cdots & H_{k(k-1)} & H_{kk} \end{vmatrix} \tag{27}$$

The determinant V_k is obtained from U_k by changing the elements of the last column to unity. In other words,

$$V_k = \begin{vmatrix} H_{11} & H_{12} & \cdots & H_{1(k-1)} & 1 \\ H_{21} & H_{22} & \cdots & H_{2(k-1)} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{k1} & H_{k2} & \cdots & H_{k(k-1)} & 1 \end{vmatrix} \tag{28}$$

Finally, the determinant W_k is obtained from U_k by changing the elements of the last column from $H_{\sigma k}$ to $H_{\sigma(k+1)}$. Thus,

$$W_k = \begin{vmatrix} H_{11} & H_{12} & \cdots & H_{1(k-1)} & H_{1(k+1)} \\ H_{21} & H_{22} & \cdots & H_{2(k-1)} & H_{2(k+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{k1} & H_{k2} & \cdots & H_{k(k-1)} & H_{k(k+1)} \end{vmatrix} \tag{29}$$

In terms of these determinants all the ladder network coefficients except for $a_{2\eta+1}$ are given by the following design equations:[51]

$$\begin{aligned} a_1 &= F_1 \qquad & a_{2k-1} &= \frac{(z_k - z_{k-1})U_{(k-1)}W_{(k-1)}}{V_{(k-1)}V_k} \\ a_2 &= \frac{-1}{U_1} \qquad & a_{2k} &= \frac{-V_k^2}{U_{(k-1)}U_k} \end{aligned} \tag{30}$$

$$k = 2 \cdots \eta$$

The coefficient $a_{2\eta+1}$ corresponds to the inductance forming the last series branch. If the function F is obtained from an open-circuit impedance, the coefficient must be determined from some other impedance. If F corresponds to a short-circuit impedance, $a_{2\eta+1}$ can be found from the value assumed by F at zero frequency, or by using the above formulae with an arbitrary additional constant $z_{\eta+1}$.[52]

The above formulae can be derived by a method of induction which is reasonably straightforward though long and tedious. The formulae for the first four coefficients are first derived from Norton's equations or directly from the behavior of the continued fraction (25) in the neighborhood of $z = z_1$ and $z = z_2$. This yields the special formulae for a_1 and a_2 indicated above and shows the formulae for a_3 and a_4 to be consistent with the general formulae indicated for a_{2k-1} and a_{2k}. The derivation is then completed by showing that if the formulae for a_{2k-1} and a_{2k} are correct, then those for a_{2k+1} and a_{2k+2} must also be correct. This is demonstrated by first using the formulae for a_{2k-1} and a_{2k} to express a_{2k+1} and a_{2k+2} in terms of the impedance that would be obtained by removing the first two branches of the network, corresponding to

[51] The formulae for a_2, a_3, a_4, involve U_1, V_1, W_1. As defined by (27), (28) and (29) these quantities appear strange in that they are determinants of only the first order. These first order determinants, however, merely represent H_{11}, 1, H_{12}, respectively.

[52] If F corresponds to a short-circuit impedance, an arbitrary shunt branch can be assumed to be connected across the short-circuit since it will not affect F. This permits $a_{2\eta+1}$ to be determined exactly as though there were a complete additional "section."

a_1 and a_2. This "reduced" impedance is then replaced by an equivalent expression in terms of the original impedance and the formulae for a_1 and a_2. Considerable manipulation of the determinants in the resulting equations finally yields the general formulae for a_{2k+1} and a_{2k+2}.

The formulae (30) are indeterminate unless all the constants $z_1, \cdots, z_\eta$ are different in accordance with the original assumption of no two identical frequencies of infinite loss. Coincident frequencies of infinite loss can be handled, however, by assuming infinitesimal differences and making use of a Taylor's series expansion of the function F representing $\frac{Z}{i\omega}$ considered as a function of $z = \frac{1}{\omega^2}$. When all the frequencies of infinite loss are identical except for the single one at infinity, the continued fraction (25) becomes a Stieltjes' fraction of a type considered by Fry (12). The known formulae for the constants in the Stieltjes' fraction expansion are undoubtedly derivable from (30) by the Taylor's series method.[53]

The effect of abandoning the original assumptions other than that of no two identical frequencies of infinite loss can best be considered after the final formulae have been developed by showing how the determinants of the first set can be expanded.

Derivation of the Final Design Formulae by Expanding the Determinants

It is readily shown that if the function F is expanded into a sum of partial fractions, this expansion will always take the following form provided the open- or short-circuit driving-point impedance from which it is derived is physically realizable:

$$F = -B_0 z + B_\infty + \frac{B_1}{z - \beta_1} + \cdots + \frac{B_\mu}{z - \beta_\mu} \tag{31}$$

In this expression the B's and β's are all positive and are also all finite except that B_0 may sometimes be zero.

In terms of the partial fraction expansion of F, the determinant element H_{qr} as defined in (26) becomes

$$H_{qr} = -B_0 - \frac{B_1}{(\beta_1 - z_q)(\beta_1 - z_r)} - \cdots - \frac{B_\mu}{(\beta_\mu - z_q)(\beta_\mu - z_r)} \tag{32}$$

[53] This derivation has not been proved rigorously but has been carried far enough to indicate the way in which the transformation of the formulae takes place.

while H_{qq} is obtained by merely equating z_r to z_q in this formula. The determinants U_k, V_k, W_k defined in (27) through (29) and appearing in the formulae (30) for the a_σ's can be expanded in terms of these partial fraction representations of the H_{qr}'s. The derivation of these expansions, however, is too long and complicated to be more than briefly outlined here.

In deriving the expansions of the determinants, the particular W_k determinant of degree identical with the number of partial fractions in the expansion of H_{qr} is first examined. It is found that this particular W_k determinant can be expressed in terms of a product of two determinants of the general form $\left|\dfrac{1}{x_i - y_j}\right|$, which are evaluated in well-known treatises on determinant theory.[54] It is then shown that W_k determinants of higher degree must vanish while those of lower degree are equivalent to sums of similar factorable determinants.[55] Each term in these sums is actually the determinant that would be obtained by using only k of the partial fractions of H_{qr}, i.e., by setting all but k of the B's in (32) equal to zero. There must be one such term for every possible choice of k partial fractions. After the expansions of the W_k determinants have been determined the U_k and V_k determinants can be expanded by treating them as certain limiting cases of W_k determinants.

When the expansions of the determinants are inserted in the equations (30) for the a_σ's, a variety of factors in differences between the infinite loss points z_1, z_2, etc., can be cancelled out. The formulae then take the form

$$\begin{aligned} a_1 &= F_1 \qquad & a_{2k-1} &= \frac{\Psi_{U(k-1)}\Psi_{W(k-1)}}{\Psi_{V(k-1)}\Psi_{Vk}} \\ a_2 &= \frac{1}{\Psi_{U1}} \qquad & a_{2k} &= \frac{\Psi_{Vk}^2}{\Psi_{U(k-1)}\Psi_{Uk}} \end{aligned} \tag{33}$$

in which the Ψ's represent the uncancelled parts of the expansions of the original determinants.

The Ψ's themselves are best expressed in terms of the quantities u_{qk}, v_{qk}, w_{qk} defined by the recursion formulae

[54] See, for instance, the chapter on functional determinants in the treatise by Scott and Mathews (13).

[55] The similar vanishing of higher degree determinants of the U_k and V_k types shows the finite nature of the continued fraction (25).

$$u_{q0} = B_q \qquad v_{qk} = \frac{u_{q(k-1)}}{(\beta_q - z_k)} \qquad u_{qk} = \frac{u_{q(k-1)}}{(\beta_q - z_k)^2} \qquad w_{qk} = \frac{u_{q(k-1)}}{(\beta_q - z_k)(\beta_q - z_{k+1})} \tag{34}$$

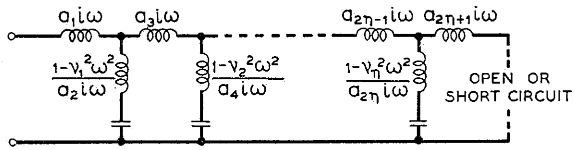
The statement of the Ψ's in terms of these quantities requires a very complicated summation and product notation in the general case. Hence, it will be better to avoid the necessity of a statement of the general case by listing enough specific cases to establish what such a statement would have to show.

The formulae for the simpler Ψ's are listed in Table I together with the additional previous relations necessary in the actual computation of ladder network coefficients. These are sufficient to indicate the general case and are also sufficient by themselves for ordinary design purposes, particularly when impedances corresponding to both ends of a network are known so that part of the elements can be determined from each end. Some additional simplifications can be obtained, however, by developing more specialized forms of the equations for specific numbers of sections. The behavior of the impedances at zero and infinity can also be used advantageously in determining one or two coefficients for which the standard formulae are most complicated or for checking purposes. Special simplified formulae have also been derived for the ladder equivalents of symmetrical lattice networks, which are normally computed first in the design of symmetrical circuits. These formulae can best be introduced at a later point, however, when the theory of lattice networks is considered.

In ordinary numerical problems, the expanded formulae do not require the extremely high precision of computation which is commonly necessary when Norton's equations are used or the formulae in terms of determinants. Similarly, the expanded formulae do not become indeterminate when coincident frequencies of infinite loss are encountered, as do the other formulae. They also lead to somewhat more straightforward numerical computations in ordinary design problems, although the extent to which this is true may vary widely from problem to problem. While the complexity of the formulae increases so rapidly with network complexity as to render them unsatisfactory for the design of ladders of more than four or possibly five sections, such complicated networks are rarely encountered. Finally, the formulae for ladders of general numbers of sections assume a form which renders them useful in such general studies as the investigation of the requirements on impedances of low-pass ladders or of the possibility of avoiding coupled coils, etc.

TABLE I

General Mid-Series Low-Pass Ladders



$$F = \frac{Z}{i\omega} = -B_0 z + B_\infty + \sum_{q=1}^{\mu} \frac{B_q}{z - \beta_q} \qquad z = \frac{1}{\omega^2} \qquad \alpha = \infty \text{ at } z = z_k = \nu_k^2$$

$$u_{q0} = B_q \qquad v_{qk} = \frac{u_{q(k-1)}}{(\beta_q - z_k)} \qquad u_{qk} = \frac{u_{q(k-1)}}{(\beta_q - z_k)^2} \qquad w_{qk} = \frac{u_{q(k-1)}}{(\beta_q - z_k)(\beta_q - z_{k+1})}$$

$$\Psi_{V1} = 1 \qquad \Psi_{V2} = \sum v_{q2} \qquad \Psi_{V3} = \sum v_{q3} v_{r3} (\beta_q - \beta_r)^2$$
$$\Psi_{V4} = \sum v_{q4} v_{r4} v_{s4} (\beta_q - \beta_r)^2 (\beta_q - \beta_s)^2 (\beta_r - \beta_s)^2$$

$$\Psi_{U1} = \sum u_{q1} + B_0 \qquad \Psi_{U2} = \sum u_{q2} u_{r2} (\beta_q - \beta_r)^2 + B_0 \sum u_{q2}$$
$$\Psi_{U3} = \sum u_{q3} u_{r3} u_{s3} (\beta_q - \beta_r)^2 (\beta_q - \beta_s)^2 (\beta_r - \beta_s)^2 + B_0 \sum u_{q3} u_{r3} (\beta_q - \beta_r)^2$$

$$\Psi_{W1} = \sum w_{q1} + B_0 \qquad \Psi_{W2} = \sum w_{q2} w_{r2} (\beta_q - \beta_r)^2 + B_0 \sum w_{q2}$$
$$\Psi_{W3} = \sum w_{q3} w_{r3} w_{s3} (\beta_q - \beta_r)^2 (\beta_q - \beta_s)^2 (\beta_r - \beta_s)^2 + B_0 \sum w_{q3} w_{r3} (\beta_q - \beta_r)^2$$

$$\sum u_q u_r = \sum_{q=1}^{\mu-1} \sum_{r=q+1}^{r=\mu} u_q u_r = u_1 u_2 + u_1 u_3 + u_2 u_3 + \cdots + u_\mu u_{\mu-1}$$

and similarly for the other sums of products of terms.

$$a_1 = B_\infty - B_0 z_1 - \sum v_{q1} \qquad a_{2k-1} = \frac{\Psi_{U(k-1)} \Psi_{W(k-1)}}{\Psi_{V(k-1)} \Psi_{Vk}}$$

$$a_2 = \frac{1}{\Psi_{U1}} \qquad a_{2k} = \frac{\Psi_{Vk}^2}{\Psi_{U(k-1)} \Psi_{Uk}}$$

$$k = 2, \cdots, \eta$$

Insertion Loss Functions and Impedances Realizable with Ladders of the Mid-Series Low-Pass Configuration

The mid-series low-pass ladder configuration turns out to be appropriate for the realization of insertion power ratios of the form

$$e^{2\alpha} = \frac{1 + \Gamma_1 \omega^2 + \Gamma_2 \omega^4 + \cdots + \Gamma_{2\eta+1} \omega^{4\eta+2}}{(1 - \nu_1^2 \omega^2)^2 (1 - \nu_2^2 \omega^2)^2 \cdots (1 - \nu_\eta^2 \omega^2)^2} \tag{35}$$

in which η represents the number of shunt branches in the ladder while the Γ_σ's and ν_σ's are arbitrary constants. Corresponding open- and short-circuit impedances realizable with mid-series low-pass ladders and subject to the previous assumptions of normal degrees for the configuration and of normal multiplicity of solutions for the element values can always be found by a straightforward method except in certain special cases corresponding to discrete choices of the Γ_σ's and ν_σ's.[56]

The first step in determining the impedances is to find a solution for the polynomials A, B, A', and B' of the general impedance theory of Part I, using the numerator and denominator of (35) as the polynomials N and P^2. It is readily shown that the multiplicity of solutions for the polynomials, which was indicated in part I, is such that the signs of A' and B' can be chosen arbitrarily as far as general realizability is concerned and also the sign of $(A + pB)$. Except in the very special cases referred to above, the impedance formulae exhibited in equations (18) in Part I yield corresponding impedances realizable with ladders of the desired type provided the signs of A', B', and $(A + pB)$ are chosen in accordance with the following conditions: $\frac{A}{P}$ must be positive at zero frequency, $\frac{B'}{B}$ must be negative at infinite frequency, and $\frac{A'}{A}$ must be positive or negative at zero frequency depending upon whether termination R_1 is greater than or less than R_2.[57]

Difficulties Encountered in Special Cases

The difficulties which can be encountered in special cases are of two types. The first type can be encountered even though the impedance functions are of the form normal for the configuration. The second

[56] By discrete choices of the constants is meant choices which can always be avoided by small changes in a single constant. It is assumed, of course, that $e^{2\alpha}$ meets the physical requirement that it must be positive and no less than $\frac{4R_1R_2}{(R_1 + R_2)^2}$ at all real frequencies.

[57] Actually all solutions are normally realizable if modified mid-series low-pass ladders are permitted. The requirement upon the sign of $\frac{A}{P}$ or $\frac{A'}{A}$ can be violated, for instance, if an ideal transformer is included in one end of the network. Similarly, the requirement upon the sign of $\frac{B'}{B}$ can normally be violated if a shunt condenser is added to one end of the ladder and if negative series inductances realizable with perfectly coupled coils are permitted.

type corresponds to the appearance of various open- or short-circuit impedances which are of reduced degree because of the coincidence of roots of the numerators and denominators of the corresponding general formulae.

It can be shown that if all the impedances are of normal degree, no finite frequency of infinite loss can coincide with a root or pole of an open- or short-circuit driving-point impedance. It follows that the corresponding values of the quantities v_{qk}, u_{qk}, w_{qk} appearing in the design formulae of Table I will all be finite. Of these quantities, u_{qk} will always be positive but v_{qk} and w_{qk} can be negative. As a result, the quantities Ψ_{Uk} formed from the u_{qk}'s in the manner indicated in Table I will all be finite but the quantities Ψ_{Vk} and Ψ_{Wk} may be zero. The vanishing of Ψ_{Wk} merely replaces a series inductance by a simple conductor but the vanishing of Ψ_{Vk} leads to the requirement of three network branches with infinite impedances at all frequencies, one shunt branch and the two adjacent series branches.[58]

If it is no longer assumed that all the multiplicity of solutions for the a_σ's are to be physical, difficulties of this type can normally be overcome by modifying the choice of the particular frequencies of infinite loss which are to be the resonances of the individual shunt branches or by choosing a different set of impedances corresponding to the same insertion loss. It is within the bounds of possibility, however, to encounter cases in which all solutions lead to the same difficulties. The ladder must then be modified to the extent of using at least one anti-resonant circuit as a series branch. This can be done by modifying properly the normal design procedure.

When there are finite frequencies of infinite loss which are coincident with roots or poles of open- or short-circuit driving-point impedances, some of the impedance functions will be of reduced degree. It is then normally possible to realize the impedances by adding terminal series or shunt branches or both to a mid-series loss-pass ladder with impedances of normal degree for its configuration. It is also usually possible, however, to obtain complete networks of the normal mid-series low-pass form by merely using impedances that are still of normal degree in computing the element values. When all impedances are of reduced degree, modifications of the normal design procedure can be used or else the general method applying to the canonical tandem section configuration.

[58] Ψ_{vk}, Ψ_{wk} may also be negative, of course, leading to negative elements, but these turn out to be realizable with coupled coils.

Elimination of Various Elements

There is no difficulty in permitting any or all of the frequencies of infinite loss to be placed at infinity. This merely requires the corresponding shunt branches to be simple condensers rather than resonant circuits. The design is carried out by setting the proper ν_σ's equal to zero in (35) and also the corresponding z_σ's representing their squares in the design formulae of Table I.

Certain power ratios of the general form of (35) also lead to the vanishing of one or more of the series inductances. One special case of this type is of particular importance. In this case, one terminal series branch vanishes while the next shunt branch is a simple capacity, which leaves a network of the type of Fig. 11. The proper form of power ratio is obtained from (35) by reducing the terms in the numerator by one and also the number of ν_σ's.[59] In other words,

$$e^{2\alpha} = \frac{1 + \Gamma_1\omega^2 + \Gamma_2\omega^4 + \cdots + \Gamma_{2\eta}\omega^{4\eta}}{(1 - \nu_1^2\omega^2)^2(1 - \nu_2^2\omega^2)^2 \cdots (1 - \nu_{\eta-1}^2\omega^2)^2} \tag{36}$$

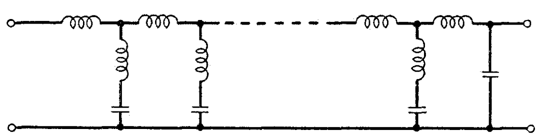


Fig. 11. A special form of mid-series low-pass ladder configuration corresponding to a special form of insertion loss function

Other Types of Ladders Commonly Encountered

Ladders of the configuration of Fig. 12, which may be described as the mid-shunt low-pass configuration, are most easily designed by redefining the coefficients $a_1, \cdots, a_{2\eta+1}$ in accordance with the designations of the branch impedances indicated in the figure. These coefficients are related to the short- and open-circuit admittances in exactly the same way that the coefficients of ladders of the mid-series type are related to the open- and short-circuit impedances. As a result, mid-series and mid-shunt ladders will produce the same insertion losses when terminated in R_1 and R_2 provided their coefficients are related by the equations

$$a_{2\sigma-1} = R_1R_2\bar{a}_{2\sigma-1} \qquad \bar{a}_{2\sigma} = R_1R_2a_{2\sigma} \tag{37}$$

[59] The vanishing of a terminal series inductance next to a resonant shunt branch, however, does not normally change the degree of the numerator in the power ratio expression in this way.

in which the a_σ's and $\bar{a}_\sigma$'s are the coefficients of the mid-series and mid-shunt ladders, respectively, and must be numbered from opposite ends in the two networks.

Negative inductances realizable with coupling in series type ladders become negative capacities in the corresponding shunt type configurations. These can be realized, however, by introducing ideal transformers in the proper way. To understand how this can be accomplished it is only necessary to note that an ideal transformer shunted by any 2-pole, such as a condenser, is equivalent to a pair of "perfectly coupled impedances" exactly similar to perfectly coupled inductances.

Other types of ladders can be designed by making use of the well-known method of frequency transformations. The previous design formulae apply to the determination of a low-pass ladder of reactance elements producing an insertion loss represented by an appropriate function of ω. Suppose some other insertion loss function is transformed

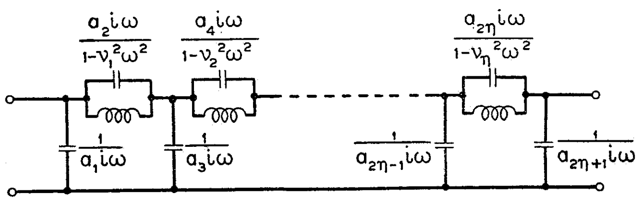


Fig. 12. The mid-shunt low-pass ladder configuration

into this same function by replacing ω by a related variable Ω. Since the reactance elements of the low-pass filter theory are merely devices for producing impedances proportional to $i\omega$ or its reciprocal, the same theory can now be used to design a corresponding ladder made up of impedances proportional to $i\Omega$ or its reciprocal. If all the elements of the original low-pass ladder are positive, the impedance branches of the transformed ladder can be realized with physical 2-poles provided $i\Omega$ represents a physical impedance function. If the original low-pass ladder includes negative inductances realizable with coupled coils, the negative elements in the transformed ladder can be realized by using ideal transformers just as in mid-shunt low-pass ladders including negative capacities.

The mid-series and mid-shunt high-pass configurations obtained by replacing inductances by capacities and vice versa in low-pass ladders can be designed by defining Ω as $-\frac{1}{\omega}$ and using power ratios represented by functions of Ω identical with the functions of ω appropriate for the low-pass configurations. Band-pass configurations can be designed by defining Ω as $\frac{\omega^2 - \omega_m^2}{\omega}$ provided their insertion loss characteristics when plotted against log (ω) are to be symmetrical about log (ω_m). They can then be realized as combinations of series and parallel resonant circuits all resonating at ω_m. The situation is exactly similar in regard to band-elimination configurations, for which Ω is $\frac{-\omega}{\omega^2 - \omega_m^2}$.[60]

The only other ladder configurations commonly encountered are the more general band-pass type indicated in Fig. 13, and its inverse. The series type illustrated can normally be designed as an equivalent simpler

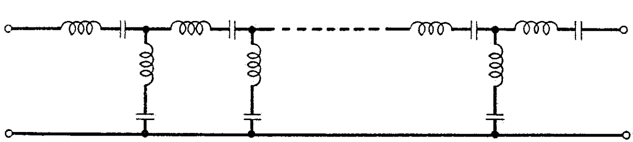


Fig. 13. The general mid-series band-pass ladder configuration

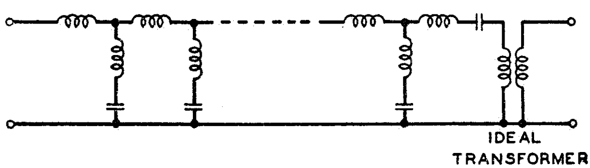


Fig. 14. An equivalent of the network of Fig. 13

network of the configuration indicated in Fig. 14.[61] It can be determined from the equivalent network by means of the so-called impedance

[60] It is sometimes more convenient, of course, to include arbitrary constant factors in the frequency transformations. In the design of a band-pass filter, for instance, it is frequently convenient to start by designing a low-pass filter with a cut-off at $\omega = 1$. Then the desired band-pass filter is obtained by replacing ω by an Ω representing

$$\frac{\omega^2 - \omega_{c1}\omega_{c2}}{\omega(\omega_{c2} - \omega_{c1})},$$

where ω_{c1}, ω_{c2} correspond to the two cut-offs of the band-pass filter.

[61] An equivalent of this type always exists except in limiting cases in which one or more shunt branches are simple inductances. A similar equivalent configuration obtained by replacing inductances by capacities and vice versa always exists except when one or more shunt branches are simple condensers. When simple inductance and capacity shunt branches are both encountered, special design methods must be used.

transformation indicated in Fig. 15A.[62] The equivalent network itself is of such a form that the formulae of Table I can be applied directly to its design. The short-circuit driving-point impedance measured at the terminals farther from the transformer, for instance, is determined by a configuration exactly the same as that determining an open-circuit impedance of a mid-series low-pass ladder with a far-end terminal shunt branch consisting of a simple condenser. The only operation not covered by the formulae of Table I is the determination of the impedance ratio of the transformer, which turns out to be determined by the behavior of the open- and short-circuit driving point impedances at zero frequency.

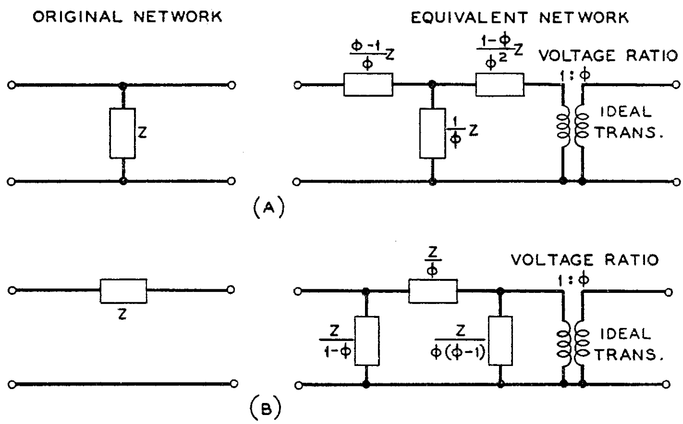


Fig. 15. Network equivalences indicating the impedance transformation principle

The configuration inverse to that of Fig. 13, which is made up of parallel resonant circuits in place of series resonant circuits, is designed by an exactly similar method with the help of the equivalence indicated in Fig. 15B.

Sufficient Conditions for Positive Elements

From the standpoint of actual construction, the perfectly coupled inductances and ideal transformers which are sometimes required for the realization of negative elements in physical ladders are highly undesirable. The two conditions stated below are sufficient to insure that

[62] The impedance transformation principle represented by Figs. 15A and 15B was discovered by Norton (14). In the determination of a network of the configuration indicated in Fig. 13 from one of the type indicated in Fig. 14 there is a considerable arbitrariness as to how the series capacity is distributed among the different series branches.

all the elements in a mid-series low-pass ladder will be positive, making unnecessary the use of coupled coils or ideal transformers in the ladder itself or in others related to it by frequency transformations or inverse relationships such as (37). While these conditions are not necessary, they turn out to be useful in demonstrating that most ladders encountered in filter design can be expected to be realizable without the use of coupling.

Assuming a mid-series low-pass configuration and impedances of the appropriate general form the first condition calls for frequencies of infinite loss which are all real (or infinite) and are greater than all the finite poles of at least one of the open-circuit driving-point impedances. The second condition requires the particular frequencies of infinite loss corresponding to the resonances of the shunt branches nearest the terminals to be also equal to or greater than all the roots of the corresponding open-circuit impedances. The sufficiency of these conditions is easily established by examining the formulae of Table I and recalling that the B's and β's are all positive.

One interesting special case is that in which all frequencies of infinite loss occur at infinity. It is seen that the above conditions are always satisfied in this special case. As a result, all power ratios of the form (35) with all ν_σ's reduced to zero are realizable with networks of the same *configuration* as constant-k low-pass image parameter filters provided $e^{2\alpha}$ meets the general physical requirement that it must be no less than $\dfrac{4R_1R_2}{(R_1+R_2)^2}$ at all real frequencies. The same situation also holds in regard to the power ratio (36) with all ν_σ's reduced to zero, the corresponding constant-k configuration merely including an odd number of "half sections." These two special power ratios include all even polynomials in ω which have unit constant terms and which satisfy the physical limit on $e^{2\alpha}$ at real frequencies.

Symmetrical and Inverse Impedance Ladders

The ladders used as filters usually have impedances and terminations meeting one of two special conditions.[63] One condition calls for an electrically symmetrical network and equal terminations. The other requires each open-circuit impedance to be the inverse of the short-circuit impedance of the other end of the network with respect to the mean of the terminations.[64]

[63] Except when they are modified to compensate for effects of dissipation.

[64] If the image impedances and transfer constants are used in the description of the networks, these conditions require equal and inverse image impedances, respectively.

If the requirement of symmetry or of inverse impedances is to be satisfied, the polynomial A' or B', respectively, must be identically zero in the formulae (18) of Part I relating the impedances to the insertion loss. An examination of the relation of A' and B' to the insertion power ratio shows that their vanishing requires expressions of the form

$$e^{2\alpha} = 1 + \left(\frac{\omega B'}{P}\right)^2 \tag{38}$$

$$A' = 0$$

in the case of symmetrical networks and equal terminations, and of the form

$$e^{2\alpha} = \frac{4R_1R_2}{(R_1 + R_2)^2} + \left(\frac{A'}{P}\right)^2 \tag{39}$$

$$B' = 0$$

in the case of inverse impedance networks.

The specification of power ratios in the forms (38) and (39) simplifies the design procedure in that no roots need be extracted in determining A' and B', the only root extraction being that involved in finding the polynomials A and B. It is readily shown that the conditions necessary for the physical realizability of these power ratios permit A', B', and P to be any even polynomials in ω with real coefficients. In other words, there will be at least one corresponding symmetrical or inverse impedance network for every power ratio of the general form

$$e^{2\alpha} = \Lambda(1 + \Phi^2) \tag{40}$$

in which Φ is any odd or even rational function of ω with real coefficients while Λ represents $\dfrac{4R_1R_2}{(R_1 + R_2)^2}$ and must be unity when Φ is an odd function.[65]

[65] When Φ is odd and Λ is different from unity, the ratio of the open-circuit driving-point impedances, which is identical with the ratio of the corresponding short-circuit impedances, will be equal to the ratio of the terminations. This permits the use of a symmetrical network combined with an ideal transformer. It is convenient to permit unequal terminations in the case of inverse impedance networks and not in the case of symmetry, or of proportional impedances, in that unequal terminations are usually necessary in the inverse impedance case if ideal transformers are to be avoided.

When all the poles of $\frac{B'}{P}$ are real or imaginary and are also finite, the power ratio (38) leading to symmetrical networks can be expressed in the form (35) appropriate for mid-series low-pass ladders of the normal configuration indicated in Fig. 9. When $\frac{A'}{P}$ obeys the same condition except for a single pole in terms of ω^2 at infinity and when also R_1 and R_2 are so chosen that α is zero at zero frequency, the power ratio (39) leading to inverse impedance networks can be expressed in the form (36) appropriate for mid-series low-pass ladders terminated at one end in shunt condensers as in Fig. 11. Exactly similar relations exist in regard to ladders of other than the mid-series low-pass configuration.

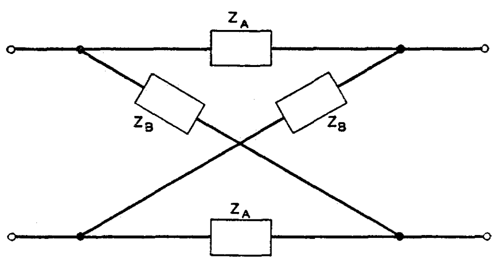


Fig. 16. The balanced lattice configuration

Lattice Networks and the Design of Symmetrical Ladders

It is well known that there is an equivalent lattice network of the type indicated in Fig. 16 corresponding to every physical network which is electrically symmetrical.[66] When the open- and short-circuit impedances are known, the impedance arms Z_A, Z_B can be very easily computed by well-known formulae. When a lattice of reactances is to be designed to produce a power ratio prescribed in the form (38), however, it is simpler to determine the impedance branches by the formulae described below rather than to first compute the open- and short-circuit impedances and then the impedance arms.

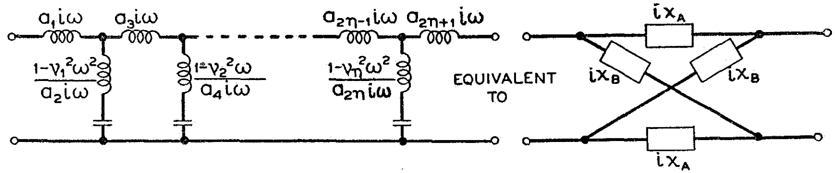
When a power ratio is prescribed in the form (38), the quantity $(P + pB')$ can obviously be expressed as the following product of two polynomials in p:

$$P + pB' = (P_1 + pB_1)(P_2 - pB_2) \tag{41}$$

[66] This was pointed out by Campbell (15).

TABLE II

Ladder Equivalents of Mid-Series Low-Pass Lattices



One Section $- \eta = 1$

$$X_A = \gamma_1\omega \qquad X_B = \frac{\gamma_2\omega^2 - \gamma_3}{\omega}$$

$$X_A = X_B \text{ at infinite loss point } \frac{1}{\omega^2} = z_1$$

$$a_1 = \gamma_1 \qquad a_3 = \gamma_1$$

$$a_2 = \frac{2}{\gamma_3}$$

Two Sections $- \eta = 2$

$$X_A = \frac{\omega(\gamma_1\omega^2 - \gamma_2)}{(\beta_1\omega^2 - 1)} \qquad X_B = \frac{\gamma_3\omega^2 - \gamma_4}{\omega}$$

$$X_A = X_B \text{ at infinite loss points } \frac{1}{\omega^2} = z_1 , z_2$$

$$a_1 = (\gamma_3 - \gamma_4 z_1) \qquad a_4 = \frac{2(\beta_1 - z_2)}{\gamma_4(2\beta_1 - z_1 - z_2)}$$

$$a_2 = \frac{2(\beta_1 - z_1)}{\gamma_4(2\beta_1 - z_1 - z_2)} \qquad a_5 = (\gamma_3 - \gamma_4 z_2)$$

$$a_3 = \gamma_4(2\beta_1 - z_1 - z_2)$$

Three Sections $- \eta = 3$

$$X_A = \frac{\omega(\gamma_1\omega^2 - \gamma_2)}{(\beta_1\omega^2 - 1)} \qquad X_B = \frac{(\gamma_3\omega^4 - \gamma_4\omega^2 + \gamma_5)}{\omega(\beta_2\omega^2 - 1)}$$

$$X_A = X_B \text{ at infinite loss points } \frac{1}{\omega^2} = z_1 , z_2 , z_3$$

$$T_1 = 2\beta_1\beta_2 - (\beta_1 + \beta_2)(z_1 + z_3) + 2z_1z_3$$

$$T_2 = 2(\beta_1 - z_3)(\beta_1 - z_1)(\beta_2 - z_1) + (z_1 - z_2)T_1$$

$$T_3 = 2(\beta_1 - z_3)(\beta_1 - z_1)(\beta_2 - z_3) + (z_3 - z_2)T_1$$

$$T_4 = \frac{(\gamma_2\beta_1 - \gamma_1)}{(\beta_1 - z_1)(\beta_1 - z_2)(\beta_1 - z_3)} = \frac{\gamma_5}{(\beta_1 - \beta_2)}$$

$$a_1 = \frac{\gamma_1 - \gamma_2 z_1}{\beta_1 - z_1} \qquad a_5 = \frac{T_3T_4(\beta_1 - z_1)}{T_1}$$

$$a_2 = \frac{2(\beta_1 - z_1)(\beta_2 - z_1)}{T_2T_4} \qquad a_6 = \frac{2(\beta_1 - z_3)(\beta_2 - z_3)}{T_3T_4}$$

$$a_3 = \frac{T_2T_4(\beta_1 - z_3)}{T_1} \qquad a_7 = \frac{\gamma_1 - \gamma_2 z_3}{\beta_1 - z_3}$$

$$a_4 = \frac{2(\beta_1 - z_2)(\beta_2 - z_2)T_1^2}{T_2T_3T_4(\beta_1 - \beta_2)}$$

where P_1, B_1, P_2, B_2 are even polynomials such that the roots of $(P_1 + pB_1)$ are all those roots of $(P + pB')$ which have negative real parts. It turns out that the impedance arms Z_A and Z_B of one lattice producing the prescribed power ratio are related to P_1, B_1, etc., by the formulae

$$Z_A = R\frac{pB_1}{P_1} \qquad Z_B = R\frac{P_2}{pB_2} \tag{42}$$

in which R represents the equal terminating resistances. The only three other lattices of reactances corresponding to the prescribed power ratio are obtained by interchanging these impedances and replacing them by their inverses with respect to R.[67]

In the design of symmetrical ladders with prescribed insertion losses, it is usually easier to determine the element values from the equivalent lattices rather than from the open- and short-circuit impedances. In the design of mid-series low-pass ladders of one, two or three shunt branches or of related networks, for instance, the special design formulae listed in Table II can be used. These formulae can be derived in much the same way as the general formulae of Table I by using an open-circuit impedance in formulating the partial fraction expansions of the determinants in (30) and noting relations between the constants required for symmetry. The relations between the constants are due to the fact that the open-circuit impedance is proportional to the sum of the two impedance arms, which requires the sums of the corresponding partial fractions to be equal at frequencies of infinite loss. Even when no other special relations are used, the initial determination of the open-circuit impedance as a sum of two functions simplifies the computation of the partial fraction expansion necessary for the use of Table I.

Part III. Special Insertion Loss Functions for Filtering Purposes

From the standpoint of filter design the general theory of Parts I and II is incomplete in two principal respects. In the first place, it gives no indication as to how to choose general insertion loss functions in such a way as to obtain efficient filter characteristics. In addition, the general design procedure is extremely complicated in numerical

[67] The polynomials B' and P are somewhat arbitrary as far as the power ratio of the form (38) is concerned. It turns out, however, that there are only four corresponding lattices in spite of this arbitrariness.

problems, involving the determination of the roots of two high degree polynomials. Part III rectifies this situation by introducing special types of realizable insertion loss functions which represent efficient filter characteristics and which also lead to relatively simple special design procedures.

Assumption of Insertion Loss Characteristics Appropriate for Symmetrical or Inverse Impedance Networks

The special insertion loss functions used for filtering purposes are appropriate for the symmetrical and inverse impedance networks described in the closing paragraphs of Part II. It turns out that any loss function representing an efficient filter characteristic at least approximates one of the types required for these particular networks. In addition, as was indicated previously, these networks can be designed with only half the root extraction which is more generally required.

Recall that insertion power ratios which are to be appropriate for symmetrical networks with equal terminations or for networks with impedances which are inverse with respect to the mean of the terminations must be of the form

$$e^{2\alpha} = \Lambda(1 + \Phi^2) \tag{43}$$

where Φ is any odd or even rational function of frequency in the two cases, respectively, and Λ represents $\dfrac{4R_1R_2}{(R_1 + R_2)^2}$ (being thus unity in the case of symmetrical networks with equal terminations). It is obvious that the poles of Φ are frequencies of infinite loss while the roots are frequencies of maximum possible insertion gain. Hence rough filter-like characteristics can be obtained by merely requiring all poles of Φ to occur in the desired attenuation bands and all roots in the desired pass bands. In order to avoid the necessity of ideal transformers or perfectly coupled inductances, however, it is best to add the restriction that Φ must be such as to make the insertion loss zero or infinite at each of the limiting frequencies zero and infinity.[68]

[68] This additional requirement is automatically satisfied when the network is to be symmetrical but not when it is to have inverse impedances. The requirement is called for in that failure to obey it always leads to the necessity of an ideal transformer or perfectly coupled inductances. On the other hand, it does not appear to be sufficient to insure that inverse impedance networks can be realized without these devices, although they are not required in the filters commonly encountered.

By way of illustration consider the special case of the low-pass filter. For a symmetrical low-pass filter

$$e^{2\alpha} = 1 + \left[S_0 \frac{\omega(\omega_1^2 - \omega^2) \cdots (\omega_\eta^2 - \omega^2)}{(1 - \nu_1^2\omega^2) \cdots (1 - \nu_\eta^2\omega^2)} \right]^2 \tag{44}$$

in which the ω_σ's, ν_σ's and S_0 are arbitrary constants. Similarly, for a low-pass filter of the inverse impedance type producing infinite loss at infinite frequency,

$$e^{2\alpha} = \frac{4R_1R_2}{(R_1 + R_2)^2} \left\{ 1 + \left[S_0 \frac{(\omega_1^2 - \omega^2) \cdots (\omega_\eta^2 - \omega^2)}{(1 - \nu_1^2\omega^2) \cdots (1 - \nu_{\eta-1}^2\omega^2)} \right]^2 \right\}. \tag{45}$$

Making each ω_σ in these expressions less than ω_c and each ν_σ less than $\frac{1}{\omega_c}$, where ω_c is the desired effective cutoff, leads to the type of low-pass filter characteristics illustrated in Figs. 17A and 17B for the case of $\eta = 3$. The insertion loss characteristics illustrated can normally be realized with the ladder networks indicated in Figs. 18A and 18B, respectively.

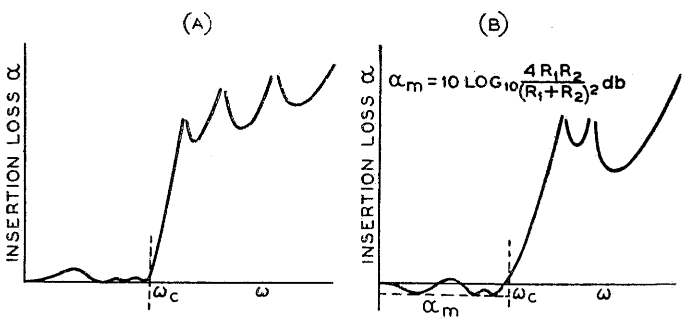


Fig. 17. Illustrations of the general form of filter characteristics obtained with symmetrical and inverse impedance networks

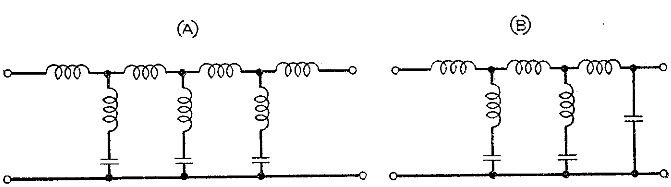


Fig. 18. Network configurations corresponding to the filter characteristics of Fig. 17

Decreasing the constant S_0 in the above expressions always decreases the pass band distortion but also decreases the suppression of attenuation band frequencies other than frequencies of infinite loss. This constant is normally made as large as is consistent with the permissible pass band distortion in order that the required suppression of unwanted frequencies may be obtained with a network of minimum complexity.[69] Zero loss at zero frequency, which is not necessarily produced by filters of the inverse impedance type, can always be obtained by properly choosing the terminations R_1 and R_2.

Tchebycheff Pass Band Parameters—Definition

Insertion loss functions of the rough filter type described above are characterized by the appearance of equal minima at arbitrary pass band frequencies and of infinite loss points at arbitrary attenuation band frequencies. Particularly useful filter characteristics and also particularly simple design procedures can be obtained by requiring the frequencies of minimum loss to be so chosen that the maxima occurring between the equal minima are themselves all equal. This leads to the type of loss characteristics illustrated in Figs. 19A and 19B for the special low-pass filters considered previously.

It is usually expedient to require also that the loss at each effective cutoff shall have the same value as at the equal maxima, as indicated in Fig. 19. In the case of inverse impedance filters transmitting zero or infinite frequencies, it is also normally advantageous to require the insertion loss to have the value zero at the equal maxima as well as at zero or infinite frequency, as in Fig. 19B. In the design of all types of filters commonly encountered, the addition of these restrictions to the requirement of equal maxima leaves the effective cutoffs and frequencies of infinite loss all arbitrary but fixes the pass band frequencies of minimum loss uniquely in terms of these parameters.[70] The characteristics obtained are the best for meeting efficiently the common type of filter requirements setting limits on permissible distortion which are constant at all pass band frequencies.

[69] The choice of S_0 is also sometimes influenced by the desire to realize the insertion loss with a ladder with no coupled inductances. Normally, however, coupled inductances are unnecessary even though S_0 is given the most advantageous value from the standpoint of loss characteristic. This will be explained later.

[70] The situation becomes more complicated only in such rare cases as multiband filters not derivable from simpler filters by means of frequency transformations.

The "equal ripple" pass band characteristics described above can be said to approximate constant losses in the Tchebycheff sense, while the corresponding pass band frequencies of minimum loss can be referred to as Tchebycheff pass band parameters. This means merely that the frequencies of minimum loss are chosen in such a way as to give the least maximum deviation from constant losses at pass band frequencies, other parameters being considered fixed.

The use of Tchebycheff parameters in network theory was first introduced by Cauer (16), who applied them to the design of image parameter filters. Filters satisfying the Tchebycheff pass band requirements of the insertion loss theory, however, are more nearly analogous to general image parameter filters than to Cauer's special Tchebycheff parameter type. A closer parallel to Cauer's use of Tchebycheff parameters is

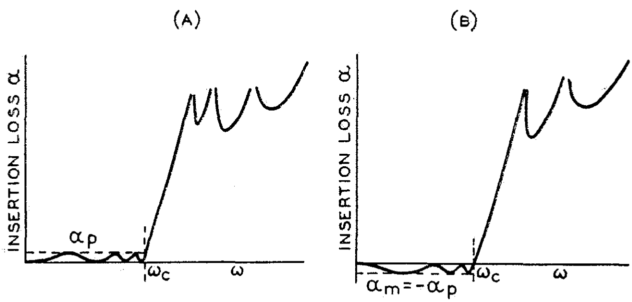


Fig. 19. Illustrations of the form of filter characteristics obtained by using Tchebycheff pass band parameters

offered by the simultaneous requirement of Tchebycheff attenuation band and pass band parameters, which will be described later.

Tchebycheff Pass Band Parameters—Theory

Tchebycheff pass band parameters can be obtained by using the following type of insertion power ratio.

$$e^{2\alpha} = \Lambda[1 + (e^{2\alpha_p} - 1) \cosh^2 (\Theta_I)]. \tag{46}$$

In this expression, Λ is $\dfrac{4R_1R_2}{(R_1 + R_2)^2}$ as before; α_p is the difference between the equal maxima and equal minima of the pass band insertion loss characteristic, as indicated in Fig. 19; and Θ_I is a function of frequency meeting the following requirements: First, Θ_I must be such that cosh (Θ_I) is an odd or even rational function of frequency, depending upon whether the power ratio is to be appropriate for symmetrical or inverse impedance filters; second, Θ_I must be pure imaginary at all pass band frequencies and must have the form $(\alpha_I + n\pi i)$ at all attenuation band frequencies, where α_I is not only real but also is infinite at various arbitrary frequencies.

It is almost obvious that the above requirements on Θ_I lead to the desired type of insertion loss characteristics. At pass band frequencies, Θ_I takes the form $i\beta_I$ leading to the power ratio

$$e^{2\alpha} = \Lambda[1 + (e^{2\alpha_p} - 1) \cos^2 (\beta_I)]. \tag{47}$$

Since the value of $\cos^2 (\beta_I)$ must lie between zero and unity, the corresponding value of α must lie in a range of amplitude α_p. On the other hand, at attenuation band frequencies

$$e^{2\alpha} = \Lambda[1 + (e^{2\alpha_p} - 1) \cosh^2 (\alpha_I)]. \tag{48}$$

This requires α to approach infinity when α_I approaches infinity. The only real problem is how to choose Θ_I in such a way that $\cosh (\Theta_I)$ is an odd or even rational function of frequency.

The forms $i\beta_I$ and $(\alpha_I + n\pi i)$ assumed by Θ_I at pass band and attenuation band frequencies, respectively, suggest that Θ_I may be similar to the image transfer constant of an image parameter filter. An analysis of the properties of general image transfer constants shows the following statements to be correct. First, if Θ_I is obtained by adding $\frac{\pi}{2}i$ to the image transfer constant of any filter (of pure reactances) with inverse image impedances, $\cosh (\Theta_I)$ will be an odd rational function of frequency. Second, if Θ_I by itself represents an image transfer constant of any symmetrical filter, $\cosh (\Theta_I)$ will be an even rational function of frequency.[71] If the theoretical pass bands of these

[71] These relations follow directly from the following equations, which can readily be derived by means of elementary network theory:

$$\cosh\left(\Theta + i\frac{\pi}{2}\right) = i\frac{\sqrt{Z_{I1}Z_{I2}}}{-Z_{O12}}$$

$$\cosh (\Theta) = \sqrt{\frac{Z_{I1}}{Z_{I2}}}\left[\frac{-Z_{O2}}{Z_{O12}}\right]$$

in which Z_{I1} and Z_{I2} are the image impedances of a filter with transfer constant Θ while Z_{O12} and Z_{O2} are corresponding open-circuit transfer and driving-point impedances. Bode exhibits equations very similar to these. See, for instance, equation (30) on page 43 of his "General Theory of Electric Wave Filters (4)."

image transfer constants coincide with the desired effective pass bands of the insertion loss filters, Θ_I will also assume the required forms $i\beta_I$ and $(\alpha_I + n\pi i)$ at pass band and attenuation band frequencies. As a result, the well established theory of the image transfer constants of image parameter filters of the inverse impedance and symmetrical types can be applied directly to the design of insertion loss filters of the symmetrical and inverse impedance types, respectively.

Tchebycheff Pass Band Parameters—Fundamental Design Procedure

In accordance with the above principles, the design of a filter with Tchebycheff pass band parameters on the insertion loss basis is carried out in terms of the image transfer constant of a hypothetical image parameter filter. It should be borne in mind, however, that there is no direct connection between the element values of the reference filter and those of the actual filter. The reference filter involves sections with matched image impedances, or the equivalent, while the actual filter does not. In addition the reference filter is normally of somewhat greater complexity than the actual filter.[72] The reference filter is introduced at all only because its transfer constant is well understood and happens to have a functional form which facilitates the determination of the insertion loss of the actual filter.

The configuration of the reference filter corresponding to an actual filter of prescribed configuration is chosen in accordance with the requirement of equal or inverse image impedances and of a one to one correspondence between the attenuation peaks of the reference filter and all the frequencies of infinite loss of the actual filter. Since the image impedances of the reference filter are only required to be properly related, a number of configurations are always possible.

By way of illustration, the two low-pass configurations used previously as illustrative insertion loss filters are shown in Fig. 20 in comparison with the simplest corresponding reference filters. It is seen that each reference filter is obtained by adding a constant-k half-section to an image parameter filter of the same configuration as the corresponding actual filter. The added half-sections supply the infinite loss points at infinity due to the corresponding poles of the series branches of the actual filters. More generally, in the case of all types of filters commonly encountered the reference filters can be obtained by adding

[72] This is because the total insertion loss of the actual filter is determined by the transfer constant of the image parameter filter, including infinite loss points corresponding to the reflection peaks of the image parameter theory.

similar constant-k half-sections of the proper types to reference filters of the same configurations as the actual filters.[73]

When it comes to the choice of the arbitrary constants of a reference filter in such a way that the corresponding actual filter will meet design specifications of the type ordinarily encountered, the procedure is actually simpler than if the reference filter itself constituted the final network. In the first place, the theoretical pass band of the reference filter is coincident with the effective pass band of the actual filter, it being unnecessary to make any allowances for ranges of high reflection losses near theoretical cut-offs. In addition, the pass band distortion α_p can be chosen directly, rather than reflection losses due to variations in the image impedances which must be corrected for interaction effects. Finally, at attenuation band frequencies at which even moderate in-

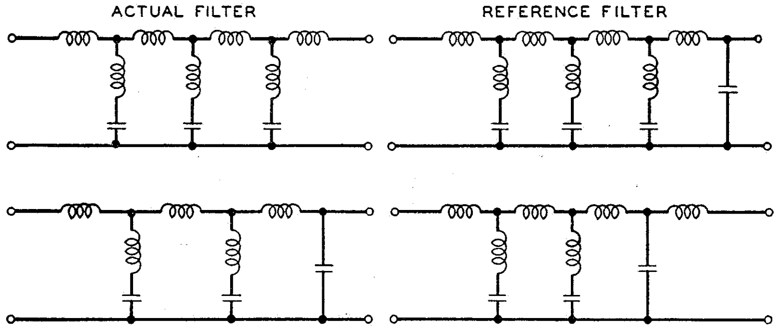


Fig. 20. Illustrations of reference image parameter filters

sertion losses are obtained, a good approximation to the loss can be computed more easily than in the case of an image parameter filter, which requires reflection losses to be added to the image attenuation. This computation is carried out by means of the following approximate formula which can be derived from equation (48) by first assuming that $\frac{1}{\Lambda}e^{2\alpha}$ and $e^{2\alpha_I}$ are both large compared with unity and then expressing α_I in terms of decibels rather than napiers:

$$\alpha - \alpha_m = [\alpha_I + 10 \log_{10} (e^{2\alpha_p} - 1) - 6.02]\, db \qquad (49)$$

where α_m is the minimum pass band insertion loss, which is negative or zero since it amounts to $10 \log_{10} (\Lambda)$ db, while α_I is identical with the image attenuation of the reference filter.

[73] Exceptions to this general rule are possible but are not apt to be encountered in ordinary filter design problems.

After the arbitrary constants have been chosen, the function cosh (Θ_I) appearing in (46) can readily be replaced by the equivalent odd or even rational function of frequency. In the case of low-pass filters, for instance, if χ is used to represent $\sqrt{1 - \frac{\omega_c^2}{\omega^2}}$ the function cosh (Θ_I) turns out to be

$$\cosh(\Theta_I) = \frac{\prod_\sigma (m_\sigma + \chi) + \prod_\sigma (m_\sigma - \chi)}{2 \prod_\sigma \sqrt{m_\sigma^2 - \chi^2}} \tag{50}$$

In this equation, ω_c represents the value of ω at the cutoff while the m_σ's represent the "m's" determining the attenuation peaks corresponding to the various half-sections of the reference image parameter filter, there being one factor in each product corresponding to each half-section. When cosh (Θ_I) is a rational function of ω, m_σ's different from unity occur in identical pairs, corresponding to halves of m-derived full sections in the reference filter. Each such pair is related to one of the ν_σ's in the expression of the form (44) or (45) for the power ratio of the actual filter by the expression

$$\nu_\sigma = (1/\omega_c)\sqrt{1 - m_\sigma^2} \tag{51}$$

Special Formulae Aiding Root Extraction

After cosh (Θ_I) has been replaced in (46) by the equivalent rational function, corresponding networks can always be designed by means of the general theory of symmetrical and inverse impedance networks described previously. The use of Tchebycheff pass band parameters, however, permits the more general design procedure to be considerably simplified by the introduction of additional special formulae. In particular, the required root extraction can be expedited by the use of special formulae giving good approximations to the required roots. How this is accomplished is described briefly below.

The required roots are those of equations of the form

$$1 \pm i\sqrt{e^{2\alpha_p} - 1}\cosh(\Theta_I) = 0 \tag{52}$$

expressed as a function of $p = i\omega$.[74] In other words, the problem is to calculate the values of p corresponding to the values of Θ_I determined by

[74] These are obviously factors of the function $\frac{1}{\Lambda} e^{2\alpha}$ determined by (46). In the case of symmetrical filters only one choice of sign need be considered, the

$$\Theta_I = \cosh^{-1}\left[\frac{\pm i}{\sqrt{e^{2\alpha_p} - 1}}\right] \tag{53}$$

If the reference image parameter filter is made up of identical m-derived sections or half-sections, Θ_I will take the form $(n\Theta_0)$ or $\left(n\Theta_0 + \frac{\pi}{2} i\right)$, in which Θ_0 represents the transfer constant of a single section or half-section. When the reference filter is not actually made up of identical sections, Θ_I can normally be approximated over a portion of the p plane including the roots of (52) by means of the Θ_I function representing a reference filter of this special type. In ordinary design problems it is only necessary to replace the "m's" describing the various sections of the reference filter with a single new "m" representing their mean.[75] Since p can normally be expressed as a simple function of Θ_0, it follows that approximations to the roots of (52) in terms of p can ordinarily be computed by replacing Θ_I by $(n\Theta_0)$ or $\left(n\Theta_0 + \frac{\pi}{2} i\right)$ in (53). When Θ_0 is the transfer constant of an m-derived low-pass half-section of "m" equal to m_0, for instance, the approximations to the roots of (52) can be calculated by means of the following formula in terms of the values of Θ_0 determined by (53):

$$p = \frac{\omega_c \sinh(\Theta_0)}{\sqrt{1 - (1 - m_0^2)\cosh^2(\Theta_0)}} \tag{54}$$

in which the sign of the square root must be such that its real part is positive.[76]

It can readily be shown that the approximations to the roots described above can be expected to be reasonably good in ordinary filter design

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corresponding roots being those of $(P + pB')$, which are required in forming (41). In the case of inverse impedance filters the roots corresponding to one choice of signs are the conjugates of those corresponding to the other.

[75] The approximating Θ_I function does not have to represent a symmetrical or inverse impedance filter as does the actual Θ_I function. In the case of low-pass or high-pass filters, the harmonic mean of the "m's" appears to have particular advantages. The same thing is true of band pass or band elimination filters if they have symmetrical loss characteristics permitting the m's to represent the derivation of confluent sections from constant-k sections.

[76] Pure imaginary values of the square root, for which the sign is not defined, are not encountered in problems in which (54) gives reasonable approximations to the required roots.

problems. Experience has shown that they are normally good enough for well-known root improvement methods to be applied directly to the determination of the actual roots to any desired precision. It also turns out that the approximate roots are of such a form that special methods can be used to obtain second approximations before the application of general root improvement methods.[77]

Other Special Formulae

Various other special formulae can be derived to facilitate the design of filters on the Tchebycheff pass band parameter basis besides those expediting the root extraction. The reflection coefficients, for instance, which measure the departure of the driving-point impedances from the corresponding terminating impedances satisfy the following limit at pass band frequencies:

$$\left|\frac{R - Z}{R + Z}\right| \leqq \sqrt{1 - e^{-2\alpha_p}} \tag{55}$$

in which Z is either driving-point impedance and R is the corresponding termination.

A rough estimate of the variation α_d of the insertion loss over the pass band which will be produced by parasitic dissipation in a low-pass or high-pass filter is usually furnished by the highly approximate relation,[78]

$$e^{\alpha_d} = 1 + \frac{2n}{Q\pi m_0^2} \coth\left[\frac{1}{n} \coth^{-1}(e^{\alpha_p})\right] \tag{56}$$

In this expression Q is the harmonic mean of the magnitudes of the reactance-resistance ratios of all the inductances and capacities, evaluated at the cut-off frequency; m_0 represents the harmonic mean of the "m's" of all the half-sections in the reference image parameter filter, and n is the total number of half sections in the reference filter.

[77] This is accomplished by using a simple transformation of variable which transforms the approximate roots into the n-th roots of a constant. One method of obtaining second approximations amounts to the use of Newton's method in conjunction with Fry's isograph. Although this machine is primarily intended for general root extraction purposes it is also convenient for applying Newton's method to the improvement of roots approximated by quantities which are all of the same magnitude.

[78] This equation furnishes even rough estimates only in the case of filters meeting ordinary specifications, particularly as regards sharpness of cutoffs and non-dissipative pass band loss.

The actual loss α_σ produced by a low-pass or high-pass ladder filter at a frequency of infinite theoretical loss not too far removed from the cutoff frequency can be estimated by means of the following formula:

$$e^{\alpha_\sigma} = \frac{2m_\sigma^2 Q}{(1 - m_\sigma^2)} e^{\alpha'} \tag{57}$$

In this equation, m_σ is the "m" corresponding to the particular peak frequency considered and α' is calculated by subtracting the attenuation of the corresponding section of the reference filter from the non-dissipative insertion loss of the actual filter and evaluating the difference at the peak frequency. Q is now best chosen as the mean "Q" of the particular elements of the ladder network whose resonance produces the peak, the ratios being evaluated at the peak frequency.

Ordinarily, equation (56) can be applied to band-pass and band-elimination filters with narrow pass or attenuation bands by merely multiplying Q by $\frac{f_{c2} - f_{c1}}{2\sqrt{f_{c2}f_{c1}}}$, where f_{c2} and f_{c1} are the cutoff frequencies and Q is now evaluated at the mean frequency of the pass or attenuation band. This assumes, however, that the loss characteristics are at last approximately symmetrical on a logarithmic frequency scale. Equation (57) can also be applied to these filters provided the peak frequencies considered are not too far removed from cutoff frequencies. This application is accomplished by multiplying Q by $\frac{|f_\infty^2 - f_{c2}f_{c1}|}{2f_\infty\sqrt{f_{c2}f_{c1}}}$, in which f_∞ is the peak frequency.

Comparison of Actual Performance of Tchebycheff Pass Band Parameter Filters with That of Image Parameter Filters

From the standpoint of actual performance, a Tchebycheff pass band parameter filter can best be compared with an image parameter filter producing the same order of magnitude of distortion at frequencies in the effective pass band. When the constant α_p measuring the pass band distortion of the insertion loss filter is of the order of tenths of a decibel, for instance, the best comparison is usually obtained by assuming the image parameter filter to have image impedances of the constant-k type. When α_p is of the order of hundredths of a decibel or smaller, on the other hand, it is usually best to assume that the reflection effects produced by the image parameter filter at pass band frequencies are reduced by the use of image impedances including impedance controlling factors.

The comparison is most clear cut when α_p is of the order of the distortion due to an image parameter filter with constant-k image impedances. Recall that the reference filter used in designing an insertion loss filter is usually obtained by adding a constant-k half-section to an image parameter filter of the same configuration as the insertion loss filter. In other words, equation (49) approximating the insertion loss of an actual filter at frequencies of high loss can usually be written in the form

$$\alpha - \alpha_m = \alpha_I' + [\alpha_0 + 10 \log_{10} (e^{2\alpha_p} - 1) - 6.02] \, db \qquad (58)$$

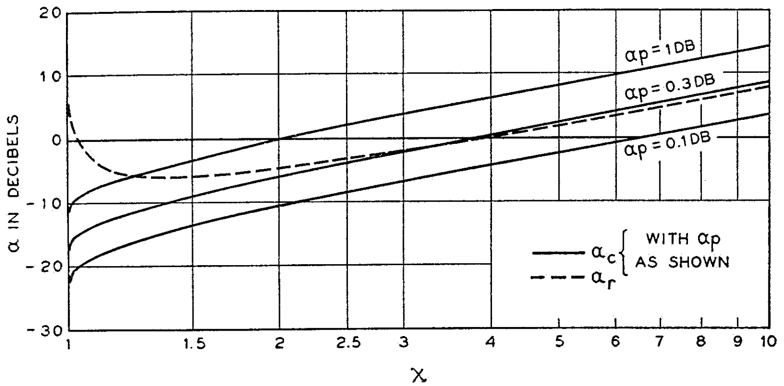
in which α_I' is the attenuation of an image parameter filter of the same configuration as the actual filter while α_0 is the attenuation of a constant-k half-section. The bracketed term in this equation is analogous to the reflection losses that would be added to the attenuation of the image parameter filter in computing its insertion loss from its attenuation α_I'. These two corrections are of the same order of magnitude when α_p is of the order of the distortion that would be produced by the image parameter filter at frequencies in its effective pass band. This is illustrated by actual curves of the two corrections in Fig. 21, assuming low-pass filters.

Although the insertion loss filter described above and the corresponding image parameter filter of the same configuration produce insertion losses of the same order of magnitude at frequencies of high loss, their pass band loss characteristics differ in one important respect. The effective cutoff of the insertion loss filter coincides with the theoretical cutoff of the image parameter filter, while the effective cutoff of the image parameter filter normally occurs at a frequency at least 10 per cent lower.[79] This means that if the image parameter filter were to be redesigned so as to have the same effective cutoff as the insertion loss filter the theoretical cutoff would have to be moved much nearer to the attenuation peaks. If the theoretical cutoff were originally within 20 per cent of an attenuation peak, for instance, it would normally have to be moved to within 10 per cent of the peak. This change would produce a marked reduction in the attenuation band loss, as is indicated by a glance at a set of attenuation curves for filters of different peak frequencies, such as that exhibited by Shea (17).

As is indicated by the above discussion, insertion loss filters of the

[79] The effective cutoff of the image parameter filter can of course be moved closer to the theoretical cutoff if impedance controlling factors are used but this requires an increase in network complexity.

Tchebycheff pass band parameter type normally represent a substantially more efficient use of the elements than image parameter filters when the pass band distortion is to be of the order of tenths of a decibel. The same general situation usually exists when the pass band distortion is to be substantially lower, calling for an image parameter filter in-



$\alpha_c = \alpha_0 + 10 \log_{10} (e^{2\alpha_p} - 1) - 6.02$ db

$\alpha_0 = \tanh^{-1} \sqrt{1 - \frac{1}{x^2}}$

= attenuation of a constant-k low-pass half-section.

$\alpha_r = 20 \log_{10} \left(\frac{x^2}{4\sqrt{x^2 - 1}} \right)$ = reflection loss between two constant-k low-pass image impedances and their nominal impedances.

$x = \frac{\omega}{\omega_c}$

Fig. 21. A comparison of the corrections involved in the determination of the insertion losses of low-pass filters of the Tchebycheff pass band parameter and image parameter types

volving impedance controlling factors, but not necessarily to the same extent.

Dependence Upon α_p of Positiveness of Ladder Elements

In Part II certain conditions were pointed out which are sufficient to insure that a mid-series low-pass ladder will not include negative inductances requiring the use of coupled coils. The first of these requires the frequencies of infinite loss to be greater than all the finite poles of at least one of the open-circuit driving-point impedances. The second requires the particular frequencies of infinite loss corresponding to the shunt branches nearest the terminals to be also equal to or greater than all the roots of the corresponding open-circuit impedances. As would be expected, the open-circuit impedances of a Tchebycheff pass band parameter filter vary in much the same way as those of a corresponding image parameter filter producing a somewhat similar loss characteristic. Specifically, a good comparison is usually obtained when the two filters produce the same order of magnitude of pass band distortion and are described by impedances which vary in the same manners at zero and infinite frequencies, rather than in inverse manners. Thus a study of the impedances of image parameter filters gives an indication of when coupled coils may be required in insertion loss filters.

The open-circuit impedances of an image parameter filter are given by

$$Z_0 = Z_I \coth(\Theta) \tag{59}$$

where Z_0 is either open-circuit impedance, Z_I is the corresponding image impedance, and Θ is the image transfer constant. Since $\coth(\Theta)$ is finite in attenuation bands, the only roots and poles of the open-circuit impedances occurring in attenuation bands are roots and poles of the image impedances.

A mid-series low-pass image impedance of the constant-k type has no root above the cutoff frequency and has no pole except at infinity. It follows that low-pass filters of the insertion loss type and filters related to them by frequency transformations can be expected to be realizable with ladders without coupled coils whenever the pass band distortion is of the order of that produced by image parameter filters with constant-k impedances.

When a single impedance controlling factor is considered, a mid-series type of low-pass insertion loss filter must be compared with an image parameter filter of the so-called mid-shunt m-terminated type, which produces an image impedance with a pole at infinity.[80] In this case,

[80] The well known ladder form of a symmetrical mid-shunt m-terminated image parameter filter has the general mid-series low-pass configuration except that the terminal series inductances do not appear. The insertion power ratio of a general mid-series low-pass ladder, when expressed in the form (35), has as many constants as there are elements in the network. It turns out that removing the terminal series branches leaves the form and degree of the power ratio unchanged, the only result being changes in the numerical values of the constants, which are greater in number than the elements of the "reduced" network. Thus the mid-shunt m-terminated image parameter low-pass filter can be described as a special case of the general mid-series low-pass configuration in which the terminal series inductances have the value zero.

the open-circuit impedances of the image parameter filter will have roots in the attenuation band but no poles except at infinity. Thus when an insertion loss filter produces a pass band distortion comparable with that obtained with image impedances each including a single impedance controlling factor, it is to be expected that no coupled coils will be required provided the frequencies of infinite loss produced by the terminal shunt branches are sufficiently high.

When two or more impedance controlling factors are included in each image impedance, there will be poles in attenuation bands and the sufficient conditions for elimination of coupling will be violated unless all infinite loss frequencies are sufficiently high. This does not mean, however, that coupling will necessarily be required whenever there are infinite loss frequencies near the cutoff, since the above conditions are sufficient but not necessary for its elimination. It merely means that the possibility of required coupling must be considered.

The general rule is obviously that reducing the pass band distortion α_p always makes the necessity of coupled coils more probable. For the range of values of α_p normally of interest it can usually be assumed that no coupling will be necessary, but unusually small values of α_p may lead to its necessity.[81]

Simultaneous Attenuation Band and Pass Band Tchebycheff Parameters

In many filter design problems, the minimum permissible suppression of unwanted frequencies is constant over prescribed effective attenuation bands at the same time that the maximum permissible distortion is constant over prescribed effective pass bands. In many such problems, the most efficient use of the arbitrary constants is obtained by requiring Tchebycheff attenuation band parameters as well as Tchebycheff pass band parameters. In other words, the frequencies of infinite loss of the previous theory are required to be so chosen that the insertion loss characteristics exhibit equal minima between the frequencies of infinite loss as well as equal maxima and equal minima in the pass bands. The simplest illustration is the design of a low-pass filter with an effective pass band extending from zero frequency to some prescribed frequency f_1 and an effective attenuation band extending

[81] The values of α_p leading to the requirement of coupling are normally extremely small, but very small values are sometimes required in order to meet restrictions on permissible impedance variations. As is indicated by (55), the impedance variations occurring at pass band frequencies approach zero only as $\sqrt{\alpha_p}$.

from a second prescribed frequency f_2 to infinity. In this special case, the simultaneous use of attenuation band and pass band Tchebycheff parameters described above leads to a loss characteristic of the type illustrated in Fig. 22 for a three-section symmetrical filter.

Mathematically, the theory of simultaneous attenuation band and pass band Tchebycheff parameters is very similar to the Tchebycheff version of the image parameter theory introduced by Cauer (16). Design formulae for the essential constants such as the frequencies of infinite loss can actually be derived by combining Cauer's theory with the theory of general Tchebycheff pass band parameters described above. Cauer's theory leads to image parameter filters characterized by equal minima of attenuation between attenuation peaks. If one of these is used as the reference image parameter filter in the design of an insertion

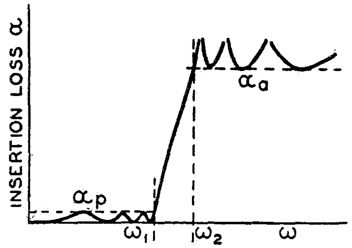


Fig. 22. An illustration of the form of filter characteristics obtained by the use of both attenuation band and pass-band Tchebycheff parameters

loss filter with Tchebycheff pass band parameters, equal minima of insertion loss between the infinite loss points are obtained. The requirement that cosh (Θ_I) must be an even or odd rational function of frequency and must be infinite at a zero or infinite attenuation band frequency, however, requires Cauer's basic theory to be modified somewhat in designing the image parameter filter.[82]

The similarity of the insertion loss filters to Cauer's image parameter filters does not extend beyond the theory involved in the choice of the design parameters. The insertion loss characteristics produced by the two types of filters are quite different. Cauer's theory would lead to insertion loss characteristics with "equal ripples" in pass bands and with equal minima between infinite loss points in attenuation bands only if

[82] The required modification can be determined by examining Cauer's various methods of modifying his theory to apply it to various image parameter problems.

interaction effects could be neglected at pass band frequencies and reflection effects at attenuation band frequencies. In actual filters, the effects of interaction and reflections are sufficient to produce marked changes in the loss characteristics.[83] The insertion loss filters, on the other hand, produce the ideal characteristics exactly, there being no corrections for such effects as reflections and interaction.[84]

Direct Derivation of the Theory

In spite of the fact that Cauer's Tchebycheff version of the image parameter theory can be used to derive the essential formulae, it is more convenient to develop the corresponding insertion loss theory by means of the more direct analysis described below. This analysis yields the same formulae as Cauer's theory and also additional useful relationships. It was used in the original derivation of the theory, which was carried out before the development of the simple theory of general Tchebycheff pass band parameters made Cauer's image parameters available for the purpose. Since other filters call for a very similar method of analysis although somewhat different formulae, only symmetrical low-pass filters will be described in detail.

Statement of Algebraic Form of the Power Ratio

It will be simplest to begin with the algebraic statement of the insertion power ratio required for simultaneous attenuation band and pass band Tchebycheff parameters, leaving its justification until later. An expression for the power ratio exhibiting a simple type of symmetry is obtained by replacing ω by a new variable Ω even though it is a low-pass filter which is under consideration. In the design of a low-pass filter, Ω is defined by

$$\Omega = \frac{\omega}{\sqrt{\omega_1 \omega_2}} \tag{60}$$

[83] At a pass band frequency at which the phase is an integral multiple of π radians, for instance, interaction effects in a symmetrical image parameter filter are such as to produce zero insertion loss whatever the reflection losses. At attenuation band frequencies the reflection effects vary from 6db reflection gain to infinite reflection loss.

[84] Even when dissipation in the elements is considered, it is frequently possible to modify the insertion loss filters in such a way that the insertion loss characteristics differ from those of non-dissipative filters only by constant losses. How this can be done will be explained in Part IV.

where ω_1 and ω_2 are the values of ω at the cutoff frequency f_1 and the limit of the effective attenuation band f_2. The values of Ω at f_1 and f_2 are then $\sqrt{k}$ and $\frac{1}{\sqrt{k}}$ where

$$k = \frac{f_1}{f_2} \tag{61}$$

The corresponding power ratio required for a symmetrical filter turns out to be of the form

$$e^{2\alpha} = 1 + \left[S_0 \frac{\Omega(\Omega^2 - \Omega_1^2) \cdots (\Omega^2 - \Omega_\eta^2)}{(1 - \Omega_1^2\Omega^2) \cdots (1 - \Omega_\eta^2\Omega^2)} \right]^2 \tag{62}$$

Consider the bracketed expression in (62). If the constant S_0 is omitted, replacing Ω by its reciprocal will replace this expression by its reciprocal. It follows that the requirement of Tchebycheff attenuation band parameters will be satisfied provided the Ω_σ's are chosen in such a way that Tchebycheff pass band parameters are obtained. As would be expected from Cauer's image parameter theory, the required values of the Ω_σ's are most simply expressed in terms of Jacobean elliptic functions. Specifically

$$\Omega_\sigma = \sqrt{k}\,\mathrm{sn}\left[\frac{2\sigma K}{2\eta + 1}, k\right] \qquad \sigma = 1, \cdots, \eta \tag{63}$$

in which the constant k representing the modulus still has the value f_1/f_2, while K is the corresponding complete integral.

Proof of Correctness of the Algebraic Expression for the Power Ratio

Suppose the power ratio (62) is transformed by replacing Ω by $\sqrt{k}\,\mathrm{sn}\,(u, k)$. The function of u obtained in this way can be simplified by replacing the Ω_σ's by their equivalents of (63) and then making use of the summation law for elliptic sines. This procedure permits the bracketed expression in (62) to be replaced by a product of elliptic sines showing the power ratio to be determined by the following pair of equations

$$e^{2\alpha} = 1 + S_0^2 \prod_{\sigma=-\eta}^{\sigma=+\eta} \left\{ k\,\mathrm{sn}^2 \left[\left(u + \frac{2\sigma K}{2\eta + 1} \right), k \right] \right\} \tag{64}$$

$$\Omega = \sqrt{k}\,\mathrm{sn}\,(u, k).$$

If u is replaced by $\left(u + \frac{2K}{2\eta + 1}\right)$ in the product in the above power ratio the net effect is to replace the factor in $\sigma = -\eta$ by one in $\sigma = \eta + 1$, the other factors being merely interchanged. The arguments of the factors in $\sigma = -\eta$ and $\sigma = \eta + 1$ differ by exactly $2K$ and must therefore be identical since this is the period of $\text{sn}^2(u, k)$. In other words, the power ratio is unchanged by replacing u by $\left(u + \frac{2K}{2\eta + 1}\right)$, which amounts to saying it is periodic in u with the period $\frac{2K}{2\eta + 1}$. The frequency variable Ω is also periodic in u but has the longer period $4K$.

For real values of u the function $\text{sn}(u, k)$ covers the range -1 to $+1$. In other words, the corresponding values of Ω coincide with the pass band and its image at negative frequencies. Thus as u increases Ω will vary cyclically over the pass band and its negative image, while in each cycle of Ω the power ratio will pass through several complete cycles, i.e., through a series of equal maxima and equal minima. A more detailed analysis of this situation shows that it requires the power ratio under consideration to represent at least a special case of Tchebycheff pass band parameters.[85] As indicated above, it then follows from the reciprocal nature of the bracketed expression in (62) that Tchebycheff attenuation band parameters are also obtained.[86]

Determination of the Roots of the Power Ratio and of Related Functions

The periodicity of the power ratio and of Ω considered as functions of u simplifies the calculation of the roots of the power ratio involved in the determination of corresponding networks. Since the period of the power ratio is a fraction of that of Ω, as soon as any one root of the

[85] The general principle of using periodic transformations in Tchebycheff parameter problems was introduced by Schelkunoff. Schelkunoff's application of the principle to Cauer's Tchebycheff theory is described by Guillemin (18).

[86] The above analysis merely proves the correctness of a particular solution for the choice of the Ω_σ's yielding attenuation band and pass band Tchebycheff parameters. The uniqueness of this solution can be demonstrated, however, by making use of the following relation, which can be shown to be necessarily satisfied:

$$\frac{\pm dy}{\sqrt{(k_1 - y^2)(1 - k_1 y^2)}} = (2\eta + 1)\frac{K_1}{K}\frac{d\Omega}{\sqrt{(k - \Omega^2)(1 - k\Omega^2)}}$$

in which y represents the bracketed expression in (62) with the constant S_0 omitted.

power ratio in terms of $i\Omega$ has been determined a number of others can be computed by merely adding multiples of the period of the power ratio to the corresponding value of u. Actually, it turns out that all the roots can be found in this way as soon as any one has been determined. If $\pm a_0$ is used to represent the only real roots, for instance, the complex roots take the form

$$i\Omega_\sigma = \frac{\pm a_0 cd_\sigma \pm iW\Omega_\sigma}{1 + a_0^2\Omega_\sigma^2} \qquad \sigma = 1, \cdots, \eta \quad (65)$$

where Ω_σ is again $\sqrt{k}\,\mathrm{sn}\left[\frac{2\sigma K}{2\eta + 1}, k\right]$ while W and cd_σ are defined by

$$\begin{aligned} W &= \sqrt{(1 + ka_0^2)\left(1 + \frac{1}{k}a_0^2\right)} \\ cd_\sigma &= \mathrm{cn}\left[\frac{2\sigma K}{2\eta + 1}, k\right]\cdot \mathrm{dn}\left[\frac{2\sigma K}{2\eta + 1}, k\right] \end{aligned} \quad (66)$$

Assuming a_0 to be the *positive* real root of the power ratio, the roots of the related polynomial $(A + pB)$ appearing in the general theory of Part I are found by using $-a_0$ as the real root and replacing $\pm a_0$ by $-a_0$ in the above formulae for the complex roots.[87] It also turns out that the roots of $(P + pB')$, which are required in the formation of equation (41) in the determination of corresponding lattices, are obtained by using $+a_0$ as the real root and replacing $\pm a_0$ by $(-)^\sigma a_0$ in the above formulae.

Introduction of a Modular Transformation on the Elliptic Functions

Formulae for computing the real zeros $\pm a_0$ and other extremely useful design relations can be obtained by the introduction of a modular transformation on the elliptic functions.[88] By means of this trans-

[87] These formulae of course give the roots in terms of $i\Omega$, defined as $\frac{i\omega}{\sqrt{\omega_1\omega_2}}$, rather than in terms of $i\omega$. The transformation from one variable to the other is carried out at any convenient point in the design procedure, commonly after the impedances of the final network branches have been determined as functions of $i\Omega$, as in the case of more complicated frequency transformations.

[88] The general theory of modular transformations (other than Landen's transformation and Jacobi's imaginary transformation) is described in very few treatises on elliptic functions. It is described in detail by Cayley (19), however, and also by Jacobi (20), who was the first to introduce it. The use of modular transformations in connection with the insertion loss theory was introduced by Norton, who also showed that their use simplifies Cauer's Tchebycheff version of the image parameter theory.

formation, the product of elliptic functions appearing in the power ratio expression (64) can be replaced by a single elliptic function of a different modulus. This transformation yields an expression of the form

$$e^{2\alpha} = 1 + (e^{2\alpha_p} - 1)\,\mathrm{sn}^2\left[(2\eta + 1)u\frac{K_1}{K}, k_1\right] \tag{67}$$

$$\Omega = \sqrt{k}\,\mathrm{sn}\,(u, k)$$

where K_1 represents the complete integral of the new modulus k_1 while α_p is again the magnitude of the equal pass band ripples of the insertion loss characteristic. The moduli k_1 and k are uniquely related in the manner described below.

Suppose K' and K_1' represent the complete elliptic integrals of moduli $\sqrt{1 - k^2}$ and $\sqrt{1 - k_1^2}$, respectively, just as K and K_1 designate the complete integrals of moduli k and k_1. Then the so-called modular constants q and q_1 are defined as $\exp\left(-\pi\frac{K'}{K}\right)$ and $\exp\left(-\pi\frac{K_1'}{K_1}\right)$, respectively. They are uniquely related to k and k_1 and are required to satisfy the equation

$$q_1 = q^{2\eta+1}. \tag{68}$$

The above relation depends upon the fact that the equivalent power ratios (64) and (67) must both have the periods $\frac{2K}{2\eta + 1}$ and $2iK'$ in terms of u. The equivalence of the power ratios can be established by showing that when the periods are the same the roots, singularities, and behavior at infinity of the functions $(e^{2\alpha} - 1)$ are identical, which requires them to be proportional in accordance with Liouville's theorem. The constant or proportionality, which relates S_0 of (64) to α_p of (67), can be found by ordinary elliptic function analysis.

Relation between Arbitrary Design Parameters

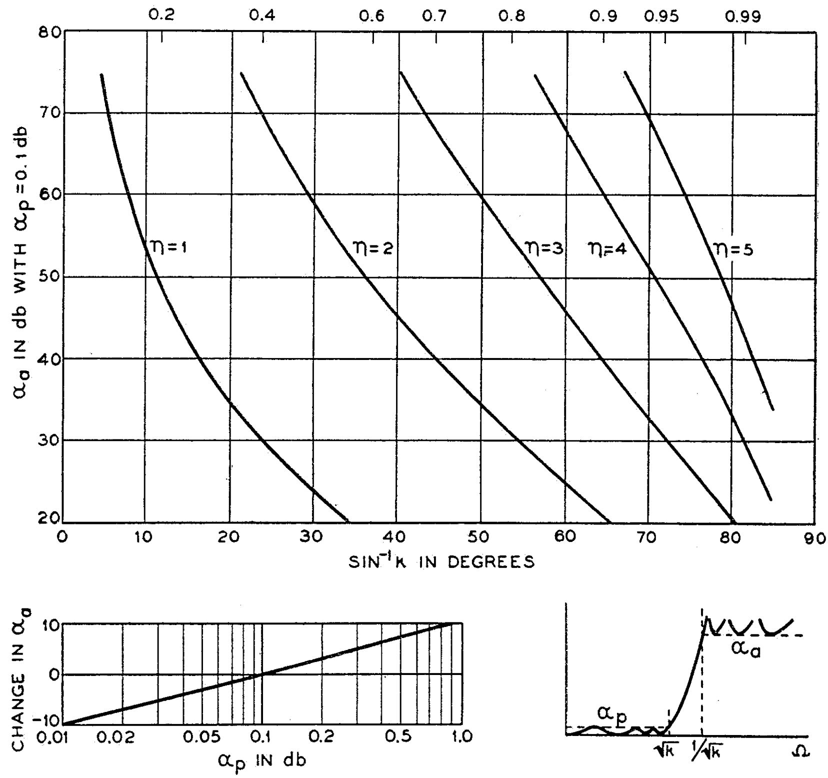
One useful formula that can be derived from (67) is a very simple relation between the arbitrary design parameters. In the first place, well-known elliptic function relations combined with the identity of q_1 and $q^{2\eta+1}$ show (67) to be equivalent to

$$e^{2\alpha} = 1 + \frac{e^{2\alpha_p} - 1}{k_1^2\,\mathrm{sn}^2\left[(2\eta + 1)u_h\frac{K_1}{K}, k_1\right]} \tag{69}$$

$$\Omega = \frac{1}{\sqrt{k}\,\mathrm{sn}\,(u_h, k)}$$

$$u_h = u + iK'.$$

For real values of u_h, the elliptic sines in these equations will vary between $+1$ and -1. This requires the corresponding values of Ω to coincide with the effective attenuation band and its image at negative



$$\alpha_a = [10 \log_{10} (e^{2\alpha_p} - 1) - 10(2\eta + 1) \log_{10} q - 12.04] \text{ db}$$

q = a function of k

η = number of ladder sections

Fig. 23. Tchebycheff attenuation band and pass band parameters—approximate relation between the design parameters

frequencies. It then follows that the loss α_a at the equal minima of the effective attenuation band will be such that

$$e^{2\alpha_a} = 1 + \frac{e^{2\alpha_p} - 1}{k_1^2} \tag{70}$$

It is obvious from this equation that k_1^2 will be quite small in the case of ordinary filters, for which α_p is small and α_a large. When k_1^2

is small, moreover, the corresponding modular constant q_1 approximates $\frac{1}{16}k_1^2$ very closely. Introducing this approximation and the identity of q_1 and $q^{2\eta+1}$ in (70), assuming $e^{2\alpha_a}$ large compared with unity, and solving for α_a gives

$$\alpha_a = [10 \log_{10} (e^{2\alpha_p} - 1) - 10(2\eta + 1) \log_{10} (q) - 12.04] \text{ db} \quad (71)$$

This equation is extremely useful in choosing the primary design parameters. The modular constant q is a measure of k, which is merely the ratio of the cutoff frequency f_1 to the attenuation band limit f_2. Thus the equation relates the sharpness of cutoff, the maximum pass band distortion α_p, the minimum attenuation band loss α_a and the number of ladder sections η. Aside from this necessary relationship these parameters are all arbitrary. The function $\log_{10} (q)$ is tabulated against $\sin^{-1} k$ in most elliptic function tables and in addition there are rapidly convergent series for computing q from k or vice versa.[89] Values of α_a determined by (71) are plotted in Fig. 23 against $\sin^{-1} k$ for various values of η, assuming α_p to be 0.1 db, together with a curve of the changes in α_a corresponding to changes in α_p.

Determination of the Real Roots of the Power Ratio

When k_1 is as small as (70) requires in ordinary design problems, $\operatorname{sn}\left[(2\eta + 1)u \frac{K_1}{K}, k_1\right]$ approximates $\sin\left[(2\eta + 1)u \frac{\pi}{2K}\right]$ for real values of u and neighboring complex values. In other words for values of Ω not too remote from the pass band, (67) can be replaced by

$$\begin{aligned} e^{2\alpha} &= 1 + (e^{2\alpha_p} - 1) \sin^2\left[(2\eta + 1)u \frac{\pi}{2K}\right] \\ \Omega &= \sqrt{k} \operatorname{sn} (u, k). \end{aligned} \quad (72)$$

The function $\operatorname{sn} (u, k)$ cannot normally be replaced by $\sin\left[u \frac{\pi}{2K}\right]$ since the modulus k represents the ratio of the cutoff frequency to the attenuation band limit and is rarely small.

The above equations can be used to compute the real roots $\pm a_0$ of the power ratio, which must be determined before the formulae (65)

[89] For a tabulation of $\log_{10} (q)$ see, for instance, Silberstein's "Synopsis of Applicable Mathematics" (21).

for the complex zeros can be used. If u is replaced by iu', the above power ratio will be zero at the real value of u' determined by

$$\sinh\left[(2\eta + 1)u' \frac{\pi}{2K}\right] = \frac{1}{\sqrt{e^{2\alpha_p} - 1}} \tag{73}$$

The real zero a_0 can be calculated from u' by the following formula derived by replacing sn (iu', k) in $i\sqrt{k}$ sn (iu', k) by Jacobi's well known equivalent:

$$a_0 = \frac{\sqrt{k}\,\mathrm{sn}\,(u', \sqrt{1 - k^2})}{\mathrm{cn}\,(u', \sqrt{1 - k^2})} \tag{74}$$

While this procedure uses the approximate formula (72) for the power ratio at an imaginary value of u, this value turns out to be such that the approximation is reasonably close in ordinary design problems. Any small error in a_0, moreover, represents merely a small change in the primary design constant α_p or α_a. A rigorous formula for u' can readily be derived from (67) but is relatively complicated in numerical applications.

Other Special Design Formulae

A large variety of additional rigorous and approximate special design formulae can be derived by similar methods. These include not only formulae for such quantities as insertion phase, impedances, etc., but also equations for determining the constants Ω_σ, a_0, etc., in terms of well-known Θ function expansions rather than from elliptic function tables. For the particular elliptic function computations required the Θ function expansions are usually easier to use than the tables.[90]

Inverse Impedance Filters with Attenuation Band and Pass Band Tchebycheff Parameters

Inverse impedance filters with attenuation band and pass band Tchebycheff parameters call for exactly the same sort of analysis as symmetrical filters. The only real difference is in the specific periodic substitution and elliptic function transformation involved. The simplest formulae are obtained by requiring zero loss at zero frequency and at the pass band minima rather than at zero frequency, at the cutoff,

[90] These Θ function expansions are the Fourier series representations of the functions. They are particularly convenient in that the arguments of the trigonometric functions involved depend only on η. The expansions are listed in various tables of formulae such as Silberstein's "Synopsis of Applicable Mathematics" (21).

and at the pass band maxima. While this leads to a somewhat less efficient use of the elements it at least permits the terminations to be equal. The corresponding power ratio turns out to be determined by the following set of equations

$$\begin{aligned} e^{2\alpha} &= 1 + (e^{2\alpha_p} - 1)\, \mathrm{sn}^2 \left[(2\eta) u \frac{K_1}{K}, k_1 \right] \\ \Omega^2 &= k\, \mathrm{sn}\,[u, k]\, \mathrm{sn} \left[\left(u + \frac{K}{\eta} \right), k \right] \\ q_1 &= q^{2\eta} \\ f_1/f_2 &= k\, \mathrm{sn}^2 \left[\frac{2\eta - 1}{2\eta} K, k \right] \end{aligned} \tag{75}$$

In these equations, Ω and f_1, f_2 are defined as in the case of symmetrical filters but k no longer represents f_1/f_2, being now defined by the last of these equations.

The more efficient requirement of zero loss at zero frequency and at the pass band maxima can be satisfied by making linear transformations on $e^{2\alpha}$ and Ω^2 in these equations. The increased efficiency is indicated by the corresponding change in the formula for f_1/f_2, which becomes

$$f_1/f_2 = k\, \mathrm{sn} \left[\frac{2\eta - 1}{2\eta} K, k \right] \tag{76}$$

Other Types of Filters

Other types of filters with attenuation band and pass band Tchebycheff parameters besides the low-pass types described above can be obtained by means of ordinary frequency transformations. These include high-pass filters and also band-pass and band-elimination filters with symmetrical loss characteristics on a logarithmic frequency scale. More general types of characteristics can be obtained by making rational transformations on the power ratio and on the square of the frequency, but this usually leads to power ratios not realizable with networks of the symmetrical or inverse impedence types. Still more general characteristics can be obtained by combining similar transformations with the use of Tchebycheff pass band parameters but general attenuation band parameters.

Instead of transforming the power ratios described previously, the analysis used in driving them can itself be modified in such a way as to lead to new types of loss characteristics. Differences between the transfer constants of reference image parameter filters with matched impedances can be used, for instance, in the design of filters on the Tchebycheff pass band parameter basis.[91] Similarly, new types of loss characteristics can be obtained by using different elliptic functions in (67). Tchebycheff attenuation band parameters can be obtained with general pass band parameters by merely replacing cosh (Θ_I) by sech (Θ_I) in (46).

Even when the power ratio is appropriate for symmetrical or inverse impedance filters, the terminations can both be chosen arbitrarily if the requirement of networks of these particular types is abandoned.[92] It is always possible, for instance, to make one termination zero or infinite.

Part IV. Dissipative Reactance Networks

The parasitic dissipation which must be present in actual reactance networks is referred to in Parts I, II and III only in connection with methods of estimating the effects of the dissipation in networks designed on a non-dissipative basis. These effects, however, frequently represent substantial differences between the characteristics actually realized and those which the non-dissipative design procedure assumes to be obtained. This makes it desirable to include in the insertion loss theory a modification applying directly to the design of dissipative reactance networks.

The logical procedure is to parallel the theory of non-dissipative networks with a synthesis theory applying to the design of networks which are made up of dissipative reactance elements and which produce prescribed insertion loss characteristics when terminated in prescribed resistances. The filter theory of Part III can then be applied to the choice of loss characteristics provided this can be done without violating the conditions necessary for the realizability of dissipative networks of the type required. Even when the configurations are required to be those which would be used in the absence of dissipation there frequently turn out to be realizable loss functions at least very closely approxi-

[91] This amounts to saying that sections described by negative "m's" can be included in the reference filters.

[92] Provided, of course, $\dfrac{4R_1R_2}{(R_1 + R_2)^2}$ is equal to or less than the smallest value assumed by $e^{2\alpha}$ at real frequencies.

mating the desired special functions of Part III except for added constant losses.[93]

The best method to use in designing a dissipative reactance network depends upon what distribution of the dissipation is required—i.e., upon the relative dissipativeness of the various elements to be used. In the networks encountered in ordinary design problems the dissipation approximates closely certain very simple distributions for which special design procedures can be developed. This situation makes it unnecessary to develop a perfectly general theory permitting the dissipation constants of all the elements to be prescribed independently. Any such general theory must obviously be undesirably complicated since it must permit the introduction of a large number of additional arbitrary constants besides those determining the insertion loss function.

Equally Dissipative Inductances and Capacities

The simplest theory is obtained by assuming a dissipative reactance network meeting the following very special requirements. Each actual inductance is required to be equivalent to an ideal inductance combined with a constant series resistance. Similarly, each actual capacity must be equivalent to an ideal capacity combined with a constant shunt resistance. Finally, all the inductance-resistance and capacity-conductance ratios are required to be equal, which amounts to requiring the phase angles of the impedances of all the reactance elements to be of equal magnitude.

Fundamental Transformation Principle

To understand the design of networks involving the uniform type of dissipation described above it is simplest to begin by examining the effect of adding such dissipation to all the reactance elements in any known network. The determination of the function of p representing any voltage ratio, impedance, or similar complex quantity describing the known network is accomplished by properly combining the im-

[93] Bode (22) discovered ways of adding resistances to image parameter filters in such a manner as to reduce the distortion due to dissipation. This procedure increases the number of reactance elements, however, by requiring certain elements to be split in two, and in addition it only partially compensates for the effects of dissipation. Bode also discovered certain general types of equalizers which can actually duplicate the discrimination characteristics of filters and in which all inductances can be in series with resistances and all capacities in parallel with resistances and can therefore be permitted to be dissipative. These equalizers are of the constant resistance type, however, and require approximately twice the number of elements appearing in non-dissipative filters.

pedances of the various elements. On a non-dissipative basis, the reactance elements produce impedances typified by Lp and $\frac{1}{Cp}$. Upon the addition of the uniform dissipation these impedances become $L(p + d)$ and $\frac{1}{C(p + d)}$, where d is a constant representing the value assumed by the equal resistance-reactance ratios at unit ω. It follows that the modification of the complex function corresponding to the addition of the dissipation amounts to the transformation represented by the substitution of $(p + d)$ for p.[94]

It follows from the above principle that the *removal* of uniform dissipation from all the reactance elements of a network will change each corresponding complex function by the transformation represented by the substitution of $(p - d)$ for p. Thus if a network in which the elements are equally dissipative is to be designed to produce a prescribed complex function it is only necessary to design any network producing the "predistorted" function obtained by replacing p by $(p - d)$ in the prescribed function. The prescribed function will then be obtained upon the addition of the required dissipation.

When the insertion loss function of a dissipative reactance network is prescribed, the first step is to find the corresponding insertion voltage ratio exactly as in the design of non-dissipative networks. The predistortion of this voltage ratio and the computation of the corresponding predistorted insertion power ratio is then followed by the design of a pure reactance network, to which the required dissipation is finally added. When the prescribed insertion loss is to be realized exactly, the corresponding voltage ratio of the form $\frac{A + pB}{P}$ yielded by the method of Part I must normally be modified by the reversal of the signs of the poles with negative real parts before predistortion, in order to obtain a predistorted voltage ratio realizable with a pure reactance network.[95] In most actual design problems, however, a more efficient use of elements is obtained by retaining $\frac{A + pB}{P}$ and only approximating the rigorous procedure in a manner to be described later, which

[94] This transformation, which was originally discovered by Bode, is described in Guillemin's "Communication Networks" (18)—Chap. VI, footnote on pp. 245, 246.

[95] It can be shown that this change in the original voltage ratio does not change the corresponding insertion loss but only the insertion phase.

usually leads to a very good approximation to the prescribed insertion loss.

Arrangement of Design Procedure to Meet Physical Conditions

In accordance with Part I, the realization of a predistorted voltage ratio with a pure reactance network calls for the following conditions. First, the predistorted voltage ratio must be expressible in the form $\frac{A + pB}{P}$, in which A, B, and P are even polynomials with real coefficients such that $(A + pB)$ has no roots with positive real parts. Second, the corresponding predistorted power ratio must be positive and no less than $\frac{4R_1R_2}{(R_1 + R_2)^2}$ at all real frequencies. Third, in order to avoid inefficiency in the use of elements it is also best to require that $(A + pB)$ and P shall have no common factors except possibly p.[96]

The usual design procedure is to choose an insertion loss function without regard to dissipation and then to require the dissipation to be small enough to yield a predistorted voltage ratio with roots with negative real parts. Replacing p by $(p - d)$ in the original voltage ratio amounts to adding d to each of the roots and poles. Thus the restriction on the dissipation requires the dissipation constant d to be smaller than the magnitude of the smallest real part of any root of the original voltage ratio, so that the addition of d can not change a negative real part into a positive real part.

In order to meet the limitation on the magnitude of the predistorted power ratio an initially unknown constant factor can be included in the original power ratio. It turns out that this constant also appears in the predistorted power ratio as nothing more than a simple multiplier and can thus easily be chosen in such a way that the predistorted power ratio is no less than $\frac{4R_1R_2}{(R_1 + R_2)^2}$.

If a prescribed insertion loss function is such that the corresponding actual voltage ratio takes the form $\frac{A + pB}{P}$ without use of common

[96] This is necessary for the exclusion of common factors from the expression $\frac{N}{P^2}$ for the corresponding power ratio. Recall that inefficient networks are usually encountered when common factors other than constants or p^2 must be combined with numerator and denominator of the simplest rational fraction expression for $e^{2\alpha}$ in order to obtain the form $\frac{N}{P^2}$.

factors other than p, the predistorted voltage ratio will not take that form.[97] Since this condition requires the finite poles of a voltage ratio to occur in pairs differing only in sign, it cannot be satisfied both before and after the addition of d to each of the poles. Modified insertion loss functions can readily be found such that the predistorted voltage ratios meet this condition rather than the actual voltage ratios. It is normally more satisfactory, however, to start with a prescribed insertion loss function of the previous type and then to realize it only to a good approximation by means of the procedure outlined below.

A power ratio is first chosen which takes the form $\frac{N}{P^2}$ without use of identical roots of N and P^2 except possibly p^2. The theory of Part I is then used to determine the corresponding voltage ratio of the form $\frac{A + pB}{P}$. The predistorted voltage ratio is next obtained by replacing p by $(p - d)$ in the numerator of this function only, the denominator P being left unchanged. A corresponding pure reactance network is then designed, after which the required dissipation is added as before. The voltage ratio actually obtained is a third function derived from $\frac{A + pB}{P}$ by replacing p by $(p + d)$ in the denominator P.

It follows from the fact that P is real at real frequencies that the actual power ratio obtained by the above procedure will approximate the original power ratio $\frac{N}{P^2}$ very closely except near the poles. This is because the first order effect of the substitution of $(p + d)$ for p upon the magnitude of a function of ω is proportional to the derivative of the phase, while the phase of P is zero at real frequencies.[98]

Special Case of Filters

The effectiveness of the design procedure described above is illustrated by the special case in which the original loss function is formed by adding a constant loss to one of the functions described in Part III as appropriate for filter purposes. In this special case, the maximum

[97] Except in the very special case in which P is a constant, i.e., in which all frequencies of infinite loss occur at infinity.

[98] The relation between the derivative of the phase and the effect of the substitution of $(p + d)$ for p upon the magnitude of a function was discovered by Bode, who used it in the derivation of his formulae for the estimation of the effect of adding dissipation to a prescribed network. It is used in exactly the same way by Guillemin (18) in deriving Mayer's solution (10) of the same Problem.

value of the dissipation constant d for which predistortion is physically possible unusually corresponds to an amount of dissipation which would produce a pass band distortion of about 6 decibels if the desired voltage ratio were not predistorted. To avoid practical difficulties it is usually necessary to keep d well below this physical limit, a margin of perhaps 30 to 50 per cent being usually needed.[99]

When d is small enough to satisfy the physical restriction, the predistortion of only the numerator of the voltage ratio frequently leads to a final loss characteristic which differs appreciably from that originally prescribed only by the substitution of rounded finite peaks for the infinite peaks in the attenuation band. It is possible, however, for the loss to be reduced substantially below the original minima between the peaks over a portion of the attenuation band.

Serious reductions in attenuation band losses can be partly compensated for by starting with complex frequencies of infinite loss so chosen that one of each conjugate pair becomes real in the final characteristic. If Tchebycheff pass band parameters are used, for instance, complex m's can be assigned to the reference image parameter filter. It turns out that these m's must be so chosen that the addition of the required dissipation to the reference filter would produce infinite attenuation peaks. This permits known methods to be used in their determination.[100] It is also possible to obtain better approximations to non-dissipative filter characteristics by including so-called compensating resistances in addition to dissipative reactances in the networks producing prescribed insertion losses, but this requires substantial changes in design procedure.

Unequally Dissipative Inductances and Capacities

Suppose now that the previous requirement of uniform dissipation is relaxed to the extent of permitting the capacities to include a different amount of dissipation than the inductances. Except in the design of such circuits as narrow-band filters, the situation immediately becomes much more complicated.[101] The predistorted voltage ratio correspond-

[99] As d approaches the physical limit the constant loss which must be added to the loss function chosen on a non-dissipative basis approaches infinity.

[100] The possibility of using complex m's to obtain infinite attenuation peaks in dissipative image parameter filters was introduced by Bode (4) and is described by Guillemin (18).

[101] A frequency transformation can be used to transform a low-pass filter of equally dissipative inductances and capacities into a band-pass filter of series and parallel resonant circuits resonating at mid-band and with associated series

ing to the removal of all dissipation from the final network can no longer be found directly from the actual voltage ratio. In addition, it now depends upon the assumption of a 4-pole including no resistances other than those produced by the equally dissipative inductances and equally dissipative capacities. It turns out, however, that a design procedure can be obtained by combining the predistortion principle appropriate for equally dissipative inductances and capacities with an additional modification of the theory of Part I.

The simplest procedure is to replace the dissipation constants of the inductances and capacities by constants d_0 and δ representing their average and one half their difference. The typical inductive and capacitive impedances then become $L(p + d_0 + \delta)$ and $\frac{1}{C(p + d_0 - \delta)}$. The predistortion method described above can be used to determine the voltage ratio that would be obtained upon the removal of the dissipation represented by d_0. The design problem is therefore solved if a method can be found for designing a network of impedances of the types $L(p + \delta)$ and $\frac{1}{C(p - \delta)}$ producing a prescribed voltage ratio. Such a design method has been found and turns out to be very similar to that described in Part I for the non-dissipative case, in which δ is zero.

Design of Networks of Oppositely Dissipative Inductances and Capacities

The fact that the above design problem can be solved depends upon the following properties of networks of impedances of the type $L(p + \delta)$ and $\frac{1}{C(p - \delta)}$. In the first place, the elements of the determinant of any network of this type can be expressed in the form.

$$Z_{jk} = \frac{L_{jk}(p^2 - \delta^2) + \frac{1}{C_{jk}}}{p - \delta} \tag{77}$$

It follows that the determinant must be equivalent to the product of $(p - \delta)^{-n}$ and an even polynomial in p, where n is the degree of the determinant. The same statement is also true of any cofactor of the

and shunt constant resistances. If the band is narrow, the resistances approximate the total effect of dissipation no matter how it is divided between the inductances and capacities.

determinant provided n is modified to take account of the change in degree. Thus each open- and short-circuit impedance of a network of this particular type is equivalent to the product of $(p - \delta)^{-1}$ and an even rational function of p.[102] In other words, the impedances differ in form from those of non-dissipative networks only by the substitution of the factor $(p - \delta)$ for p.

The important design formulae are those for determining the open- and short-circuit impedances as functions of frequency. After these have been determined the impedances obtained upon the removal of all dissipation can easily be computed. In accordance with (77), all that is necessary is to replace the factor $(p - \delta)^{-1}$ by p^{-1} in each impedance and to replace p^2 by $(p^2 + \delta^2)$ in the even rational function multiplying this factor. Formulae for computing the impedances of the dissipative network are listed below. The derivation will be omitted as in the case of the non-dissipative networks of Part I since the formulae can be checked by simple mesh computations.

Particularly simple formulae are obtained by expressing the insertion voltage ratio in the form

$$\frac{V_{20}}{V_2} = \frac{p}{p - \delta} \frac{A + pB}{P} \tag{78}$$

in which A, B and P are even polynomials as before. The additional polynomials A' and B' are then found exactly as in the determination of non-dissipative networks producing the voltage ratio $\frac{A + pB}{P}$. In other words, A' and B' are determined from the following pair of equations formed by combining (12) and (13) of Part I:

$$\begin{aligned} A^2 - p^2B^2 - \frac{4R_1R_2}{(R_1 + R_2)^2} P^2 &= J'^2(p_1'^2 - p^2) \cdots (p_n'^2 - p^2) \\ A' + pB' &= J'(p_1' - p) \cdots (p_n' - p) \end{aligned} \tag{79}$$

[102] This theorem can also be derived from the following theorem included in Guillemin's "Communication Networks" (18)—Vol. II, Chap. X, p. 445. Suppose each element in a network produces an impedance proportional to $(p + 2\alpha)$ or $\frac{1}{(p + 2\beta)}$. Then any impedance of the network takes the form of the product of $\sqrt{\frac{p + 2\alpha}{p + 2\beta}}$ and an odd rational function of $\sqrt{(p + 2\alpha)(p + 2\beta)}$. This theorem was preceded by a still more general theorem due to Bode which states that if each element of a network produces an individual impedance proportional to Z_1 or Z_2 any impedance of the complete network will be the product of Z_2 and a rational function of $\frac{Z_1}{Z_2}$.

In terms of these polynomials, the impedances of the dissipative network are given by

$$
\begin{aligned}
Z_{O1} &= R_1 \frac{(A - A') + \delta(B + B')}{(p - \delta)(B + B')} \\
Z_{O2} &= R_2 \frac{(A + A') + \delta(B + B')}{(p - \delta)(B + B')} \\
Z_{O12} &= \frac{-2R_1R_2}{(R_1 + R_2)} \frac{P}{(p - \delta)(B + B')} \\
Z_{S1} &= R_1 \frac{p^2(B - B') + \delta^2(B + B') + 2\delta A}{(p - \delta)[(A + A') + \delta(B + B')]} \\
Z_{S2} &= R_2 \frac{p^2(B - B') + \delta^2(B + B') + 2\delta A}{(p - \delta)[(A - A') + \delta(B + B')]} \\
Z_{S12} &= \frac{R_1 + R_2}{2} \left(\frac{p^2(B - B') + \delta^2(B + B') + 2\delta A}{(p - \delta)P} \right)
\end{aligned}
\tag{80}
$$

A very similar alternative set of formulae expressing the short-circuit impedances more simply rather than the open-circuit impedances can be obtained by starting with a voltage ratio derived from (78) by reversing the sign used with δ.

The specific type of voltage ratio function required depends more upon the particular configuration than in the case of non-dissipative networks. A mid-series high pass ladder, for instance, will produce an infinite loss point at $p = +\delta$ while the corresponding mid-shunt ladder will produce one at $p = -\delta$. The exact type of voltage ratio function required can ordinarily be determined by means of conventional circuit analysis. Since the dissipation associated with one type of element is negative, it may be possible for the corresponding power ratio to be less than $\frac{4R_1R_2}{(R_1 + R_2)^2}$ at certain real frequencies. In ordinary design problems, however, it is to be expected that there will be a minimum permissible value somewhere in the near neighborhood of $\frac{4R_1R_2}{(R_1 + R_2)^2}$.

The determination of $(A' + pB')$ by (79) is also more restricted than in the case of non-dissipative networks. This is because it is the multiplicity of solutions for $(A' + pB')$ which leads to the various non-equivalent configurations producing a given voltage ratio, while different configurations which can produce the same voltage ratio on a non-dissipative basis may no longer have this property after the addition of the dissipation.

Useful information bearing on the choice of B' is obtained by examining the driving-point impedances of a network determined by (80). Simple circuit analysis shows the corresponding reflection coefficients to be

$$\frac{R_1 - Z_1}{R_1 + Z_1} = \frac{p - \delta}{p}\left(\frac{A' + pB'}{A + pB}\right) - \frac{\delta}{p}$$
$$\frac{R_2 - Z_2}{R_2 + Z_2} = \frac{p - \delta}{p}\left(\frac{-A' + pB'}{A + pB}\right) - \frac{\delta}{p} \qquad (81)$$

These formulae indicate that at least one of the driving-point impedances will be the negative of the corresponding termination at zero frequency unless $\frac{A'}{pB'}$ and $\frac{A}{pB}$ approach zero at that point and unless $\frac{B'}{B}$ approaches -1. It can be shown that $\frac{A'}{pB'}$ and $\frac{A}{pB}$ will approach zero and $\frac{B'}{B}$ will approach ± 1 provided zero frequency represents an even order pole of the voltage ratio (78), or else a point at which it is finite. The above conditions will then be satisfied by the proper choice of the sign of B', which (79) leaves arbitrary.

Other Types of Uniformly Dissipative Networks

Similar methods can be used in the design of dissipative reactance networks in which each element is equivalent to an inductance or capacity combined with both series and shunt constant resistances, all elements of each kind being equally dissipative at all frequencies. This permits the simulation of the variation of dissipation with frequency encountered in actual elements.

The effect of removing both series and shunt types of dissipation from both kinds of elements in like amounts can be computed by means of a bilinear predistortion transformation. The removal of the proper amount of dissipation of each kind leaves a network in which the capacitive dissipation is the negative of the inductive dissipation. The impedances are then represented by products of $(p - \delta - \gamma p^2)^{-1}$ and even rational functions of p, where δ and γ depend only upon the dissipation constants. Methods similar to those described previously show that the corresponding impedances can be determined by replacing δ by $(\delta + \gamma p^2)$ in (78) and (80).

Theoretically, networks of still more complicated elements can be designed in exactly the same way. The only requirement is the existence of a predistortion transformation producing a predistorted voltage ratio which can be realized with a network of two kinds of elements such that the ratio of their impedances is proportional to an even rational

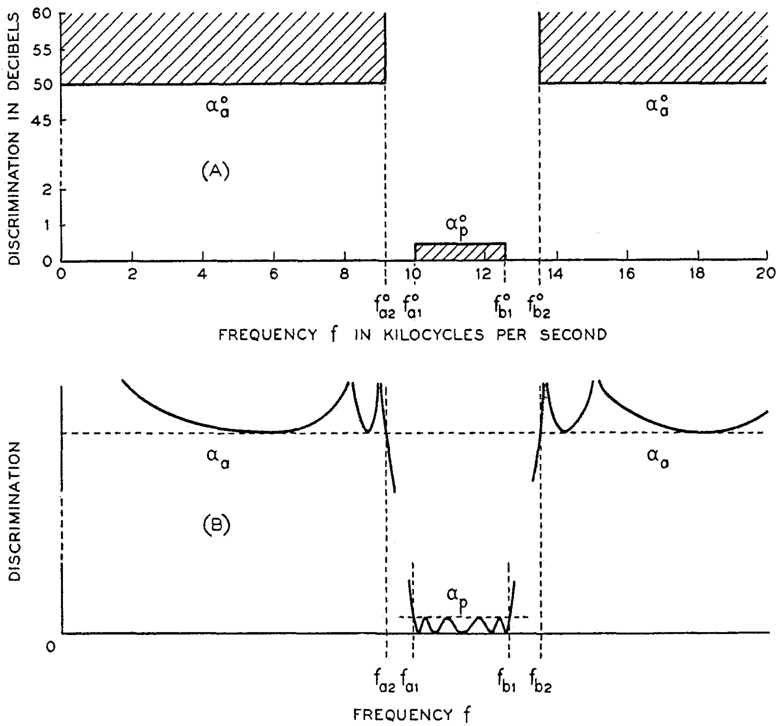


Fig. 24. A) Design specifications—When $0 < f < f^0_{a2}$, $f^0_{a1} < f < f^0_{b1}$, or $f^0_{b2} < f < \infty$, the discrimination characteristic must lie in a shaded area. (Discrimination = insertion loss minus an arbitrary constant loss.)

B) Form of discrimination characteristic meeting the design specifications.

function of frequency and such that the inverse of the predistortion transformation leaves them physical impedances. The open- and short-circuit impedances are computed by replacing δ by the proper even rational function. The principal difficulty is the determination of the required form of voltage ratio and the advantageous choice of a specific function of this form.

Concentrated Dissipation

The only non-uniformly dissipative networks which are commonly encountered approximate non-dissiptative networks to which constant resistances have been added at only a few points. Methods similar to those described above do not apply to these networks. Successive approximation methods for correcting for the effect of adding the resistances have been used successfully but are too complicated to be included here.

Illustrative Numerical Design

In the introduction, the specific design procedure appropriate for filters of a special illustrative type was outlined in order to motivate the development of the general theory. By way of conclusion, an illustrative numerical design will now be described briefly in order to clarify further the way in which the theory can be applied to actual design problems. The filters considered in the introduction were low-pass filters having the mid-series low-pass ladder configuration and satisfying the requirement of simultaneous attenuation band and pass band Tchebycheff parameters. The numerical example which will be considered now is the design of a band-pass filter which is related to one of these low-pass filters by the frequency transformation principle described in Part III.

Suppose that a band-pass filter is to be designed to meet the following specifications:

A) Frequencies to be transmitted—10.0 kc to 12.5 kc
B) Frequencies to be effectively eliminated—0 to 9.2 kc, 13.5 kc to ∞
C) Pass-band distortion—<0.4 db
D) Discrimination against unwanted frequencies—>50 db (82)
E) Dissipation compensation to be sufficient to lead to readily realizable elements
F) Final network to be a ladder involving no transformers or coupled coils

Requirements A through D are indicated graphically in Fig. 24A.

Using the notation indicated in Fig. 24A the critical frequencies f_{a1}^0, f_{a2}^0, f_{b1}^0, f_{b2}^0 satisfy approximately the equation

$$f_{a1}^0 f_{b1}^0 = f_{a2}^0 f_{b2}^0 \tag{83}$$

while the discrimination requirement in (82) is the same for both attenuation bands. This amounts to saying that the above loss requirements are symmetrical on a logarithmic frequency scale about the mid-band frequency $\sqrt{f_{a1}^0 f_{b1}^0}$. This symmetry makes it possible to simplify the design procedure by first designing a low-pass filter and then converting it to a band-pass filter by means of a frequency transformation. Due to the uniformity of the discrimination requirement at all attenuation band frequencies, the low-pass filter can be designed by the straightforward method of simultaneous attenuation band and pass band Tchebycheff parameters. The final loss characteristic then takes the form illustrated qualitatively in Fig. 24B for the special case in which the low-pass filter includes two sections.

The first operation in the design procedure is the choice of the arbitrary constants of the low-pass filter theory subject to the specificatons to be satisfied by the final band-pass filter. This choice is guided by the following considerations:

The maximum pass band distortion α_p and the minimum attenuation band discrimination α_a are the same for both the low-pass and the band-pass filters since their loss characteristics differ only as to the frequency scales. The number of sections η is also the same for the two filters provided sections of the 6-element type of configuration are considered in the band-pass case. Finally, it can be shown that the constant k measuring the sharpness of cut-off obeys the relation

$$k = \frac{f_{b1} - f_{a1}}{f_{b2} - f_{a2}} \tag{84}$$

in which f_{a1}, f_{b1}, etc., are defined by Fig. 24B and are themselves related by

$$f_{b1} f_{a1} = f_{b2} f_{a2}\,. \tag{85}$$

In accordance with Part III, the constants α_p, α_a, k, η of the low-pass theory are not all arbitrary but must be related by (71)—i.e., by

$$\alpha_a = [10 \log_{10}(e^{2\alpha_p} - 1) - 10(2\eta + 1)\log_{10}(q) - 12.04] \text{ db} \tag{86}$$

in which $\log_{10}(q)$ is a quantity which is tabulated against $\sin^{-1} k$ in most elliptic function tables.

A choice of constants which is consistent with (85) and (86) and with the original specifications in (82) is as follows:

$$
\begin{aligned}
k &= 0.62 \\
\eta &= 2 \\
\alpha_p &= 0.30 \text{ db} \\
\alpha_a &= 52.4 \text{ db} \\
f_{a1} &= 9.96 \text{ kc} \\
f_{b1} &= 12.54 \text{ kc} \\
f_{a2} &= 9.2872 \text{ kc} \\
f_{b2} &= 13.4484 \text{ kc}
\end{aligned}
\tag{87}
$$

These constants fix the final loss characteristic except for changes due to the requirement of dissipative elements.

After the constants of the Tchebycheff theory have been chosen, the corresponding voltage ratio $\frac{A + pB}{P}$ is evaluated. This is easily accomplished by means of the special formulae for the roots and poles which are included in the Tchebycheff theory. If p is used to represent $i\Omega$ rather than $i\omega$, the voltage ratio corresponding to the choice of constants indicated in (87) turns out to be

$$
\frac{V_{20}}{V_2} = \frac{A + pB}{P} = \frac{(a_0 + p)(\rho_1^2 + 2a_1 p + p^2)(\rho_2^2 + 2a_2 p + p^2)}{a_0 \rho_1^2 \rho_2^2 (1 + \Omega_1^2 p^2)(1 + \Omega_2^2 p^2)} \tag{88}
$$

in which the constants have the following values:

$$
\begin{aligned}
a_0 &= 0.37766 \\
a_1 &= 0.25943 \\
a_2 &= 0.077333 \\
\Omega_1^2 &= 0.24902 \\
\Omega_2^2 &= 0.57282 \\
\rho_1^2 &= 0.37822 \\
\rho_2^2 &= 0.66141
\end{aligned}
\tag{89}
$$

The low-pass type of voltage ratio is obtained by defining Ω as $\frac{\omega}{\sqrt{\omega_1 \omega_2}}$

as in equation (60) of Part III. The corresponding band-pass type of voltage ratio is obtained by redefining Ω in accordance with the relation

$$\Omega = \frac{\omega^2 - \omega_{b1}\omega_{a1}}{\omega\sqrt{(\omega_{b1} - \omega_{a1})(\omega_{b2} - \omega_{a2})}} \tag{90}$$

in which ω_{a1}, ω_{b1}, etc. are the values assumed by ω at the frequencies f_{a1}, f_{b1}, etc., defined by Fig. (24B).

The next operation is the predistortion of the voltage ratio (88) in order to compensate for the dissipation required in the elements of the final network. The predistortion is accomplished by replacing p by $(p - d)$ in the numerator of (88). The constant d appearing in this transformation is related to the dissipativeness of the final band-pass filter by

$$d = \frac{\sqrt{f_{a1}f_{b1}}}{\sqrt{(f_{b1} - f_{a1})(f_{b2} - f_{a2})}}(d_L + d_C) \tag{91}$$

in which d_L represents the mean of the resistance-reactance ratios of the coils, evaluated at mid-band frequency, and d_C represents the corresponding mean of the conductance-susceptance ratios of the condensers. In order to obtain a physically realizable design, d_L and d_C must be sufficiently small to render d substantially smaller than a_2 of (89). A suitable pair of values is

$$\begin{aligned} d_L &= 0.0100 \\ d_C &= 0.0025 \end{aligned} \tag{92}$$

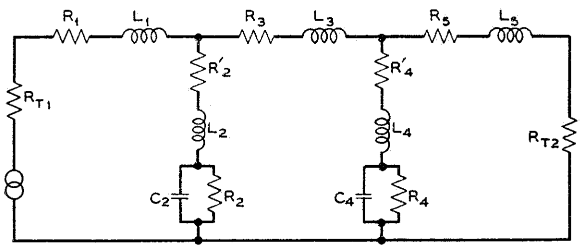
which yields

$$d = 0.04263 \tag{93}$$

In addition to the modification of the voltage ratio (88) by the predistortion transformation described above, a temporarily unknown constant factor is added to represent an added constant loss. The general theories of Parts I and II are then applied to the design of a corresponding pure reactance network, using the frequency variable Ω in place of ω.[103] By choosing the added constant loss and the ratio of terminations in the proper way, a ladder type of network is obtained

[103] If Ω is defined as $\frac{\omega}{\sqrt{\omega_1\omega_2}}$, as for a low-pass filter, it is proportional to ω and can obviously be used in place of it in the formulae of Parts I and II. If it is defined by (90), as for a band-pass filter, the resulting network is related to the low-pass network by the frequency transformation principle.

which includes no transformers or negative impedance branches and which produces a minimum transducer loss.[104] The addition of the



$$R_{T2} = 0.084427\ R_{T1}$$

$$L_1 = 1.1834 \frac{R_{T1}}{\sqrt{\omega_1 \omega_2}} \qquad R_1 = 0.050448\ R_{T1}$$

$$C_2 = \frac{1.8849}{R_{T1}\sqrt{\omega_1 \omega_2}} \qquad R_2 = 12.445\ R_{T1}$$

$$L_2 = 0.13211 \frac{R_{T1}}{\sqrt{\omega_1 \omega_2}} \qquad R_2' = 0.0056318\ R_{T1}$$

$$L_3 = 2.3227 \frac{R_{T1}}{\sqrt{\omega_1 \omega_2}} \qquad R_3 = 0.099017\ R_{T1}$$

$$C_4 = \frac{1.7650}{R_{T1}\sqrt{\omega_1 \omega_2}} \qquad R_4 = 13.290\ R_{T1}$$

$$L_4 = 0.32454 \frac{R_{T1}}{\sqrt{\omega_1 \omega_2}} \qquad R_4' = 0.013835\ R_{T_1}$$

$$L_5 = 0.85255 \frac{R_{T1}}{\sqrt{\omega_1 \omega_2}} \qquad R_5 = 0.036344\ R_{T1}$$

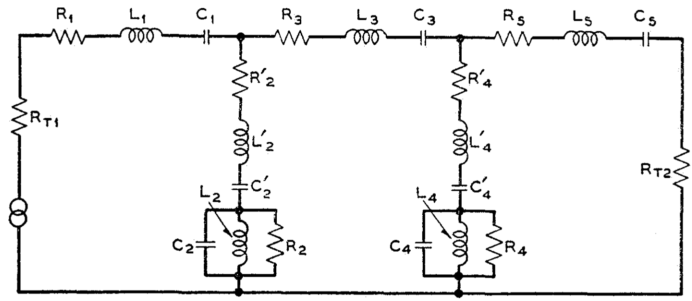
Provided R_{T1} is in ohms and ω_1 , ω_2 are in radians per second:
Inductances are in henries
Capacities are in farads
Resistances are in ohms

Fig. 25. Preliminary low-pass filter

proper resistances to this network, corresponding to the substitution of $(p + d)$ for p in the impedance functions representing the different

[104] Recall that transducer loss was defined in the introduction as the difference in level between the received power and the maximum power obtainable from the generator with any passive network.

branches, leads to a good approximation to the loss determined by (88) plus an added constant loss.



$$R_{T2} = 0.084427\ R_{T1}$$

$L_1 = 0.057481\ R_{T1}$	$C_1 = \dfrac{3.5282}{R_{T1}}$	$R_1 = 0.050448\ R_{T1}$
$L_2 = 0.0022151\ R_{T1}$	$C_2 = \dfrac{91.556}{R_{T1}}$	$R_2 = 12.445\ R_{T1}$
$L_2' = 0.0064170\ R_{T1}$	$C_2' = \dfrac{31.605}{R_{T1}}$	$R_2' = 0.0056318\ R_{T1}$
$L_3 = 0.11282\ R_{T1}$	$C_3 = \dfrac{1.7976}{R_{T1}}$	$R_3 = 0.099017\ R_{T1}$
$L_4 = 0.0023656\ R_{T1}$	$C_4 = \dfrac{85.732}{R_{T1}}$	$R_4 = 13.290\ R_{T1}$
$L_4' = 0.015764\ R_{T1}$	$C_4' = \dfrac{12.865}{R_{T1}}$	$R_4' = 0.013835\ R_{T1}$
$L_5 = 0.041411\ R_{T1}$	$C_5 = \dfrac{4.8974}{R_{T1}}$	$R_5 = 0.036344\ R_{T1}$

Provided R_{T1} is in ohms:
Inductances are in millihenries
Capacities are in microfarads
Resistances are in ohms

Fig. 26. Band-pass filter derived from the preliminary low-pass filter by a frequency transformation

If a mid-series type of ladder configuration is assumed and if Ω is defined as $\dfrac{\omega}{\sqrt{\omega_1 \omega_2}}$, the above procedure leads to the low-pass filter indicated in Fig. 25. Transforming this network by redefining Ω in terms of (90), with f_{a1}, f_{a2}, etc., fixed by (87), yields the band-pass design indicated in Fig. 26. The computed loss characteristic of this network is plotted in Fig. 27. The resistances associated with the resonant circuits can be approximated with dissipation resistances. Better approximations can be obtained, however, by replacing the shunt branches by configurations of the type indicated in Fig. 28, which can be made very nearly equivalent to them.

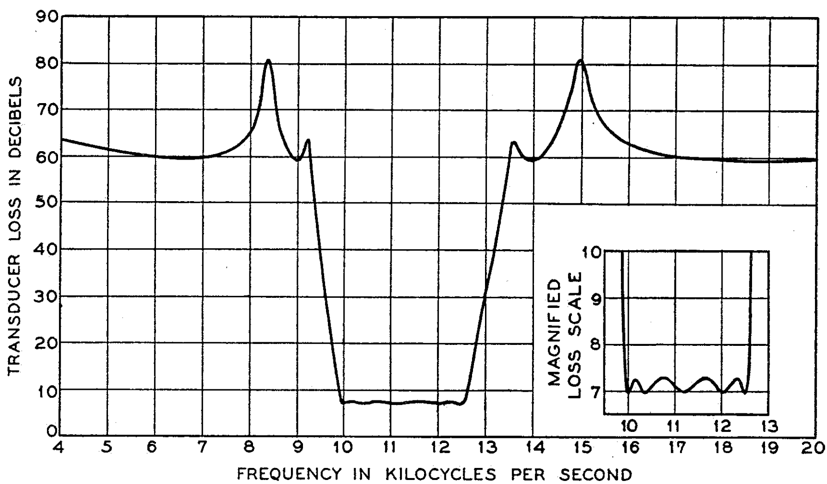


Fig. 27. Transducer loss of the network of Fig. 26 (i.e., insertion loss plus reflection loss corresponding to inequality of terminations)

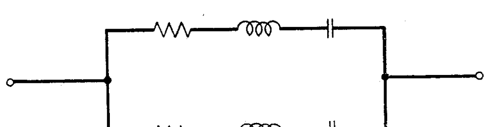


Fig. 28. Alternative shunt branch configuration

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